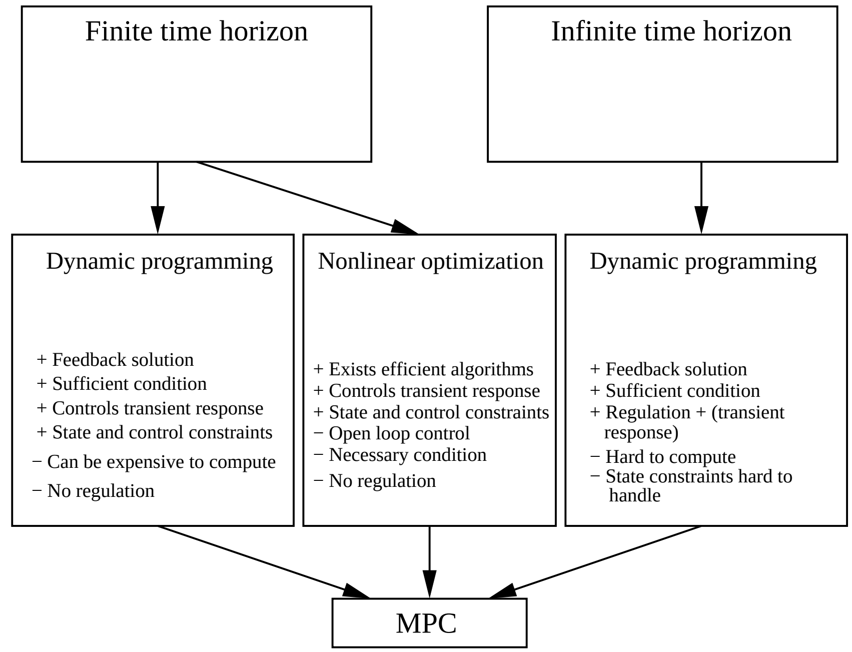
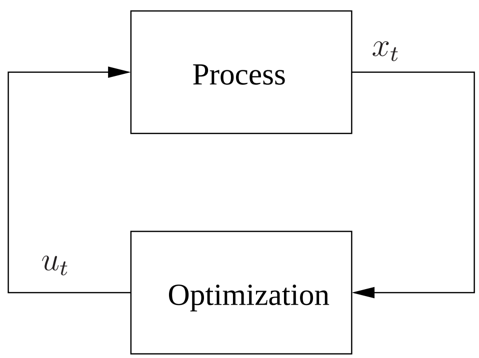
Model Predictive Control

- Overview of discrete optimal control methods
- Model predictive control (MPC)
- Online optimization vs explicit MPC
- Stability of MPC

Overview of Discrete Time Optimal Control Methods



Model Predictive Control



- The idea behind MPC is to use an optimization algorithm as controller.
- The optimization is done based on predicted state variables.
- The prediction is done based on a model which is the reason for the term "model predictive control"

The Algorithm

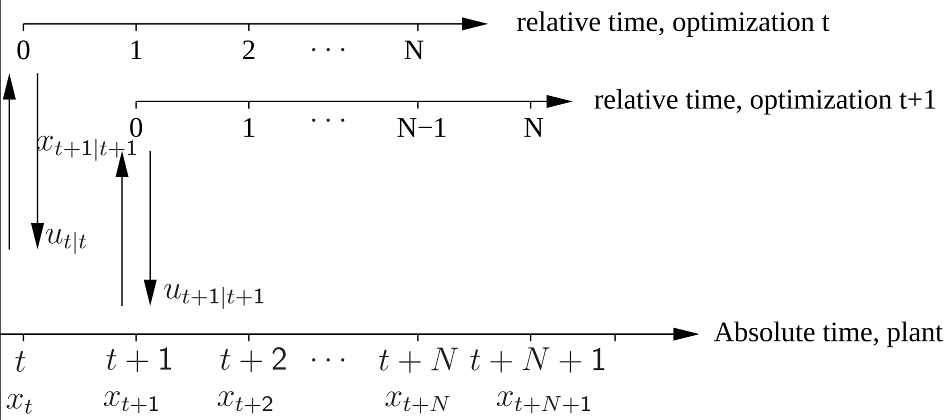
1. Measure $x_{t|t} := x_t$.
2. Determine $U_t^* = (u^*_{t|t}, u^*_{t+1|t}, \dots, u^*_{t+N-1|t})$ by solving

$$\min \sum_{k=0}^{N-1} f_0(x_{t+k|t}, u_{t+k|t}) \quad \text{subj. to} \quad \begin{cases} x_{t+k+1|t} = f(x_{t+k|t}, u_{t+k|t}), \\ x_{t+k|t} \in \mathbf{X}, \ \ u_{t+k|t} \in \mathbf{U} \\ x_{t|t} = x_t \end{cases}$$

3. Apply $u_t := u^*_{t|t}$
4. Let $t := t + 1$ and go to 1.

$x_{t+k+1|t} = f(x_{t+k|t}, u_{t+k|t})$ is the predicted state given $x_{t|t}$.

Absolute and Relative Time Axis



Properties of MPC

(+) Feedback solution

(+) Can handle state constraints and control constraints

(+) May consider mixed continuous/discrete variables (Hybrid control)

(−) Stability and feasibility cannot be guaranteed in general. However, MPC gives closed loop stability under certain reasonable assumptions.

(−) Computional complexity may be severe in general. However, MPC is tractable in many applications.

