Low SNR – When Only Decoding Will Do

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***Abstract*—We investigate the issue of distributed receiver cooperation in a multiple-relay network with memoryless independent fading channels, where the channel state information can't be obtained. The received signals at distributed receiving nodes are first compressed or quantized before being sent to the decoder via rate-limited cooperation channels for joint processing. We focus on the low SNR regime and analyze the capacity bounds using network equivalence theory and a multiple-layer binning peaky frequency shift keying (FSK). When the received signals at the relaying nodes are in low SNR regime and the cooperation rates are not sufficiently high, compressed/quantized observations at relaying nodes become useless and only decoding can help.**

***Index Terms*—Low SNR, distributed receivers, network equivalence, peaky FSK, MMSE**

I. Introduction

Wireless communication in the wideband regime experiences dispersive fading channels both in time and in frequency. The large bandwidth but finite power constraint leads to low signal-to-noise ratio (SNR) which makes the system essentially power/noise limited. Given the same average received power constraint, the capacity of infinite-bandwidth fading channel without channel state information and the capacity of the infinite-bandwidth Gaussian channel are equal [1] [2, Sec. 8.6] for the case of Rayleigh fading. The key to achieve this is the low duty-cycle (hence peaky) frequency shift keying (FSK) transmission with non-coherent detection. This result has been extended by Telatar and Tse [3] to general multipath fading channels by using a threshold decoder instead of the maximum likelihood decoder [2, Sec. 8.6]. Given average receive power constraint P, single-side noise spectral density N_0, the achievable rate by peaky FSK is [3, Theorem 1]

$$R < \frac{P}{N_0}\left(1 - \frac{2T_d}{T_c}\right), \quad \text{[nats]}, \tag{1}$$

where T_d is the channel delay spread, T_c is the coherence time, and $C = P/N_0$ is the capacity of infinite-bandwidth Gaussian channel. This achievable rate is improved in [4] to

$$R < \frac{P}{N_0}\left(1 - \frac{T_d}{T_c}\right), \quad \text{[nats]}, \tag{2}$$

where a multi-tone FSK with narrower frequency separation is proposed to greatly improve the error exponent. In [5] the boundaries of peaky signaling and coherent detection are discussed and the concept of *optimal spreading bandwidth* is illustrated in [5, Fig. 1] in the context of DS-CDMA over time and frequency selective fading channels. A lower bound of the achievable rates by non-peaky signaling is presented by Lozano and Porrat in [6], where the concept of *critical bandwidth* is proposed to identify the boundary of power limited and bandwidth limited regions.

If the channel is doubly dispersive, i.e., selective both in time and in frequency, Medard and Gallager point out in [7] that without channel state information and without feedback, the achievable rate goes to zero as bandwidth grows if the signal energy spreads over the whole bandwidth. With such "spread spectrum" like signals, Telatar and Tse [3] shows that the mutual information is inversely proportional to the number of resolvable paths. The critical importance of channel knowledge has been discussed in [3], [6], [8]–[10], where the loss of mutual information due to imperfectness of channel estimation is upper and lower bounded in [8], and a flashing signalling with unbounded amplitude is proposed in [9] for the case with imperfect channel knowledge. If the signal peakedness is constraint both in time and frequency, the non-coherent capacity bounds have been characterized in [10].

In this paper, we focus on the wideband regime (low SNR) and investigate the issue of distributed receiver cooperation over memoryless fading channels, where channel state information is not available. The cooperation among distributed receivers are realized via rate-limited orthogonal channels, either noisy or noiseless. It has been shown [11] in the context of a degraded relay network that the relay's observation/estimation passed through rate-limited relay-destination channel is useless unless the relay can decode the message. In this paper we show that it is also true in more general settings.

The rest of the paper is organized as follows. We describe the channel model and revisit the peaky FSK scheme in Sec. II. The capacity upper and lower bounds of a network with distributed receiver cooperation are investigated in Sec. III. Conclusion and discussions are in Sec. IV.

II. Channel model and the peaky FSK

The low duty-cycle FSK scheme, which is peaky both in time and in frequency, is briefly described here based on the work in [3, Theorem 1] and [4]. Assuming the channel is dispersive both in time and in frequency, with a delay spread T_d and a coherence time[1] T_c, the channel impulse response (without noise) can be written as

$$h(t) = \textstyle\sum_l a_l(t)\delta(t - \tau_l(t)), \tag{3}$$

[1]It is widely adopted that the coherence bandwidth $B_c = 1/T_d$ and the coherence time $T_c \simeq 1/f_d$ where f_d is the Doppler spread.

