A Brief Introduction to Valuations on Lattice Polytopes

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These notes are based on a five-lecture summer school course given by the author at the "Summer Workshop on Lattice Polytopes" at Osaka University in 2018. We give a short introduction to the theory of valuations on lattice polytopes. Valuations are a classical topic in convex geometry. The volume plays an important role in many structural results, such as Hadwiger's famous characterization of continuous, rigid-motion invariant valuations on convex bodies. Valuations whose domain is restricted to lattice polytopes are less well-studied. The Betke-Kneser Theorem establishes a fascinating discrete analog of Hadwiger's Theorem for lattice-invariant valuations on lattice polytopes in which the number of lattice points — the discrete volume — plays a fundamental role. From there, we explore striking parallels, analogies and also differences between the world of valuations on convex bodies and those on lattice polytopes with a focus on positivity questions and links to Ehrhart theory.

Keywords: lattice polytopes, valuations, Betke-Kneser Theorem, translation-invariance, combinatorial positivity, combinatorial mixed valuations, Ehrhart theory.

1. Lattice-invariant valuations

1.1. *Hadwiger's Characterization Theorem*

A **valuation** is a map φ from a family of convex bodies $\mathcal{P}$ in $\mathbb{R}^d$ containing the empty set into an abelian group G such that $\varphi(\emptyset) = 0$ and

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q)$$

for all $P, Q \in \mathcal{P}$ for which $P \cup Q, P \cap Q \in \mathcal{P}$. The prototypical example of a valuation is the d-dimensional Euclidean volume $\mathrm{Vol}(P)$ which has many desirable properties. Besides being a valuation, it is rigid-motion invariant, positive, **d-homogeneous** (that is, $\mathrm{Vol}(tP) = t^d \mathrm{Vol}(P)$ for all convex bodies $P \subseteq \mathbb{R}^d$ and $t \geq 0$), and continuous with respect to the Hausdorff metric. A natural question is to determine all real-valued valuations with the same properties. Questions of that kind are a classical theme in valuation theory and an answer to this particular one was given by Hadwiger [12] who proved the following foundational result.

Theorem 1.1 (Hadwiger's Characterization Theorem [12]). *The family of continuous, real-valued, rigid-motion invariant valuations on convex bodies is a $(d+1)$-dimensional vector space spanned by the quermassintegrals.*

The quermassintegrals are the valuations $W_0, W_1, \ldots, W_d$ that are related to the volume via the **Steiner polynomial**

$$\mathrm{Vol}(tP + \mathcal{B}_d) \ = \ \sum_{i=0}^{d} \binom{d}{i} W_i(P) t^i \,.$$

Here, $\mathcal{B}_d$ denotes the unit ball, and "+" denotes the **Minkowski sum** which for two convex bodies P and Q is defined by $P + Q := \{p + q \colon p \in P, q \in Q\}$. It can furthermore be seen that all W_i are positive and, as coefficients of the Steiner polynomial, i-homogeneous.

We continue this section by considering another interesting valuation and recalling fundamentals from Ehrhart theory that serve us as base point and motivation in the following. We state the Betke-Kneser Theorem for lattice-invariant valuations and provide the main proof ideas given in [6]. Section 2 is devoted to translation-invariant valuations and their behavior under dilation. Our main objective is to recover a polynomiality and a reciprocity result due to McMullen [19]. In Section 3, we present a notion of positivity for translation-invariant valuations introduced in [14] that aligns with fundamental results in Ehrhart theory. In Section 4 we introduce combinatorial mixed valuations extending the notion of mixed volumes and address questions of positivity and monotonicity [13]. The purpose of these notes is to give an overview over the content of the summer school course. The focus is on results and proof ideas rather than giving full details (for which references are provided). No specific prerequisites are needed but familiarity with combinatorial concepts, in particular, with (lattice) polytopes is assumed. For further reading we recommend [1, 11, 21, 26].

1.2. *Ehrhart theory and the Betke-Kneser Theorem*

Of central interest in the following is the valuation $\mathrm{E}(P) := |P \cap \mathbb{Z}^d|$ counting the number of lattice points in a polytope $P \subset \mathbb{R}^d$. It is also called the **discrete volume** as it exhibits some strikingly parallel behavior to the volume, as we will see. For example, the discrete volume is certainly not homogeneous; we leave it to the reader to check small examples. However, if we view homogeneity as polynomiality in the dilation factor then, restricted to the class of lattice polytopes, this carries over to counting lattice points. The following result is due to Ehrhart [10] and constitutes the foundation of a field called *Ehrhart theory.*

Theorem 1.2 (Ehrhart [10]). *Let $P \subset \mathbb{R}^d$ be a lattice polytope. Then $|nP \cap \mathbb{Z}^d|$ is given by a polynomial $\mathrm{E}_P(n)$ of degree $\dim P$ for integers $n \geq 0$.*

The polynomial $\mathrm{E}_P(n)$ is called the **Ehrhart polynomial** of P. Since the discrete volume is a valuation on lattice polytopes, also the Ehrhart polynomial $\mathrm{E}_P(n) = \mathrm{E}_0(P) + \mathrm{E}_1(P)n + \cdots + \mathrm{E}_d(P)n^d$ itself as well as its coefficients $P \mapsto \mathrm{E}_i(P)$ define valuations. A fundamental question in Ehrhart theory is to characterize these coefficients. The coefficients E_i are homogeneous, however, in contrast to the quermassintegrals, they can be negative (see, e.g., [1]). Towards a characterization of Ehrhart polynomials, Stanley [23] showed that the coefficients of the Ehrhart polynomial of an r-dimensional lattice polytope with respect to the basis $\binom{n+r}{r}, \binom{n+r-1}{r}, \ldots, \binom{n}{r}$ are always nonnegative integers.

Theorem 1.3 (Stanley's Nonnegativity Theorem [23]). *Let P be a lattice polytope of dimension r. Then there are natural numbers $h_0^*(P), h_1^*(P), \ldots, h_r^*(P)$ such that*

$$\mathrm{E}_P(n) \;=\; h_0^*(P)\binom{n+r}{r} + h_1^*(P)\binom{n+r-1}{r} + \cdots + h_r^*(P)\binom{n}{r}.$$

The vector $h^*(P) = (h_0^*(P), \ldots, h_d^*(P))$ is called the h^*-vector, where $h_i^*(P) := 0$ for $i > r$ and $h_P^*(t) = h_0^*(P) + h_1^*(P)t + \cdots h_d^*(P)t^d$ is called the h^*-polynomial of P. Notice that, in contrast to the coefficients $\mathrm{E}_i(P)$ of the monomial basis, the coefficients $h_i^*(P)$ are not valuations in general. This is due to the fact that the chosen basis depends on the dimension of the polytope.

