CHAPTER 2

Linear State Equations

2.1. Systems of linear differential equations

Consider the homogeneous system of first-order differential equations

$$\dot{x}(t) = A(t)x(t) \quad ; \quad x(t_0) = a \tag{8}$$

where A is an $n \times n$ matrix valued continuous function of time t. One can show that (8) has a unique solution through $x(t_0) = a$ on every bounded interval containing t_0 and for all $a \in \mathbb{R}^n$.

Let $\Phi_k(t, t_0)$, $k = 1, 2, \ldots, n$, be the unique solutions of (8) with initial conditions

$$x(t_0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, x(t_0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, x(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and define the $n \times n$ transition matrix

$$\Phi(t, t_0) = [\Phi_1(t, t_0), \Phi_2(t, t_)), \ldots, \Phi_n(t, t_0)].$$

Then we have

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t, s) = A(t)\Phi(t, s) \\ \Phi(s, s) = I \end{cases} .$$

Since

$$a = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

by the superposition principle, the system (8) with $x(t_0) = a$ has the solution

$$x(t) = a_1\Phi_1(t,t_0) + a_2\Phi_2(t,t_0) + \cdots + a_n\Phi_n(t,t_0)$$

that is

$$x(t) = \Phi(t,t_0)a.$$

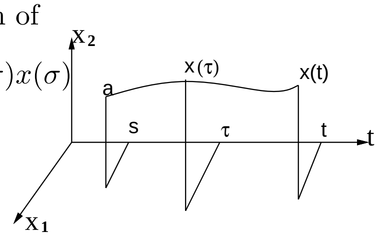
Let us list some properties of the transition matrix function Φ:

(1) $\Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s)$ for all (t,s,τ) as illustrated by the commutative diagram

$$\begin{array}{ccc} & X \xrightarrow{\Phi(\tau,s)} X & \\ \Phi(t,s) & \searrow \quad \downarrow & \Phi(t,\tau) \\ & X & \end{array}$$

Proof. Consider the unique solution of

$$\begin{cases} \dot{x}(\sigma) = A(\sigma)x(\sigma) \\ x(s) = a \end{cases}$$



Then, as illustrated in the figure $x(t) = \Phi(t,s)a$, $\quad x(t) = \Phi(t,\tau)x(\tau)$ and $x(\tau) = \Phi(\tau,s)a$, and hence

$$\Phi(t,\tau)\Phi(\tau,s)a = \Phi(t,s)a.$$

that is

$$[\Phi(t,\tau)\Phi(\tau,s) - \Phi(t,s)]a = 0$$

Since this must hold for all $a \in \mathbb{R}^n$ property (1) follows. □

(2) $\Phi(t,s)$ is nonsingular, and $\Phi(t,s)^{-1} = \Phi(s,t)$.

Proof. This follows immediately from

$$\Phi(t,s)\Phi(s,t) = I \tag{9}$$

which is a consequence of property (1). □

(3) $\frac{\partial \Phi}{\partial s}(t,s) = -\Phi(t,s)A(s)$.

Proof. Differentiating (9) with respect to s yields that

$$\frac{\partial \Phi}{\partial s}(t,s)\Phi(s,t) + \Phi(t,s)A(s)\Phi(s,t) = 0$$

Since $\Phi(s,t)$ is nonsingular, property (3) follows. □

Let us next consider the solution of the control system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_0) = a \end{cases}.$$

Set $z(t) := \Phi(t_0, t)x(t)$, i.e $x(t) = \Phi(t, t_0)z(t)$. Then,

$$\dot{z} = -\Phi(t_0, t)A(t)x(t) + \Phi(t_0, t)\dot{x}(t)$$

$$\text{so that } \begin{cases} \dot{z} = \Phi(t_0, t)B(t)u(t) \\ z(t_0) = a \end{cases}$$

and therefore,

$$z(t) = a + \int_{t_0}^{t} \Phi(t_0, s)B(s)u(s)ds,$$

or, premultiplying by $\Phi(t, t_0)$,

$$x(t) = \Phi(t, t_0)a + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)ds.$$

Notice that this equality also holds for $t \leq t_0$.