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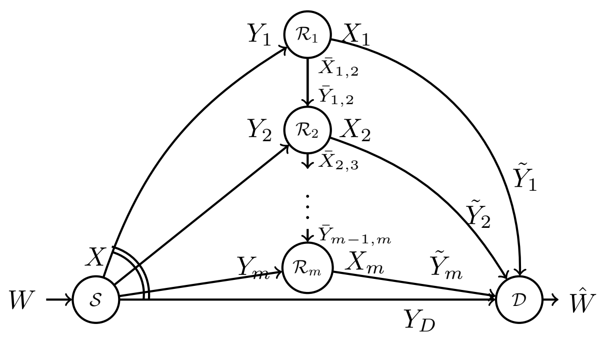

Fig. 1. Wireless relay network where the source $\mathcal{S}$ transmit a message W to destination node $\mathcal{D}$ via the wideband fading channel with average transmit power constraint P. This transmitted signal is over heard by m relay nodes. Channels between relay nodes and from relay to destination are orthogonal.

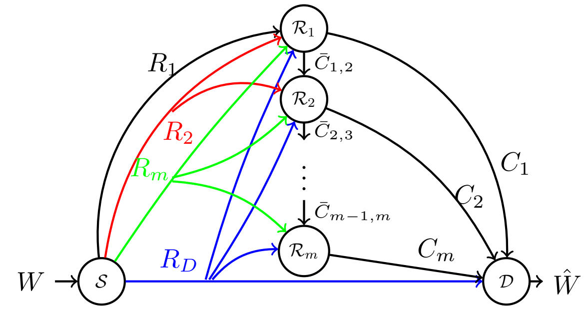

Fig. 2. The upper bounding model from network equivalence theory where the orthogonal channels from relays to the destination are replaced by bit-pipes with rate $C_i{=}\max_{P_{X_i}} I(X_i;\tilde{Y}_i)$, channels from $\mathcal{R}_i{\to}\mathcal{R}_j$ are replaced by bit-pipes with rate $\bar{C}_{i,j}{=}\max_{P_{\bar{X}_{i,j}}} I(\bar{X}_{i,j};\bar{Y}_{i,j})$, and the BC from the source is replaced by hyper-arcs with rates $R_1, R_2, \ldots, R_m$ and R_D.

where $a_l(t)$ represents the channel gain of path l and $\tau_l(t){\geq}0$ is the corresponding delay. We further assume that $a_l(t)$ and $\tau_l(t)$ remains constant during time $[nT_c, nT_c{+}T_c)$ for all $n{\in}\mathbb{Z}$ but changes independently after each duration of T_c. By the definition of delay spread we have $\max_l \tau_l(t) \leq T_d$ and the maximum number of resolvable paths equals BT_d, where B is the signal bandwidth since we can only differentiate signals that arrive with interval larger than $1/B$.

Given a message m that is uniform over $\{0, 1, \ldots, M{-}1\}$, i.e., $\log(M)$ nats per message, the peaky FSK chooses a sinusoid waveform

$$x_m(t) = \begin{cases} \sqrt{P/\theta}\exp(j2\pi f_m t), & \text{for } t \in [0, T_s], \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $j{=}\sqrt{-1}$, $T_s \leq T_c$ is the symbol duration, and f_m is the carrier frequency chosen from $\{\frac{k}{T_s - T_d} : k{=}1, \ldots, m\}$ to send message m. P is the average power constraint and $\theta \in (0, 1]$ is the duty-cycle parameter for transmission in time such that the transmission only occurs at a fraction θ of time at power P/θ to satisfy the average power constraint. A repetition code is use to transmit the same signal N times, which helps to smooth out the channel uncertainty. Based on [3, Theorem 1] and choosing T_s arbitrarily close to T_c, peaky FSK can achieve the rate in (2) as $N{\to}\infty$ and $\theta{\to}0$. If $T_d \ll T_c$, it achieves the wideband capacity P/N_0.

In [12] peaky FSK is combined with structured binning where the message set is evenly split into bins of the same size. Operation at the source node remains unchanged and the bin index (the "base") is decoded at the relay node. Then the bin index is forwarded to the destination node via an orthogonal channel using another peaky FSK. Upon the reception of the bin index, the destination node retrieves the "satellite" message out of that bin based on the received signal from the source.