Unlike the volume, the discrete volume is not rigid-motion invariant. However, it is **lattice-invariant**, that is, invariant under transformations preserving the integer lattice $\mathbb{Z}^d$ (that is, **unimodular transformations**). This property carries over to the Ehrhart polynomial and its coefficients. Again, a natural question is to characterize all such valuations. The Betke-Kneser Theorem gives a characterization of lattice-invariant valuations and explains the particular role of the Ehrhart polynomial in valuation theory.

Theorem 1.4 (Betke-Kneser Theorem [6]). *The family of real-valued, lattice-invariant valuations on lattice polytopes is a $(d{+}1)$-dimensional vector space spanned by the coefficients of the Ehrhart polynomial.*

In the remainder of this section we outline the approach taken in [6] to prove this theorem.

1.3. *Valuations and groups*

A union $P = P_1 \cup \cdots \cup P_m$ of d-dimensional polytopes $P_1, \ldots, P_m$ is a **dissection** of a d-dimensional polytope P if $\dim(P_i \cap P_j) < d$ for all $i \neq j$. Let $(\tilde{F}^d, +)$ be the free abelian group generated by

$$\{[\![P]\!] \colon P \quad d-\text{dimensional lattice polytope in } \mathbb{R}^d\}$$

and let $\tilde{R}^d$ be the collection of the following two types of relations:

- $[\![P]\!] - [\![T(P)]\!]$ for any d-dimensional lattice polytope P and any unimodular transformation T.
- $[\![P]\!] - \sum_{i=1}^m [\![P_i]\!]$ for any dissection $P = P_1 \cup \cdots \cup P_m$ into lattice polytopes $P_1, \ldots, P_m$.

It turns out that $\tilde{\pi}^d := \tilde{F}^d/\tilde{R}^d$ has a very simple structure, namely that of a cyclic group. Let $\Delta_d = \operatorname{conv}(0, e_1, \ldots, e_d)$ be the d-dimensional standard simplex.

Theorem 1.5 ([6]). *The group $\tilde{\pi}^d$ is the infinite free cyclic group generated by $[\![\Delta_d]\!]$.*

For the proof the concept of visibility can be employed. A face F of a d-dimensional polytope P is **visible** from a point $q \in \mathbb{R}^d$ if $[q, p) \cap P = \emptyset$ for all points $p \in F$. Here, $[q, p)$ denotes the half-open segment $\{tq + (1-t)p \colon 0 < t \leq 1\}$. A face that is not visible is called **invisible**. A point $q \in \mathbb{R}^d$ is **general** with respect to P if q is not contained in any facet defining hyperplane of P. Visible and invisible facets can be used to dissect $\operatorname{conv}(P \cup \{q\})$ in two different ways. For that, let $F_1, \ldots, F_m$ be the facets of P and let $I_q(P) = \{i \in [m] \colon F_m \text{ visible}\}$. Then

$$\operatorname{conv}(P \cup \{q\}) = P \cup \bigcup_{i \in I_q(P)} \operatorname{conv}(F_i \cup \{q\}), \tag{1}$$

$$= \bigcup_{i \notin I_q(P)} \operatorname{conv}(F_i \cup \{q\}), \tag{2}$$

and the right hand side of (1) and (2) define dissections of $\operatorname{conv}(P \cup \{q\})$.

Proof idea of Theorem 1.5. Since every lattice polytope can be triangulated into empty lattice simplices (that is, simplices whose only lattice points are their vertices), it suffices to show that for every empty lattice simplex S, $[\![S]\!] = \mathrm{V}(S)[\![\Delta_d]\!]$ where $\mathrm{V}(S) = d!\mathrm{Vol}(S)$ denotes the normalized volume. The proof is by induction on $\mathrm{V}(S)$. If $\mathrm{V}(S) = 1$ then there is a unimodular transformation T with $T(\Delta_d) = S$, and we are done. If S is an empty lattice simplex with facets $F_1, \ldots, F_m$ and $\mathrm{Vol}S > 1$ then from (1) and (2) it follows that

$$[\![S]\!] = \sum_{i \notin I_q(S)} [\![\operatorname{conv}(F_i \cup \{q\})]\!] - \sum_{i \in I_q(S)} [\![\operatorname{conv}(F_i \cup \{q\})]\!] .$$

If now $q \in \mathbb{Z}^d$ is chosen in such a way that $\mathrm{Vol}(\operatorname{conv}(F_i \cup \{q\}q)) < \mathrm{Vol}(S)$ for all facets F_i then the claim follows by induction. In [6] such a q was explicitly constructed. $\square$

Let $(F^d, +)$ be the free abelian group with generators

$$\{[\![P]\!] \colon P \text{ lattice polytope in } \mathbb{R}^d\}$$

and let R^d be the collection of relations:

- $[\![P]\!] - [\![T(P)]\!]$ for any lattice polytope P and any unimodular transformation T.
- $[\![P]\!] - \sum_{\emptyset \neq I \subseteq [m]} (-1)^{|I|-1} [\![\bigcap_{i \in I} P_i]\!]$ for any union $P = P_1 \cup \cdots \cup P_m$ such that $\bigcap_{i \in I} P_i$ is a lattice polytope for all $I \neq \emptyset$.

Let $\pi^d := F^d/R^d$. The second condition corresponds to the **inclusion-exclusion** property that is satisfied by any valuation φ on lattice polytopes [5, 20, 25]; for every union $P = P_1 \cup \cdots \cup P_m$ such that $\bigcap_{i\in I} P_i$ is a lattice polytope for all $I \neq \emptyset$

$$\varphi(P) = \sum_{\emptyset \neq I \subseteq [m]} (-1)^{|I|-1} \varphi\left(\bigcap_{i\in I} P_i\right) .$$

It follows that every lattice-invariant valuation φ corresponds to a unique homomorphism of abelian groups $\bar{\varphi}\colon \pi^d \to G$ defined by $\bar{\varphi}([\![P]\!]) = \varphi(P)$. It turns out that also π^d has a very simple structure which can be seen by similar arguments as in Theorem 1.5.

Theorem 1.6 ([6]). *The group π^d is a free abelian group with generators $\{[\![\Delta_i]\!]\}_{i=0,1,\dots,d}$.*

An immediate corollary is the following.

Corollary 1.1 ([6]). *Every lattice-invariant valuation is uniquely determined by its values on the standard simplices $\{[\![\Delta_i]\!]\}_{i=0,1,\dots,d}$.*

Putting the pieces together we are now ready for the proof of the Betke-Kneser Theorem.

