A Small Tour of Information Theory

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§Part I of the lecture given at ACROPOLIS Winter School, Barcelona.

2013-02-14

A theory to **quantify** the processing of information.

Entropy: a quantitative measure of information/uncertainty

$H(X)$: the amount of information you get by observing X.
$H(X,Y)$: - - - - - - - - - : - - - - - - - - - by observing X and Y.
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$H(X,Y) = H(X|Y) + H(Y) = H(Y|X) + H(X)$.
$H(X,Y) \leq H(X) + H(Y)$: equality holds iff X and Y are independent.
$H(X|Y) \leq H(X)$: condition reduces entropy!

Mutual information $I(X;Y) = I(Y;X)$

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Entropy and Mutual Information

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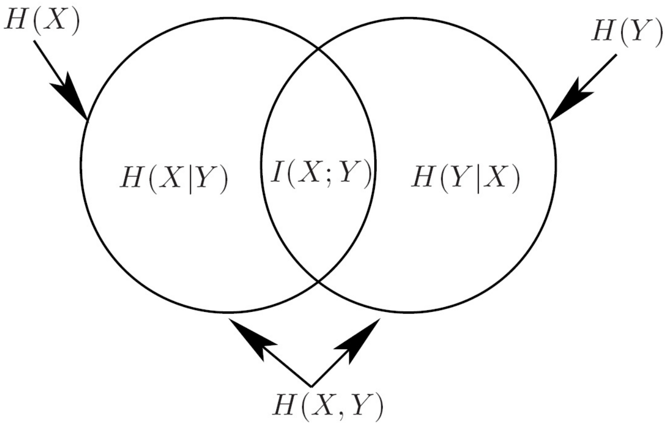
Relationship among entropy and mutual information

$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) = H(X) + H(Y) - H(X,Y)$$



Where's information theory

- Lossless source compression: rate $R > H$
- Lossy source compression: rate-distortion theorem
- Channel coding: $0 < R < C$ is possible (Shannon'48)
- Separation theorem: first source coding, then channel coding
- Separation theorem II: separate/joint channel-network coding

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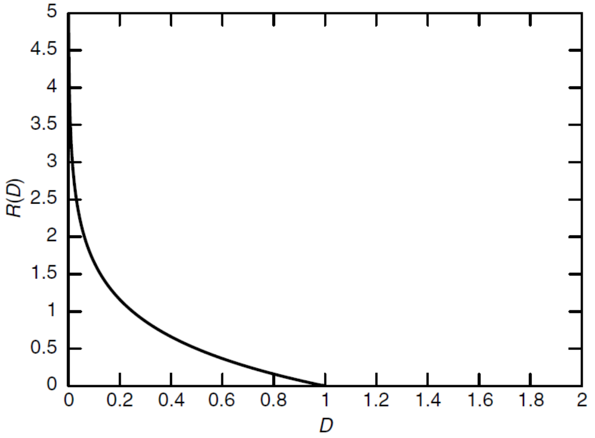

[CT'06]

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  Examples of Gaussian channel capacity:
  discrete real-valued Gaussian channel: $C = \frac{1}{2}\log(1 + \frac{P}{N})$ [bpcu]
  discrete complex-valued Gaussian channel: $C = \log(1 + \frac{P}{N})$ [bpcu]
  band-limited Gaussian channel: $C = W\log(1 + \frac{P}{WN_0})$ [bps]
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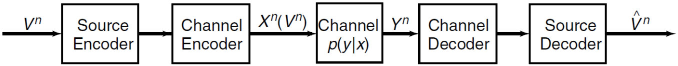

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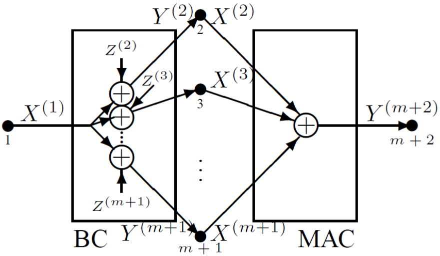

[KEM'10]