III. Distributed receiver cooperation

We consider a relay network illustrated in Fig. 1 where the source $\mathcal{S}$ broadcasts message W to the destination node $\mathcal{D}$ and m relay nodes $\mathcal{R}_i$, $i{=}1, \ldots, m$. Cooperation among relay nodes and from relay to destination are realized via orthogonal rate-limited channels. All channels are memoryless block fading as described in (3) with coherence time T_c and delay spread T_d ($<T_c$). Given abundant bandwidth and limited power, we are interested in transmission schemes that can approach the capacity, although the capacity itself may not be fully characterized. A serial of network equivalence tools have been developed in [13]–[15] and we use these tools to develop capacity upper bounds, which help us to evaluate the performance of transmission schemes.

A. Network equivalence upper bound

In [13] it has been shown that the capacity of a network remains unchanged if we replace an independent point-to-point noisy channel by a noiseless bit-pipe with rate equals the capacity of the original noisy channel. Therefore, we can replace the orthogonal channels from relay node $\mathcal{R}_i$ to $\mathcal{D}$ and to $\mathcal{R}_j$ by bit-pipes with rate, respectively,

$$C_i = \max_{P_{X_i}} I(X_i; \tilde{Y}_i),\ \bar{C}_{i,j} = \max_{P_{\bar{X}_{i,j}}} I(\bar{X}_{i,j}; \bar{Y}_{i,j}). \tag{5}$$

Based on the framework developed in [14], [15], the upper bounding model for the broadcast channel originated at the source node can be constructed by utilizing hyper-arc — a single-tail multiple-head bit-pipe. One such model is illustrated in Fig. 2 where the rates of hyper-arcs are given by

$$\begin{aligned} R_1 &= \max_{p_X} I(X; Y_1 \cdots Y_m Y_D) - I(X; Y_2 \cdots Y_m Y_D), \\ R_k &= \max_{p_X} I(X; Y_k \cdots Y_m Y_D) - I(X; Y_{k+1} \cdots Y_m Y_D), \\ R_m &= \max_{p_X} I(X; Y_m Y_D) - I(X; Y_D), \\ R_D &= \max_{p_X} I(X; Y_D). \end{aligned} \tag{6}$$

Note that the upper bounding model represents a layered cooperation structure since $\forall k = 1, \ldots, m$,

$$R_D + \sum_{i=k}^{m} R_i = \max_{p_X} I(X; Y_k \cdots Y_m Y_D). \tag{7}$$

The upper bounding model in Fig. 2 only contains point-to-point(s) bit-pipes, and we can calculate its capacity C_{upp} by the *max-flow min-cut* principle and evaluate the value of all possible cuts (keeping the hyper-arcs in mind). Naturally, we have (only focusing on two cuts)

$$C_{upp} \leq R_D + \min\left\{\textstyle\sum_{i=1}^{m} R_i, \sum_{i=1}^{m} C_i\right\}. \tag{8}$$

Since the wideband channels in the relay network are memoryless independent fading, channel state information is not available at either the transmitter or receiver side. Assuming perfect channel knowledge at the receiver, the capacity of infinite bandwidth fading channel equals the capacity of an infinite bandwidth Gaussian channel with the same received SNR. We can therefore upper bound the rate constraints in (6) by their corresponding Gaussian channel capacities. Denoting the average received power at relay node $\mathcal{R}_i$ as $\alpha_i P$, where $\alpha_i > 0$, $i = 1, \ldots, m$, represents the average power gain (or, loss). Without loss of generality, assuming

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq \alpha_D > 0, \tag{9}$$

the rate constraints in (6) in the wideband (i.e., low SNR) regime can be written as (in nats)

$$R_D = \alpha_D \frac{P}{N_0}, \; R_k = \alpha_k \frac{P}{N_0}, k = 1, \ldots, m. \tag{10}$$

B. Multi-layer binning Peaky FSK lower bound

We generalize the peaky FSK with binning [12] to many-layer binning and build a lower bound. This generalization is based on the observation that decoding the bin index alone requires the same rate as decoding the whole message. This is due to the nature of FSK where signal randomness comes from its potential frequency carrier positions and the fact that threshold decoding is done for every possible frequency carriers. The encoding and decoding process are described as follows. For convenience of description, we assume the average channel gain from source to relay nodes as in (9), and we denote the average received SNR on relay-to-relay link $\mathcal{R}_i \to \mathcal{R}_j$ as $\bar{\alpha}_{i,j} P/N_0$ and the average received SNR on relay-to-destination link $\mathcal{R}_i \to \mathcal{D}$ as $\tilde{\alpha}_i P/N_0$.