Proof of the Betke-Kneser Theorem. By Corollary 1.1, the space of real-valued rigid-motion invariant valuations is a vector space of dimension $d+1$. Thus, by observing that the coefficients of the Ehrhart polynomial are homogeneous of degrees $0, 1, \dots, d$ and therefore linearly independent, the proof is complete. $\square$

2. Translation-invariant valuations

2.1. *Polynomiality*

In the following let Λ denote $\mathbb{R}^d$ or $\mathbb{Z}^d$ and let $\mathcal{P}(\Lambda)$ be the family of polytopes with vertices in Λ called **Λ-polytopes**. A valuation $\varphi\colon \mathcal{P}(\Lambda) \to G$ is called **translation-invariant** (or a **Λ-valuation**) if $\varphi(P + t) = \varphi(P)$ for all $P \in \mathcal{P}(\Lambda)$ and all $t \in \Lambda$. Examples of $\mathbb{R}^d$-valuations include the volume and the **Euler characteristic** χ which evaluates to 1 on non-empty polytopes. An important example of a $\mathbb{Z}^d$-valuation is the discrete volume. McMullen [19] generalized Ehrhart's polynomiality theorem [10] to translation-invariant valuations.

Theorem 2.1 (McMullen [19]). *Let $\varphi\colon \mathcal{P}(\Lambda) \to G$ be a translation-invariant valuation and let $P \in \mathcal{P}(\Lambda)$ be a Λ-polytope. Then the function $\varphi(nP)$ agrees with a polynomial $\varphi_P(n)$ of degree at most* $\dim P$ *for all integers $n \geq 0$.*

Here, a polynomial is defined in the following combinatorial way. Let $G^{\mathbb{Z}}$ denote the collection of functions from $\mathbb{Z}$ to G. The shift operator $S\colon G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ is defined by $(Sf)(n) = f(n+1)$ for all $f\colon \mathbb{Z} \to G$. The difference operator Δ is defined through

$(\Delta f)(n) = f(n+1) - f(n)$, that is, $\Delta = S - I$. A function $\mathbb{Z} \to G$ is a **polynomial** of degree at most d if and only if $\Delta^{d+1} f \equiv 0$. In terms of generating polynomials this can be characterized in the following way (see, e.g., Stanley [24]).

Theorem 2.2. *A function $f\colon \mathbb{Z} \to G$ is a polynomial of degree at most d if and only if*

$$\sum_{n \geq 0} f(n) t^n \;=\; \frac{h(t)}{(1-t)^{d+1}}$$

as rational functions where $h(t) = h_0 + h_1 t + \cdots + h_d t^d \in G[t]$ is a polynomial with $\deg h \leq d$. *Equivalently,*

$$f(n) \;=\; h_0 \binom{n+d}{d} + h_1 \binom{n+d-1}{d} + \cdots + h_d \binom{n}{d}$$

for all $n \geq 0$.

In particular, if f is the Ehrhart polynomial of a lattice polytope, then h is its h^*-polynomial. We will outline a proof of Theorem 2.1 given in [14] that uses Theorem 2.2 and provide an interpretation of the coefficients $h_0, h_1, \ldots, h_d$ in the case that $f(n) = \varphi_P(n)$ for arbitrary translation-invariant valuations φ and Λ-polytopes P.

Since every Λ-polytope can be triangulated into Λ-simplices, by the inclusion-exclusion property it is sufficient to prove polynomiality of $\varphi(nP)$ for arbitrary Λ-simplices P. However, to avoid inclusion-exclusion and thus considering lower dimensional polytopes, we consider half-open polytopes and half-open decompositions. This will come in handy in regards to positivity questions in later sections. Let P be a polytope with facets $F_1, \ldots, F_m$ and let $q \in \operatorname{aff} P$ be a point in general position to P. We obtain the **half-open polytope** $\mathrm{H}_q P$ by removing all visible faces of P:

$$\mathrm{H}_q P \;=\; P \setminus \bigcup_{i \in I_q(P)} F_i \,.$$

Proposition 2.1 ([15]). *Let $P = P_1 \cup \cdots \cup P_m$ be a dissection and let $q \in \mathbb{R}^d$ be general with respect to P_i for all $1 \leq i \leq m$. Then*

$$\mathrm{H}_q P \;=\; \mathrm{H}_q P_1 \sqcup \cdots \sqcup \mathrm{H}_q P_m$$

is a partition. In particular, if $q \in \operatorname{relint} P$, then P can be decomposed into half-open polytopes.

Any valuation on Λ-polytopes can be extended to half-open polytopes in a natural way by using the inclusion-exclusion property [20], namely

$$\varphi(\mathrm{H}_q P) := \varphi(P) - \sum_{\emptyset \neq J \subseteq I_q(P)} (-1)^{|J|-1} \varphi(F_J) \,,$$

where $F_J := \bigcap_{i \in J} F_i$. The following is an immediate consequence of Proposition 2.1.

Corollary 2.1. *Let $P = P_1 \cup \cdots \cup P_m$ be a triangulation into Λ-simplices and let $q \in \operatorname{relint} P$ be general with respect to P_i for all $1 \le i \le m$. Then*

$$\varphi(P) \;=\; \varphi(\mathrm{H}_q P_1) + \cdots + \varphi(\mathrm{H}_q P_k)\,.$$

Thus, since every Λ-polytope can be triangulated into Λ-simplices, Theorem 2.1 is a direct consequence of the following result.

Theorem 2.3. *Let $\tilde{S}$ be an half-open simplex in $\mathcal{P}(\Lambda)$ and φ be a translation-invariant valuation. Then $\varphi(n\tilde{S})$ agrees with a polynomial $\varphi_{\tilde{S}}$ of degree at most d for integers $n \ge 0$.*

Proof idea. We illustrate the argument given in [14] on a half-open triangle. We partition the dilated half-open triangle $\tilde{S}$ into congruent half-open triangles $\tilde{S}$ and $\hat{S}$ as shown in Figure 1.

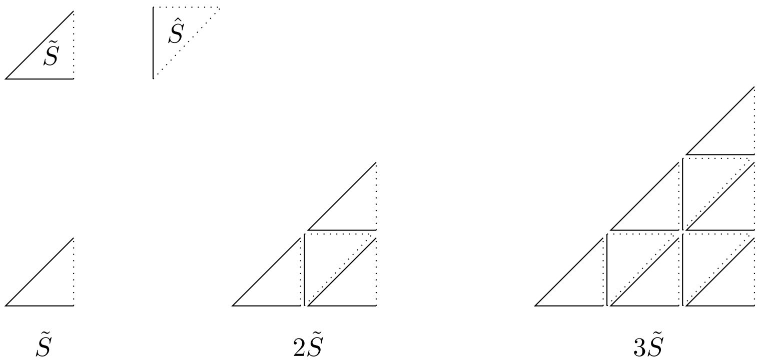

Fig. 1. Decomposition of integer dilates of $\tilde{S}$ into translates of $\tilde{S}$ and $\hat{S}$.