- Power systems and power electronics
- Process control
- Active suspension

Online Optimization Versus Explicit MPC

Online optimization: Solve on-line in run-time the optimization $\min_{U_t} J(x_{t|t}, U_t)$, where

$$U_t = (u_{t|t}, u_{t+1|t}, \ldots, u_{t+N-1|t})$$

$$J(x_{t|t}, U_t) = \sum_{k=0}^{N-1} f_0(x_{t+k|t}, u_{t+k|t})$$

$$\text{subj. to} \quad \begin{cases} x_{t+k+1|t} = f(x_{t+k|t}, u_{t+k|t}), \\ x_{t+k|t} \in \mathbf{X}, \ \ u_{t+k|t} \in \mathbf{U} \end{cases}$$

Find explicit solution

$$U_t^* = (\mu(0, x_{t|t}), \ldots, \mu(N-1, x_{t+N-1|t})) = \text{argmin}_{U_t} J(x_{t|t}, U_t)$$

and use $u_{t|t}^* = \mu(x_{t|t}) := \mu(0, x_{t|t})$ as state feedback function.

- The explicit solution is possible to compute in case of
  1. Linear cost, linear dynamics, and linear constraints.
  2. Quadratic cost, linear dynamics, and linear constraints (harder)
- The optimal solution is a piecewise linear map or look-up table.

Two Tractable Cases

Linear dynamics and quadratic cost

$$\min \sum_{k=0}^{N-1} x_{t+k|t}^T Q x_{t+k|t} + u_{t+k|t}^T R u_{t+k|t}$$

$$\text{subj. to} \quad \begin{cases} x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t}, \\ x_{t+k|t} \in \mathbf{X}, \;\; u_{t+k|t} \in \mathbf{U} \end{cases}$$

where $Q \geq 0$, $R > 0$, $\mathbf{X} = \{x : -1 \leq Cx \leq 1\}$, $\mathbf{U} = \{\alpha \leq u \leq \beta\}$.

- Solved using quadratic programming
  - ETH slides (Löfberg et. al.)

2. Linear dynamics and linear cost

$$\min \sum_{k=0}^{N-1} \|Qx_{t+k|t}\|_p + \|Ru_{t+k|t}\|_p$$

$$\text{subj. to} \quad \begin{cases} x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t}, \\ x_{t+k|t} \in \mathbf{X}, \;\; u_{t+k|t} \in \mathbf{U} \end{cases}$$

where $p = 1$ or $p = \infty$, i.e. if $y = \begin{bmatrix} y_1 & y_2 & \ldots & y_n \end{bmatrix}^T$ then

(a) $\|y\|_1 = \sum_{k=1}^n |y_k|$

(b) $\|y\|_\infty = \max_{k=1,\ldots,n} |y_k|$

- Solution can be obtained using linear programming
  - ETH slides (Löfberg et. al.)

Basic Stability Result

$$U_t = \begin{bmatrix} u_{t|t}, u_{t+1|t}, \ldots, u_{t+N-1|t} \end{bmatrix}$$

$$J(x_t, U_t) = \sum_{k=0}^{N-1} f_0(x_{t+k|t}, u_{t+k|t}) \quad \text{subj. to} \quad \begin{cases} x_{t+k+1|t} = f(x_{t+k|t}, u_{t+k|t}), \\ x_{t+k|t} \in \mathbf{X}, \;\; u_{t+k|t} \in \mathbf{U} \\ x_{t+N|t} = 0 \end{cases}$$

Then the MPC algorithm can be formulated as

1. Measure $x_{t|t} := x_t$.
2. Let $U_t^* = \begin{bmatrix} u_{t|t}^*, u_{t+1|t}^*, \ldots, u_{t+N-1|t}^* \end{bmatrix} = \min_{U_t} J(x_{t|t}, U_t)$
3. Apply $u_t := u_{t|t}^*$
4. Let $t := t + 1$ and go to 1.

Theorem 1. *Suppose*

(i) $f_0(0,0) = 0$ *and there exists* $\epsilon > 0$ *such that*

$$f_0(x,u) \geq \epsilon(\|x\|^2 + \|u\|^2)$$

(ii) $f(0,0) = 0$

(iii) $0 \in \mathbf{X}$ *and* $0 \in \mathbf{U}$

(iv) $\min_{U_t} J(x_{0|0}, U_t) < \infty$ *(initial feasibility)*

then the MPC algorithm on the previous slide gives a feasible and convergent closed loop trajectory (stability) $(x_t, u_t) \to (0,0)$ *as* $t \to \infty$

- For a proof see the MPC slides (Löfberg et. al.).
- For an example see the hand-out on MPC problems.

Example

ose

$$x_{t|t}, U_t) = \sum_{k=0}^{1} x_{t+k|t}^T Q x_{t+k|t} + u_{t+k|t}^2$$

$$\text{subject to } \begin{cases} x_{t+k+1|t} = A x_{t+k|t} + B u_{t+k|t} \\ -1 \le C x_{t+k|t} \le 1, \ k = 0, 1 \\ x_{t+2|t} = 0, \quad -1 \le u_{t+k|t} \le 1, \ k = 0, 1 \end{cases}$$

e

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

rmine the stability region.

Stability Region

In a separate handout we show that the stability region is

$$\mathbf{X}_0 = \{x : -1 \le x_1 + x_2 \le 1; -1 \le -2x_2 - x_1 \le 2\}$$