The following proposition provides us with a procedure to determine $\Phi(t, s)$.

Proposition 2.1.1. *Let X be an arbitrary $n \times n$ matrix solution of*

$$\dot{X}(t) = A(t)X(t) \quad ; \quad X(t_0) = C$$

where C is nonsingular, this is called a regular matrix solution*. Then $X(t)$ is nonsingular for all t and*

$$\Phi(t, s) = X(t)X(s)^{-1}.$$

Proof. Since C and $\Phi(t, s)$ are nonsingular (Property 2), then so is $X(t) = \Phi(t, t_0)C$ and hence

$$X(t)X(s)^{-1} = \Phi(t, t_0)CC^{-1}\Phi(t_0, s) = \Phi(t, s)$$

The proof is complete. □

For time-invariant systems determining the transition matrix function becomes much simpler in that it can be expressed in terms of the matrix exponential.

Definition 2.1.2. $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

Note that, since $\sum_{k=0}^{N} \frac{1}{k!} \|A\|^k \leq e^{\|A\|} < \infty$, the sum converges.

We collect some properties of matrix exponentials. The proofs are left for readers as exercises.

(1) If $D = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $e^D = \mathrm{diag}\,(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n})$;
(2) $e^{P^{-1}AP} = P^{-1}e^A P$;
(3) If $AB = BA$, then, $e^A e^B = e^{A+B}$; *Warning:* In general, $e^A e^B \neq e^{A+B}$.
(4) $(e^A)^{-1} = e^{-A}$;
(5) $\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$;

It follows from property (5) above that

$$X(t) = e^{At}$$

is a regular matrix solution of the time-invariant system

$$\dot{x} = Ax$$

where A is constant. Then, by Proposition 2.1.1

$$\Phi(t,s) = X(t)X(s)^{-1} = e^{At}(e^{As})^{-1} = e^{At}e^{-As},$$

that is

$$\Phi(t,s) = e^{A(t-s)} \tag{10}$$

Therefore the solution of the time-invariant system

$$\dot{x} = Ax + Bu$$

becomes

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}Bu(s)ds.$$

Remark 2.1.3. One might ask whether (10) can be generalized to the time-varying case, i.e. is it true that

$$\Phi(t,s) = \exp\{\int_s^t A(\tau)d\tau\} \tag{11}$$

when $A(t)$ is time varying? The answer is that a sufficient condition for (11) to hold is that $A(t)$ and $\int_s^t A(\tau)d\tau$ commute. □

2.2. Systems of linear difference equations

The corresponding analysis for discrete-time system is quite analogous. In fact, considering the following discrete-time system

$$x(t+1) = A(t)x(t),$$

the transition matrix is generated by

$$\Phi(t+1,s) = A(t)\Phi(t,s)$$
$$\Phi(t,t) = I.$$

It is not hard to see that

$$\Phi(t,s) = A(t-1)A(t-2)\cdots A(s) \quad \text{for } t > s$$

and $\Phi(t,s)$ is defined for $t < s$ only if $\Phi(s,t)$ is invertible, i.e. $A(k)^{-1}$ for $k = s, s+1, \ldots, t-1$. In this case, $\Phi(t,s) = \Phi(s,t)^{-1}$. In addition $\Phi(t,s)$ has the following properties.

(1) $\Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s)$;
(2) $\Phi(t,s-1) = \Phi(t,s)A(s-1)$

Hence, the solution of the control system

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

becomes

$$x(t) = \Phi(t,s)x(s) + \sum_{\sigma=s}^{t-1} \Phi(t,\sigma+1)B(\sigma)u(\sigma).$$

For time-invariant systems, the transfer matrix

$$\Phi(t,s) = A^{t-s}$$

is invertible if and only if A^{-1} exists. Note that e^{At} in continuous time corresponds to A^t in discrete time. Here lies one of the fundamental differences between the continuous-time and discrete-time setting. Indeed e^{At} is never singular, whereas A^t might be.