By translation-invariance and Corollary 2.1,

$$\varphi_{\tilde{S}}(n) \;=\; \varphi(\tilde{S})\binom{n+1}{2} + \varphi(\hat{S})\binom{n}{2}\,.$$

This method can be generalized to higher dimensions. More precisely, let $\tilde{S} = \mathrm{H}_q S$ for some Λ-simplex S and general point q with respect to S and let $I = I_q(S)$. Let $F_1, \ldots, F_{d+1}$ be its facets and $v_1, \ldots, v_{d+1}$ be the vertices labeled in such a way that $v_i \notin F_i$. Furthermore, let $\bar{v}_i := (v_i, 1)^T$ for all vertices v_i. Extending a standard method in Ehrhart theory we consider the half-open parallelepiped

$$\Pi \;=\; \{\mu_1 \bar{v}_1 + \cdots + \mu_{d+1}\bar{v}_{d+1} \colon 0 \le \mu_i < 1 \text{ if } i \notin I, 0 < \mu_i \le 1 \text{ otherwise}\} \subset \mathbb{R}^{d+1}\,.$$

Then the coefficients can be expressed in terms of the values of φ on the half-open hypersimplices $\Pi_i = \Pi \cap \{x \in \mathbb{R}^{d+1} \colon x_{d+1} = i\}$; see [14] for further details. We collect the full statement in the next corollary for later reference. $\square$

Corollary 2.2 ([14]). *Let $\tilde{S}$ be an half-open Λ-simplex and φ be a translation-invariant valuation. Then with the notation as in the proof of Theorem 2.3,*

$$\varphi(n\tilde{S}) = \sum_{r=0}^{d} \varphi(\Pi_r) \binom{n+d-r}{d}.$$

2.2. *Reciprocity*

By Theorem 2.1, $\varphi(nP)$ agrees with a polynomial $\varphi_P(n)$ for all integers $n \geq 0$. It is a natural question to ask for an interpretation for evaluating this polynomial at negative integers. A fundamental result in Ehrhart theory is the Ehrhart-Macdonald reciprocity theorem [10, 16] which relates the evaluation at negative integers to counting interior lattice points. The following theorem due to McMullen [19] generalizes the reciprocity to translation-invariant valuations.

Theorem 2.4 (McMullen [19]; Ehrhart-Macdonald reciprocity [10, 16]). *Let P be a lattice polytope and φ be a Λ-valuation. Then*

$$\varphi_P(-n) \; = \; (-1)^{\dim P} \varphi(\mathrm{relint}(-nP)) .$$

Here, $\varphi(\mathrm{relint}P) := \sum_{F \subseteq P} (-1)^{\dim P - \dim F} \varphi(F)$.

Proof. We illustrate the proof idea given in [3, Chapter 5] considering the case that P is a lattice polygon. By considering the polygon $Q := mP$ we invite the reader to convince herself that it is sufficient to prove that $\varphi_P(-1) = (-1)^{\dim P} \varphi(\mathrm{relint}(-P))$. Let again $\tilde{S}$ be a half-open lattice triangle and let $\hat{S}$ be the half-open triangle as in the proof of Theorem 2.3 (see Figure 1). We saw that

$$\varphi_{\tilde{S}}(n) \; = \; \varphi(\tilde{S}) \binom{n+1}{2} + \varphi(\hat{S}) \binom{n}{2} .$$

Evaluating the polynomial on the right hand side at -1 yields $\varphi(\hat{S})$. Assuming that $\tilde{S}$ was obtained from the triangle S by removing all faces visible from a point q we then observe that $\hat{S}$ is obtained from $-S$ by removing all facets that are invisible from $-q$. Now let $P = \tilde{S}_1 \cup \ldots \cup \tilde{S}_m$ be a decomposition into half-open Λ-triangles. Then

$$\varphi_P(-1) = \sum \varphi_{\tilde{S}_i}(-1) = \sum (-1)^d \varphi(\hat{S}_i) = \varphi(\mathrm{relint}(-P)) ,$$

as illustrated in Figure 2. This argument can be generalized to higher dimensions. □

3. Combinatorial positive valuations

3.1. *Combinatorial positivity and combinatorial monotonicity*

In the following, let G always be an ordered abelian group. In the last section we have seen that for every translation-invariant valuation $\varphi\colon \mathcal{P}(\Lambda) \to G$ and every

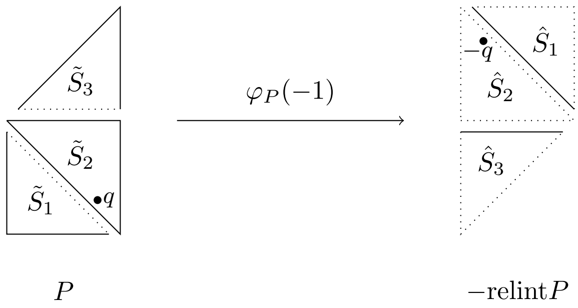

Fig. 2. Half-open partitions of P and $\mathrm{relint}(-P)$.

Λ-polytope P of dimension r the function $\varphi(nP)$ is given by a polynomial $\varphi_P(n)$ of degree at most r for all integers $n \geq 0$. Thus, by Theorem 2.2 there exist coefficients $h_0^\varphi(P), h_1^\varphi(P), \ldots, h_r^\varphi(P) \in G$ such that

$$\varphi_P(n) \;=\; h_0^\varphi(P)\binom{n+r}{r} + h_1^\varphi(P)\binom{n+r-1}{r} + \cdots + h_r^\varphi(P)\binom{n}{r}.$$

The vector $h^\varphi(P) = (h_0^\varphi(P), h_1^\varphi(P), \ldots, h_d^\varphi(P))$ is called the **h^*-vector of P with respect to φ** where we set $h_i^\varphi(P) := 0$ for $i > r$. The polynomial $h_P^\varphi(t) = \sum_{i=0}^r h_i^\varphi(P)t^i$ is called the **h^*-polynomial of P with respect to φ**. By Stanley's Nonnegativity Theorem (Theorem 1.3), the h^*-polynomial has only nonnegative coefficients when considering the discrete volume of lattice polytopes. Moreover Stanley [22] showed the following.

Theorem 3.1 ([22]). *For all lattice polytopes $P, Q \in \mathcal{P}(\mathbb{Z}^d)$ satisfying $P \subseteq Q$*

$$h_i^*(P) \;\leq\; h_i^*(Q) \quad \textit{for all } i = 0, \ldots, d\,.$$

That is, the (Ehrhart) h^*-vector is componentwisely monotone with respect to inclusion.