For a given set of positive integers $\{M_1, \ldots, M_m, M_D\}$, the messages are chosen uniformly at random from the message set $\mathcal{W} \triangleq \{0, 1, \ldots, M-1\}$ where

$$M = M_D \prod_{i=1}^{m} M_i, \tag{11}$$

and a message $w \in \mathcal{W}$ is represented by a vector of $(m+1)$ non-negative integers $bin(w) = [k_1, k_2, \cdots, k_m, k_D]$ such that

$$0 \leq k_D \leq M_D - 1, \; 0 \leq k_i \leq M_i - 1, \; i = 1, \ldots, m,$$
$$w = k_D + k_m M_D + \sum_{i=1}^{m-1} k_i M_D \prod_{j=i+1}^{m} M_j. \tag{12}$$

The encoding at the source node is the same as described in Sec. II where a waveform $x_w(t)$ with symbol length T_s as defined in (4) is used at low duty-cycle $\theta \in (0, 1]$ with a length-N repetition coding. Since the total transmission time is NT_s/θ, the information rate is

$$r = \frac{\theta}{NT_s} \log(M) = r_D + \textstyle\sum_i r_i, \; \text{[nats]}, \tag{13}$$

where $r_i = \frac{\theta}{NT_s} \log(M_i)$ represents the information rate carried by the ith-layer bin index.

The decoding process at relaying nodes are described as follows. At relay node $\mathcal{R}_1$, the average received SNR is $\alpha_1 \frac{P}{N_0}$. M parallel matched filters are applied to the received signal Y_1 over all the M frequency carriers and all the N repeating copies. We denote the output at frequency f_k and the nth repetition copy as

$$A_{k,n}^{(1)} = \frac{1}{\sqrt{N_0(T_s - T_d)}} \int_{nT_c+T_d}^{nT_c+T_s} \exp(-j2\pi f_k t) y(t) dt.$$

Then we take the average power of all the N outputs $A_{k,n}^{(1)}$,

$$S_k^{(1)} = \frac{1}{N} \sum_{n=1}^{N} |A_{k,n}^{(1)}|^2, \; 0 \leq k \leq M-1,$$

and evaluate them against a predefined threshold

$$B^{(1)} = 1 + (1 - \epsilon_1) \frac{\alpha_1 P (T_s - T_d)}{\theta N_0},$$

for some $\epsilon_1 \in (0, 1)$. If there exist a unique $\hat{w} \in \mathcal{W}$ such that $S_{\hat{w}}^{(1)}$ exceeds the threshold $B^{(1)}$, we declare $\hat{w}$ is received; otherwise we declare an error. Following the similar error analysis as in [3], as $N \to \infty$ and $\theta \to 0$, the probability of error can be made arbitrarily small if

$$r = r_D + \sum_{i=1}^{m} r_i < (1 - \epsilon_1) \frac{\alpha_1 P}{N_0} \left(1 - \frac{T_d}{T_s}\right). \tag{14}$$

The relay node $\mathcal{R}_1$ then transmits the layer-1 bin index k_1 to the destination and to relays by peaky FSK via orthogonal channels. Reliable transmission of k_1 is possible if

$$r_1 < (1 - \tilde{\epsilon}_1) \frac{\tilde{\alpha}_1 P}{N_0} \left(1 - \frac{T_d}{T_s}\right), \tag{15}$$

$$r_1 < (1 - \bar{\epsilon}_{1,j}) \frac{\bar{\alpha}_{1,j} P}{N_0} \left(1 - \frac{T_d}{T_s}\right), \; j{=}2, \ldots, m, \tag{16}$$

for some $\tilde{\epsilon}_1, \bar{\epsilon}_{1,j} \in (0, 1)$.

After successfully receiving the layer-1 bin index k_1 from $\bar{Y}_2$, the relay node $\mathcal{R}_2$ applies M/M_1 parallel matched filters to Y_2 over the frequency carriers inside the layer-1 bin k_1, and compares their N-copy averages to a predefined threshold

$$B^{(2)} = 1 + (1 - \epsilon_2) \frac{\alpha_2 P (T_s - T_d)}{\theta N_0},$$

for some $\epsilon_2 \in (0, 1)$. Following the similar error analysis as in [12], the probability of error can be arbitrarily small if

$$r_D + \sum_{i=2}^{m} r_i < (1 - \epsilon_2) \frac{\alpha_2 P}{N_0} \left(1 - \frac{T_d}{T_s}\right). \tag{17}$$

Then $\mathcal{R}_1$ transmits the layer-2 bin index k_2 to the destination and to relays. Reliable transmission of k_2 is possible if

$$r_2 < (1 - \tilde{\epsilon}_2) \frac{\tilde{\alpha}_2 P}{N_0} \left(1 - \frac{T_d}{T_s}\right), \tag{18}$$

$$r_2 < (1 - \bar{\epsilon}_{2,j}) \frac{\bar{\alpha}_{2,j} P}{N_0} \left(1 - \frac{T_d}{T_s}\right), \; j{=}3, \ldots, m, \tag{19}$$

for some $\tilde{\epsilon}_2, \bar{\epsilon}_{2,j} \in (0, 1)$.