TX power P, independent AWGN noise with power N

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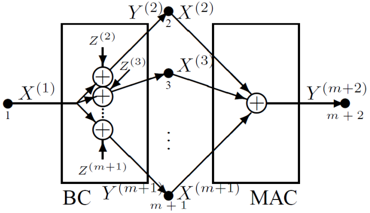

[KEM'10]

Separate channel-network coding: $R = \log(1 + P/N)$

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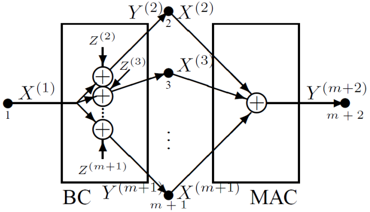

[KEM'10]

Separate channel-network coding: $R = \log(1 + P/N)$

Joint channel-network coding: $R = \log(1 + mP/N)$

Where's information theory: BC

Two-user broadcast channel: P, $|h_2| > |h_1|$

orthogonal TX: $t \in [0,1]$, $[0,t]x = x_1$; $[t,1]x = x_2$,
$R_1 = t\log(1 + P|h_1|^2)$, $R_2 = (1-t)\log(1 + P|h_2|^2)$
superposition: $x = \sqrt{\alpha}x_1 + \sqrt{1-\alpha}x_2$, $\alpha \in [0,1]$
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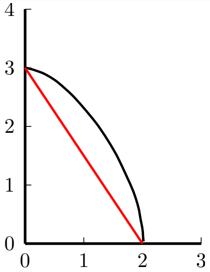

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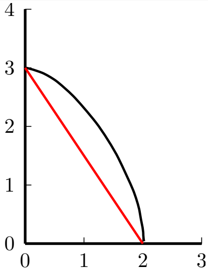

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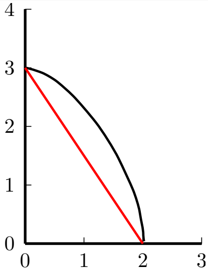

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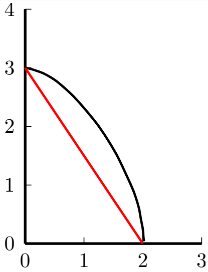

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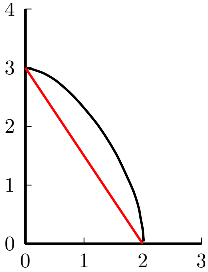

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capacity region: $y = x_1 + x_2 + n$, successive interference cancellation

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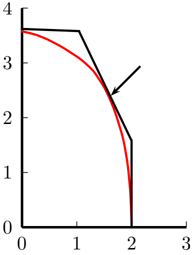

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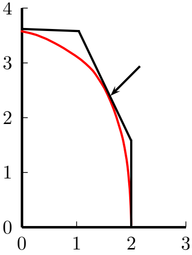

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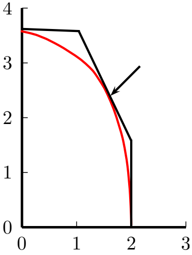

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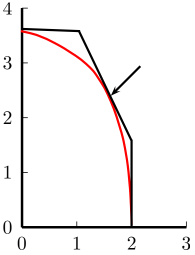

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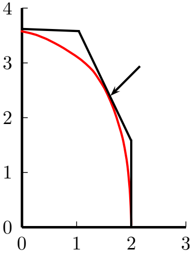

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HARQ: soft combining

repetition redundance: $\log(1 + P_1 + P_2 + \ldots)$

incremental redundance: $\log(1 + P_1) + \log(1 + P_2) + \ldots$

Overheard signal in wireless transmission

interference: has to be mitigated

resource: when cooperation among nodes is allowed

Diversity order, Water-filling, Outage probability, and more ...

Where's information theory: HARQ, and more

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the existence of schemes to achieve the predicted performance

Insights from information theory results

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gap from the theoretical limit indicates possibility of improvement
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How to use information theory

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- the existence of schemes to achieve the predicted performance

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Want to know more about information theory?

Shannon'48 C. E. Shannon,
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