In accordance with Stanleys results (Theorem 1.3 and Theorem 3.1) we define a translation-invariant valuation φ to be **combinatorially positive** if $h_i^\varphi(P) \geq 0$ for all $P \in \mathcal{P}(\mathbb{Z}^d)$ and all i, and **combinatorially monotone** if $h_i^\varphi(P) \leq h_i^\varphi(Q)$ for all i whenever $P \subseteq Q$. Examples of combinatorial positive and combinatorially monotone valuations are, of course, the discrete volume, the volume as it can be seen that the corresponding h^*-polynomial equals (up to a scalar) the Eulerian polynomial (see, e.g., [1]), and the so-called solid-angle polynomials [2]. It is left as an exercise to the reader that the Euler characteristic is not combinatorially positive.

In the spirit of the classification results in Section 1 we would like to characterize combinatorial positive and combinatorial monotone valuations. It turns out that both notions are equivalent and a simple characterization can be given [14].

Theorem 3.2 ([14]). *Let φ be a translation-invariant valuation. Then the following are equivalent:*

(i) *φ is combinatorially positive.*
(ii) *φ is combinatorially monotone.*
(iii) *$\varphi(\mathrm{relint}\Delta) \geq 0$ for all simplices $\Delta \in \mathcal{P}(\Lambda)$.*

Proof idea. We outline the proof given in [14]. The h^*-vector with respect to a valuation can be naturally extended to half-open Λ-polytopes. If $\tilde{S}$ is a half-open simplex of dimension r. Then, by Corollary 2.2, $h_r^\varphi(\tilde{S}) = \varphi(\Pi_r)$. In particular, if $S = \tilde{S}$ is a closed simplex, then $h_r^\varphi(S) = \varphi(-\mathrm{relint}S)$ from which we see $(i) \Rightarrow (iii)$. The implication $(ii) \Rightarrow (i)$ is clear since the empty polytope is contained in any other polytope. If $P = \mathrm{H}_q P_1 \sqcup \cdots \sqcup \mathrm{H}_q P_m$ is a half-open decomposition, by the inclusion-exclusion principle it follows that $h^\varphi(P) = h^\varphi(\mathrm{H}_q P_1) + \cdots + h^\varphi(\mathrm{H}_q P_m)$. In particular, to show positivity of $h^\varphi(P)$ it is sufficient to assume that $P = \tilde{S}$ is an half-open simplex and $h_i^\varphi(\tilde{S}) = \varphi(\Pi_i)$ for all i. We observe that Π_i is a partially open polytope, (that is a polytope with certain faces removed). Using a triangulation of the corresponding closed polytope into Λ-simplices it can therefore be partitioned as $\Pi_i = \bigsqcup_l \mathrm{relint}T_l$ where T_l are simplices in the triangulation contained in Π_i. Thus, $\varphi(\Pi_i) = \sum_l \varphi(\mathrm{relint}T_l)$ and, assuming (iii), we obtain combinatorial positivity. To get combinatorial monotonicity a slightly more refined argument is needed and we refer the reader to [14]. $\square$

3.2. *Combinatorially positive lattice-invariant valuations.*

In case of lattice-invariant valuations on lattice polytopes combinatorial positivity has an even simpler characterization.

Theorem 3.3 ([14]). *A lattice-invariant valuation $\varphi\colon \mathcal{P}(\mathbb{Z}^d) \to G$ is combinatorially positive if and only if $\varphi(\mathrm{relint}\Delta_i) \geq 0$ for all $0 \leq i \leq d$.*

Proof. The proof will become apparent from the next two results; namely Theorem 3.4 and Lemma 3.1. $\square$

That is, in case of lattice-invariant valuations condition (iii) in Theorem 3.2 needs only to be checked on the standard unimodular simplices. In particular, the cone of combinatorial positive valuations in polyhedral. We can describe this cone more concretely by considering the coefficients of the Ehrhart polynomial with respect to the basis $\binom{n-1}{0}, \binom{n-1}{1}, \ldots, \binom{n-1}{d}$.

$$\mathrm{E}_P(n) \;=\; f_0^*(P)\binom{n-1}{0} + f_1^*(P)\binom{n-1}{1} + \cdots + f_d^*(P)\binom{n-1}{d}.$$

These coefficients $f_i^*(P)$ where first considered by Breuer [7] who coined the name **f^*-vectors** and showed that they are always nonnegative on relatively open complexes. The following theorem completely characterizes combinatorially positive lattice-invariant valuations.

Theorem 3.4 ([14]). *Let φ be a lattice invariant valuation. Then φ is combinatorially positive if and only if*

$$\varphi \;=\; \alpha_0 f_0^* + \alpha_1 f_1^* + \cdots + \alpha_d f_d^*$$

for some $\alpha_0, \alpha_1, \ldots, \alpha_d \geq 0$.

This is parallel to the characterization of positive and monotone continuous rigid-motion invariant valuations which is a direct consequence of Hadwiger's Characterization [12].

Theorem 3.5. *Let φ be a real-valued continuous rigid-motion invariant valuation on convex bodies. Then φ is positive or monotone if and only if*

$$\varphi = \alpha_0 W_0 + \alpha_1 W_1 + \cdots + \alpha_d W_d$$

for some $\alpha_0, \alpha_1, \ldots, \alpha_d \geq 0$.

The following lemma clarifies the role of the f^*-vector and is left as an enjoyable exercise to the reader.

Lemma 3.1 ([14]). *For all $0 \leq i, j \leq d$,*

$$f_j^*(\operatorname{relint}(\Delta_i)) \;=\; \delta_{i,j}\,.$$

In particular, $f_0^*, f_1^*, \ldots, f_d^*$ form a basis of the vector space of lattice-invariant valuations and for any lattice-invariant valuations φ,

$$\varphi \;=\; \varphi(\operatorname{relint}\Delta_0) f_0^* + \varphi(\operatorname{relint}\Delta_1) f_1^* + \cdots + \varphi(\operatorname{relint}\Delta_d) f_d^*\,.$$

From Lemma 3.1 and Theorem 3.2, it follows that Theorem 3.4 is equivalent to showing that $f_0^*, \ldots, f_d^*$ are combinatorially positive valuations. We will show even more: For all translation-invariant valuations φ consider $f_0^\varphi(P), \ldots, f_d^\varphi(P)$ such that

$$\varphi_P(n) \;=\; \sum_{i=0}^{d} f_i^\varphi(P) \binom{n-1}{i},$$

Theorem 3.6 ([14]). *Let φ be a translation-invariant valuation. Then the following are equivalent:*

(i) φ is combinatorially positive.
(ii) f_i^φ is combinatorially positive for all $0 \leq i \leq d$.

