SF3828 Large scale convex optimization and monotone operators, 2024.

Home assignments, collection number 1.

Solve all the problems. Hand in the solutions to problems 2, 6 and 7. **Due date: March 13, 2024.**

Note: You may discuss the problems with other students, but you should write your own solutions, "in your own words".

Throughout assume that $\mathbf{R}^n$ is a Hilbert space with inner produce $\langle x, y\rangle = \sum_i x_i y_i$.

1. Compute the subdifferentials of the following functionals $f : \mathbf{R}^n \to \overline{\mathbf{R}}$:

- $f(x) = \frac{1}{2}\|x\|_2^2$,
- $f(x) = \frac{1}{2}\|x\|_2$,
- $f(x) = \frac{1}{2}\|x\|_1$.

2. Let $f : \mathbf{R}^n \to \mathbf{R}$ be the max function given by $f(x) = \max(x_1, x_2, \ldots, x_n)$.

- Show that f is convex.
- Compute the directional derivative

  $$f'(x, d) = \lim_{\alpha \to 0^+} \frac{f(x + \alpha d) - f(x)}{\alpha}$$

  for all $x, d \in \mathbf{R}^n$.
- Calculate $\partial f(x)$ for all $x \in \mathbf{R}^n$.

3. Compute the conjugate functions of the following functionals $f : \mathbf{R}^n \to \overline{\mathbf{R}}$

- $f(x) = \frac{1}{2}\|x\|_2^2$
- $f(x) = \frac{1}{2}\|x\|_2$
- $f(x) = \frac{1}{2}\|x\|_1$
- Let $L : \mathbf{R}^n \to \mathbf{R}^m$ be a linear operator. Let $b \in \mathbf{R}^m$, and let $V = \{x \mid Lx = b\}$ be nonempty. Compute the conjugate of $f(x) = \delta_V(x)$.
- Let $C = \{x \mid x \leq 0\}$. Compute the conjugate of $f(x) = \delta_C(x)$.

Here δ_C denotes the indicator function, defined by

$$\delta_C(x) := \begin{cases} 0 & \text{if} x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

4. Let $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ be CCP and let $x \in \operatorname{ri}\operatorname{dom} f$. Show that f is subdifferentiable at x. (*Hint*: You may use for example an argument building on separating hyperplanes. Note what part of the argument that do not hold when $x \in \operatorname{dom} f \setminus \operatorname{ri}\operatorname{dom} f$)

5. Let $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ be CCP. Show that the following two definitions of μ-strongly convex functions (with $\mu > 0$) are equivalent

- $f(x) - \frac{\mu}{2}\|x\|_2^2$ is convex
- $\langle \partial f(x) - \partial f(y), x - y \rangle \geq \mu \|x - y\|_2^2$ for all x, y.

6. Let $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ and suppose that f is CCP function (and that $\operatorname{ri}\operatorname{dom} f \neq \emptyset$). Further suppose, if nothing else is stated, that f is μ-strongly convex with $\mu > 0$.

1. Show that the (nonempty) level-sets of f are bounded.
2. Show that $f(y) \to \infty$ as $\|y\| \to \infty$.
3. Show that the infimum of f is attained, i.e., show that $\operatorname{argmin}_x f(x)$ exists.
4. Find a counter-example that shows $\operatorname{argmin}_x f(x)$ need not exist if f is only strictly convex.

7. For $f, g : \mathbf{R}^n \to \overline{\mathbf{R}}$ proper and convex functions, the Fenchel dual problem to

$$\inf_{x \in \mathbf{R}^n} f(x) + g(x) \tag{P}$$

is defined as

$$\sup_{y \in \mathbf{R}^n} -f^*(-y) - g^*(y). \tag{D}$$

- For $f, g : \mathbf{R}^2 \to \overline{\mathbf{R}}$ defined by

$$f(x) = \begin{cases} \max(-1, -\sqrt{x_1 x_2}), & \text{if } x_1 \geq 0, x_2 \geq 0 \\ +\infty & \text{otherwise,} \end{cases}$$
$$g(x) = \delta_{\{x | x_1 = 0\}}(x),$$

  show that there is a duality gap. In particular, $\inf(\text{P}) = 0$, while $\sup(\text{D}) = -1$.

- For $f, g : \mathbf{R} \to \overline{\mathbf{R}}$ defined by

$$f(x) = \begin{cases} x(\log(x) - 1), & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ +\infty & \text{otherwise,} \end{cases}$$
$$g(x) = \begin{cases} \frac{1}{2}x^2, & \text{if } x \leq 0 \\ +\infty & \text{otherwise,} \end{cases}$$

  show that $\inf(P) = 0$, while $\sup(D) = 0$, but that there is no optimal solution for the dual problem.

Good luck!