We continue this process for $\mathcal{R}_3, \ldots, \mathcal{R}_m$. The destination node successfully recovers bin indexes $[k_1, \ldots, k_m]$ from the relaying nodes, and it applies $M/(M_1 M_2 \cdots M_m) = M_D$ parallel matched filters to Y_D over the frequency carriers inside the layer-m bin given by $[k_1, k_2, \ldots, k_m]$, and evaluates their average against a predefined threshold

$$B^{(D)} = 1 + (1-\epsilon_D)\frac{\alpha_D P(T_s - T_d)}{\theta N_0},$$

for some $\epsilon_D \in (0,1)$. As before, the probability of error can be made arbitrarily small if

$$r_D < (1-\epsilon_D)\frac{\alpha_D P}{N_0}\left(1 - \frac{T_d}{T_s}\right). \tag{20}$$

By choosing T_s arbitrarily close to T_c and all ϵ arbitrarily small ($\to 0$), the achievable rate is given by

$$R_{FSK} = r_D + \sum_{i=1}^{m} r_i, \tag{21}$$

$$r_D < \frac{\alpha_D P}{N_0}\left(1 - \frac{T_d}{T_c}\right), \quad r_i < \frac{\tilde{\alpha}_i P}{N_0}\left(1 - \frac{T_d}{T_c}\right), \tag{22}$$

$$r_i < \frac{\bar{\alpha}_{i,j} P}{N_0}\left(1 - \frac{T_d}{T_c}\right), \quad \forall j = i{+}1, \ldots, m, \tag{23}$$

$$r_D + \sum_{k=i}^{m} r_k < \frac{\alpha_i P}{N_0}\left(1 - \frac{T_d}{T_c}\right). \tag{24}$$

Note that the upper and lower bounds meet when the channels from source/relay to the destination are the bottleneck.

C. When relay can't decode

It has been known since [7] that, without channel state information and without feedback, the achievable rate tends to zero as bandwidth grows if the signal energy spreads over the whole bandwidth. Instead, as suggested in [6], if signals concentrate on a fraction of available bandwidth such that that signal bandwidth is large enough to enjoy the benefit of linear scaling in the low SNR regime, but small enough to enable channel estimation at low cost, the achievable rate can approach the wideband Gaussian capacity within a small gap.

However, if a relay node can't decode, as shown in [11], even an Minimum Mean Square Error (MMSE) estimator can't help, resulting in a distortion with power approaching the signal power if Gaussian codewords are used for transmission. Even if we use peaky FSK instead, as in Sec. III-B and [12], the relay can't decode the bin index without decoding the whole message. Since the received signals at relay nodes are in low SNR regime with large bandwidth, we can claim that a relay's contribution is useless if it can't decode, as long as the rate of relay-destination channel is not sufficiently high, i.e., the received signal at relay node can't be forwarded to the destination without being contaminated by extra noise.

IV. Conclusion and Discussion

In this work, we have characterized the capacity upper and lower bounds of a multiple-relay network in the wideband regime over memoryless independent fading channels, which channel state information is not available. The upper bound is obtained by network equivalence tools and the lower bound is based on a multiple-layer binning peaky FSK. If a relay note can't decode, its contribution is useless as long as the rate of the cooperation channel is not sufficiently high.

It is interesting to compare out work to the distributed transmitter/receiver cooperation proposed in [16] where signals at distributed receivers are first quantized into fixed Q bits and then forwarded to the destination for decoding. For extended networks, as the number of nodes n scales up, the distance between clusters increases such that all the nodes in the receiving cluster end up in the low SNR regime. If Gaussian codebooks are used, they can either decode all the bits, which is not realistic, or they can't decode and thus useless. To solve this problem, [16] uses bursty transmission over frequency-flat channels (fixed known channel gain but random phase). Since the low duty-cycle goes to zero as n scales up, it is challenging to fulfill the channel model assumption.

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