Proof idea. One direction follows from the observation that $\varphi = f_0^\varphi$. For the other direction one can prove for all lattice polytopes P of dimension r that

$$f_{r-k}^\varphi(\operatorname{relint} P) \;=\; \sum_{i=k}^{r} h_i^\varphi(-P) \binom{i}{k}$$

for all $0 \leq k \leq r$ by applying Theorem 2.4. Since φ is combinatorially positive the right hand side of the equation is nonnegative and thus f_{r-k}^φ is combinatorially positive by Theorem 3.2. Further details may be found in [14]. □

Proof of Theorem 3.2. The discrete volume is combinatorially positive by Stanley's Nonnegativity Theorem (Theorem 1.3). Thus $f_0^*, \ldots, f_d^*$ are combinatorially positive by Theorem 3.6. □

3.3. *Weak h^*-monotone valuations*

As will become apparent in the next section, it is also very natural to consider a weaker notion of combinatorial monotonicity that takes into account the dimension of Λ-polytopes. A translation-invariant valuation is called **weakly h^*-monotone** if

$$h_i^\varphi(P) \;\leq\; h_i^\varphi(Q)$$

for all $0 \leq i \leq d$ whenever $P \subseteq Q$ and $\dim P = \dim Q$. Using similar techniques as in the proof of Theorem 3.2, the following characterization and classification results were obtained in [14]:

Theorem 3.7 ([14]). *Let φ be a translation-invariant valuation. The following are equivalent:*

(i) φ *is weakly* h^**-monotone.*
(ii) $\varphi(\mathrm{relint}\Delta) + \varphi(\mathrm{relint}F) \geq 0$ *for every simplex* $\Delta \in \mathcal{P}(\Lambda)$ *and any facet* F *of* Δ*.*
(iii) $\varphi(\tilde{S}) \geq 0$ *for every half-open* Λ*-simplex* $\tilde{S}$*.*

Restricted to the class of lattice-invariant valuations weak h^*-monotonicity can be characterized by considering the coefficients of the Ehrhart polynomial in (yet another) basis. Let

$$\mathrm{E}_P(n) \;=\; \tilde{f}_0(P)\binom{n}{0} + \tilde{f}_1(P)\binom{n}{1} + \cdots + \tilde{f}_d(P)\binom{n}{d}.$$

Theorem 3.8 ([14]). *Let φ be a lattice-invariant valuation. Then φ is weakly h^*-monotone if and only if*

$$\varphi \;=\; \alpha_0\tilde{f}_0 + \alpha_1\tilde{f}_1 + \cdots + \alpha_d\tilde{f}_d$$

for some $\alpha_0, \alpha_1, \ldots, \alpha_d \geq 0$.

We conclude this section by discussing the relationship of combinatorial positivity and weak h^*-monotonicity and monotonicity ($\varphi(P) \leq \varphi(Q)$ whenever $P \subseteq Q$) and positivity ($\varphi(P) \geq 0$). Clearly, combinatorial monotonicity/positivity implies weak h^*-monotonicity. It can be furthermore seen from Theorem 3.7 by using half-open decomposition that weak h^*-monotonicity implies **monotonicity**. Since every valuation is 0 on the empty polytope, monotonicity implies **positivity**. Figure 3 summarizes the chain of implications. We note that the reverse implications do not hold [14].

$$\begin{array}{c} \boxed{\text{combinatorial positivity}} \\ \Downarrow \\ \boxed{\text{weak } h^*\text{-monotonicity}} \\ \Downarrow \\ \boxed{\text{monotonicity}} \\ \Downarrow \\ \boxed{\text{positivity}} \end{array}$$

Fig. 3. Combinatorial positivity and its relatives.

4. Combinatorial mixed valuations

A fundamental notion in convex geometry is the mixed volume $\mathrm{MV}_d(P_1, \ldots, P_d)$ of convex bodies $P_1, \ldots, P_d$. It can be defined in different equivalent ways and each definition comes with certain advantages and disadvantages. A classical result is Minkowski's Identity (see, e.g., [11]) which states that $\mathrm{Vol}(t_1P_1 + \cdots + t_mP_m)$ is given by a homogeneous polynomial of degree d for convex bodies $P_1, \ldots, P_m$ and $t_1, \ldots, t_m \geq 0$. More precisely, with the definition $\mathrm{MV}(P_1, \ldots, P_d) := \frac{1}{d!}[t_1 \cdots t_d]\mathrm{Vol}(t_1P_1 + \cdots + t_dP_d)$, (that is, $\mathrm{MV}(P_1, \ldots, P_d)$ is defined as the coefficient of the monomial $t_1 \cdots t_d$ divided by $d!$) Minkowski's Identity states that

$$\mathrm{Vol}(t_1P_1 + \cdots + t_mP_m) \;=\; \sum_{j_1,\ldots,j_d=1}^{m} \mathrm{MV}(P_{j_1}, \ldots, P_{j_d})t_{j_1} \cdots t_{j_d}\,.$$

In particular, MV is symmetric in its arguments and, by homogeneity, $\mathrm{MV}(P_1, \ldots, P_m) = 0$ if $m < d$. Furthermore, it can be seen that the mixed volume is multilinear in each argument.

An alternative definition is the following:

$$\mathrm{MV}(P_1, \ldots, P_m) := \frac{1}{m!} \sum_{I \subseteq [m]} (-1)^{d-|I|}\mathrm{Vol}(P_I)$$

where $P_\emptyset = \{0\}$ and $P_I = \sum_{i \in I} P_i$ for all $\emptyset \neq I \subseteq [m]$. This definition has the advantage that it is not necessarily trivial on less than d arguments and can be extended to valuations. We define the **combinatorial mixed valuation** [13] associated to a translation-invariant valuation φ of $P_1, \ldots, P_r \in \mathcal{P}(\Lambda)$ to be

$$\mathrm{CM}_r\varphi(P_1, \ldots, P_r) \;:=\; \sum_{I \subseteq [r]} (-1)^{r-|I|}\varphi(P_I)\,,$$

where in the following we suppress the index r in $\mathrm{CM}_r\varphi$ whenever the number of arguments is clear from the context. Similar to mixed volumes in Minkowski's Identity, combinatorial mixed volumes can be interpreted as coefficients of a polynomial. The Bernstein-McMullen Theorem [4, 17, 19] states that for any translation-invariant valuation φ and arbitrary Λ-polytopes $P_1, \ldots, P_r$, $\varphi(n_1P_1 + \cdots n_rP_r)$ is given by a polynomial for integers $n_1, \ldots, n_r \geq 0$. The following theorem is a discrete version of Minkowski's Identity.

Theorem 4.1 ([13]). *Let $P_1, \dots, P_r \in \mathcal{P}(\Lambda)$ and φ be a translation-invariant valuation. Then*

$$\varphi(n_1P_1 + \cdots + n_rP_r) \;=\; \sum_{\mathbf{k}\in\mathbb{Z}^r_{\geq 0}} \mathrm{CM}\varphi(P_1[k_1], \cdots, P_r[k_r]) \binom{n_1}{k_1} \cdots \binom{n_r}{k_r}$$

where $P_i[k_i]$ denotes the k_i-fold appearance of P_i.

In particular, since the coefficients are uniquely determined,

$$P_i \mapsto \varphi(P_1[k_1], \cdots, P_r[k_r])$$

defines a translation-invariant valuation for all i and all $\mathbf{k}$.

A desirable, non-trivial property of the mixed volume is that it is always non-negative and, moreover, **monotone** with respect to inclusion, that is

$$0 \leq \mathrm{MV}(P_1, \dots, P_r) \;\leq\; \mathrm{MV}(Q_1, \dots, Q_r)$$

whenever $P_i \subseteq Q_i$ for all $1 \leq i \leq r$. It turns out, that this is true for combinatorial mixed valuations as well under the assumption of weak h^*-monotonicity.

Theorem 4.2 ([13]). *Let φ be a weakly h^*-monotone valuation. Then*

$$0 \leq \mathrm{CM}\varphi(P_1, \dots, P_r) \leq \varphi(Q_1, \dots, Q_r)$$

whenever $P_1 \subseteq Q_i$ for all $1 \leq i \leq r$.

The goal of this section is to proof Theorem 4.2. We will use the language of the polytope algebra introduced by McMullen [18].

Let $\mathbb{Z}\mathcal{P}(\Lambda)$ be the free abelian group with generators $[\![P]\!]$ for all $P \in \mathcal{P}(\Lambda)$. Let U be the subgroup generated by elements of the form

- $[\![P \cup Q]\!] + [\![P \cap Q]\!] - [\![P]\!] - [\![Q]\!]$ for all $P, Q \in \mathcal{P}(\Lambda)$ for which $P \cap Q, P \cup Q \in \mathcal{P}(\Lambda)$ and $[\![\emptyset]\!] = 0$, and
- $[\![P + t]\!] - [\![P]\!]$ for all $P \in \mathcal{P}(\Lambda)$ and $t \in \Lambda$.

Then the **polytope algebra** is defined as $\Pi(\Lambda) := \mathcal{P}(\Lambda)/U$. The polytope algebra has the universal property that for every translation-invariant valuation $\varphi\colon \mathcal{P}(\Lambda) \to G$ there is a unique homomorphism of abelian groups $\bar{\varphi}\colon \Pi(\Lambda) \to G$ such that $\bar{\varphi}([P]) = \varphi(P)$, and conversely. The product structure on $\Pi(\Lambda)$ is defined by the Minkowski sum of polytopes, that is, for $P, Q \in \mathcal{P}(\Lambda)$ we have $[\![P]\!] \cdot [\![Q]\!] := [\![P + Q]\!]$. Even though it is not relevant in the following, it allows us to express the combinatorial mixed valuation in a particularly nice form. The following corollary is obtained by applying Theorem 4.1 to the universal valuation $P \mapsto [\![P]\!]$.

Corollary 4.1 ([13]). *Let $P_1, \dots, P_r$ be Λ-polytopes. Then*

$$[\![n_1P_1 + \cdots + n_rP_r]\!] \;=\; \sum_{\mathbf{k}\in\mathbb{Z}^r_{\geq 0}} \mathrm{CM}[\![P_1[k_1], \dots, P_r[k_r]]\!] \binom{n_1}{k_1} \cdots \binom{n_r}{k_r},$$

where $\mathrm{CM}[\![P_1, \dots, P_r]\!] := \sum_{I\subseteq[r]} (-1)^{r-|I|}[\![P_I]\!] = \Pi_{i=1}^r([\![P_i]\!] - 1)$.

In order to proof Theorem 4.2 we will need to interpret combinatorial mixed volumes geometrically. A Minkowski sum $P = P_1 + \cdots + P_r$ is called **exact** if $\dim P = \dim P_1 + \cdots + \dim P_r$. If $P_1, \ldots, P_r$ are simplices then their Minkowski sum is called a **cylinder** whenever it is exact. It is further called a k-cylinder if $\dim P_i > 0$ for exactly k-many summands P_i. For $k = 0, \ldots, d$ let

$$\tilde{Z}_k \ := \ \mathbb{Z}_{\geq 0}\{[\![S]\!] \colon S \text{ half-open } k\text{-cylinder } \}$$

be the cone generated by half-open k-cylinders, and let

$$W = \mathbb{Z}_{\geq 0}\{[\![\mathrm{relint} S]\!] + [\![\mathrm{relint} F]\!] \colon S \in \mathcal{P}(\Lambda) \text{ simplex}, F \subseteq S \text{ facet}\}\,.$$

Then, by Theorem 3.7, every weakly h^*-monotone φ evaluates nonnegatively on W. Thus, by the following lemma which is left as an exercise to the reader, every such valuation also evaluates nonnegatively on cylinders.

Lemma 4.1 ([13]).

$$\tilde{Z}_d \subseteq \tilde{Z}_{d-1} \subseteq \cdots \subseteq \tilde{Z}_1 = W\,.$$

It therefore suffices to prove that combinatorial mixed valuations are contained in $\tilde{Z}_k$ for some k.

Proposition 4.1 ([13]). *Let $\tilde{S}$ be a half-open simplex. Then*

$$[\![n\tilde{S}]\!] \ = \ \zeta_0 + \zeta_1\binom{n}{1} + \cdots + \zeta_d\binom{n}{d}$$

where $\zeta_k \in \tilde{Z}_k$ for all $k = 0, \ldots, d$.

Proof. We illustrate the proof idea on a 2-dimensional half-open triangle. Similar to the proof of Theorem 2.3, the key idea is to partition the integer dilates in a suitable way with congruent pieces. The n-th dilate of the half-open triangle $\tilde{S}$ in Figure 4 can be dissected into $\binom{n}{1}$ translates of $\tilde{S}$ and $\binom{n}{2}$ translates of a half-open parallelepiped which is exact since it is a Minkowski sum of two segments. With the notation as in Figure 4, we then have

$$\varphi \ = \ \varphi(\tilde{S})\binom{n}{1} + \varphi(T)\binom{n}{2}\,.$$

In higher dimensions, one can assume that $\tilde{S}$ is obtained by removing facets from a simplex $S = \{x \in \mathbb{R}^d \colon 0 \leq x_1 \leq \cdots \leq x_d \leq 1\}$. For a generic point $p \in nS$, let $\bar{p} = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \ldots, \lfloor x_d \rfloor)$. Then $p - \bar{p}$ is contained in a cylinder of the form

$$\left\{ x \in \mathbb{R}^d : \begin{array}{c} 0 \leq x_1 \leq \cdots \leq x_{b_1} \leq 1 \\ 0 \leq x_{b_1+1} \leq \cdots \leq x_{b_2} \leq 1 \\ \vdots \\ 0 \leq x_{b_{n-1}+1} \leq \cdots \leq x_{b_n} \leq 1 \end{array} \right\},$$

for some $0 \leq b_1 \leq \cdots \leq b_{n-1} \leq b_n = d$. These cylinders dissect nS, and counting the number of occurrences up to translation of every such cylinder yields the result. For full details see [14]. $\square$

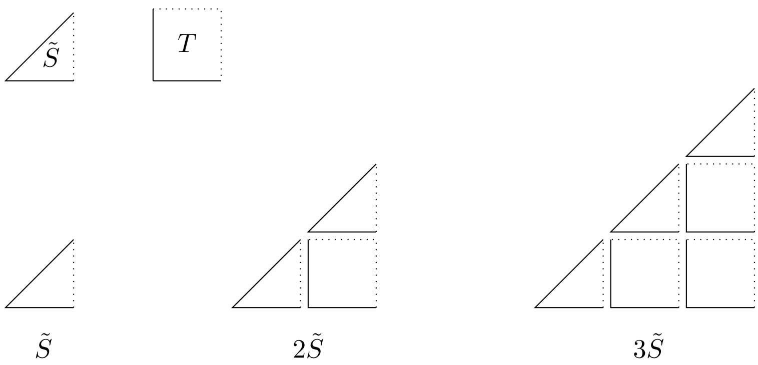

Fig. 4. Decomposition of integer dilates of $\tilde{S}$ into translates of $\tilde{S}$ and T.

More generally, by applying Proposition 4.1 to every Minkowski summand in a cylinder we obtain the following.

Corollary 4.2 ([13]). *Let $S_1 + \cdots + S_r$ be a half-open cylinder. Then there exist $\zeta_{\mathbf{k}} \in \tilde{Z}_{|k|}$ such that*

$$[\![n_1 S_1 + \cdots + n_r S_r]\!] \;=\; \sum_{\mathbf{k} \in \mathbb{Z}^k_{\geq 0}} \zeta_{\mathbf{k}} \binom{n_1}{k_1} \cdots \binom{n_k}{k_r}$$

where $\zeta_{\mathbf{k}} \in \tilde{Z}_{|\mathbf{k}|}$ for all $\mathbf{k}$.

By Corollary 4.2 it is therefore sufficient to see that every Minkowski sum $n_1 P_1 + \cdots + n_r P_r$ has a dissection into cylinders $R_i = R_{i_1} + \cdots + R_{i_r}$ with $R_{i_j} \subseteq P_j$. This can be achieved by the **Cayley-trick** (see [9]): the **Cayley polytope** of polytopes $P_1, \ldots, P_r \subset \mathbb{R}^d$ is defined as

$$\operatorname{Cay}(P_1, \ldots, P_r) \;:=\; \operatorname{conv}\left(\bigcup_i P_i \times \{e_i\} \right) \subseteq \mathbb{R}^d \times \mathbb{R}^r \,.$$

One observes that $\frac{1}{r}(P_1 + \cdots + P_r) = \operatorname{Cay}(P_1, \ldots, P_r) \cap W$ where $W = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^r \colon y_i = \frac{1}{r} \text{ for all } i\}$. Now it can be shown that every triangulation of $\operatorname{Cay}(P_1, \ldots, P_r)$ restricts to a subdivision of $P_1 + \cdots + P_r$ into cylinders as we wanted.

Theorem 4.3 ([13]). *Let $P_1, \ldots, P_r, Q_1, \ldots, Q_r \subseteq \mathcal{P}(\Lambda)$ such that $P_i \subseteq Q_i$ for all $1 \leq i \leq r$. Then*

$$[\![n_1 Q_1 + \cdots + n_r Q_r]\!] - [\![n_1 P_1 + \cdots + n_r P_r]\!] \;=\; \sum_{\mathbf{k} \in \mathbb{Z}^r_{\geq 0}} \zeta_{\mathbf{k}} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r}$$

with $\zeta_{\mathbf{k}} \in \tilde{Z}_{|\mathbf{k}|}$.

Proof idea. Since $P_i \subseteq Q_i$ for all $1 \leq i \leq r$ it follows that $\mathrm{Cay}(P_1, \ldots, P_r) \subseteq \mathrm{Cay}(Q_1, \ldots, Q_r)$. If $\dim(P_1 + \cdots + P_d) = \dim(Q_1 + \cdots + Q_r)$ then both Cayley polytopes have the same dimension. Triangulating $\mathrm{Cay}(P_1, \ldots, P_r)$ and extending it to a triangulation of $\mathrm{Cay}(Q_1, \ldots, Q_r)$ yields a subdivision into cylinders of $P_1 + \cdots + P_r$ that gets extended to a subdivision into cylinders of $Q_1 + \cdots + Q_r$. Choosing a general point in $P_1 + \cdots + P_r$ yields a partition of $(Q_1 + \cdots + Q_r) \setminus (P_1 + \cdots + P_r)$ into half-open cylinders and the result follows with Lemma 4.2. If $\dim(P_1 + \cdots + P_d) < \dim(Q_1 + \cdots + Q_r)$ a slightly more refined argument has to be applied. See [14] for the full details. ☐

We are now ready to proof the main theorem of this section (Theorem 4.2).

Proof of Theorem 4.2. By Theorem 4.1, $\zeta_{\mathbf{k}} = \mathrm{CM}[\![Q_1[k_1], \ldots, Q_r[k_r]]\!] - \mathrm{CM}[\![P_1[k_1], \ldots, P_r[k_r]]\!]$ in Theorem 4.3 which is evaluated to a nonnegative number for every weakly h^*-monotone valuation, by Lemma 4.1. ☐

5. Outlook

The route taken in this course only showed a glimpse of the past and current research on valuations on lattice polytopes and is strongly biased by the authors own research. To learn more about current research on valuations on lattice polytopes we recommend [8].

Acknowledgements: The author was supported by the Knut and Alice Wallenberg Foundation. She would like to thank Takayuki Hibi and Akiyoshi Tsuchiya for the organization and the invitation to speak at the "Summer Workshop on Lattice Polytopes" at Osaka University, and Sebastian Manecke, Raman Sanyal and Liam Solus for many helpful comments on this manuscript.

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