Cross-Layer Design of Network-Coded Transmission with a Delay Constraint

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***Abstract*—We investigate the cross-layer design of wireless networks where end-to-end data transmission is subject to delay constraint and there is no end-to-end feedback. The transmission is coded by random linear network coding (RLNC) on packet level to recover from packet erasures and by forward error-correction coding (FEC) on bit level to combat channel distortions. Based on the two-layer model developed by Adams et al. where the end-to-end coded transmission is characterized by a throughput-reliability function, we formulate the cross-layer design as a goodput optimization problem relax the integrality constraint. We show that for single-hop transmissions there exists a globally optimal operating point for the relaxed problem. For multiple-hop transmissions, the goodput function is component-wise concave with respect to the physical layer data rate over each individual link.**

***Index Terms*—Wireless networks, coded transmission, delay constraint, cross-layer design, optimization**

I. Introduction

Data transmission over wireless networks suffers from packet losses caused by decoding errors on the physical layer as well as impairment from higher layers such as congestion control and buffer overflow. Traditionally, forward error-correction coding (FEC) implemented on the physical layer is designed primarily for reliability transmission over a single link, aiming to provide a very low error rate at the cost of data rates. Since wireless channels are polluted by noise and interference and further debased by time/frequency dispersion, the cost to provide reliable link-by-link wireless connection can be prohibitive in scenarios where data delivery service is enabled by mobile, airborne relays. Besides, even if link-by-link reliable transmission is feasible, it might not be the most resource efficient way to provide reliable end-to-end transmission due to erasures caused by high layers effects.

The inefficiency of the traditional wireless architecture for data transmission in a large network drives the research area of cross-layer modeling and design. For example, random codes are used in both physical and network layers in [1] to investigate the throughput-delay tradeoff over a single link. Perfect FEC and perfect packet erasure coding are employed in [2] to explore the tradeoff between spectral efficiency and reliability, whereas joint channel/packet coding are analyzed in [3] over fast block fading channels with a delay constraint. The works in [1]–[3], elucidate different facets of the cross-layer problem, consider only single-link networks where generalization to larger topologies is nontrivial. We also recognize here some work on multiple hops, albeit not necessarily wireless ones. Random linear network coding (RLNC) [4] with re-encoding at intermediate nodes is employed in [5] to translate a lossy multicast network into a lossless packet network with hyper-arcs, and the min-cut of the lossless packet network determines the capacity of the original lossy network. Wireless networks with correlated erasure links are analyzed in [6] and it is shown that linear network coding achieves its capacity. The delay in packet erasure networks is investigated in [7] where RLNC is used in a rateless fashion, and the delay is optimized based on the tradeoff between codeword lengths on physical layer and on network layer.

In most of previous work, cross-layer optimization has been investigated, but generally depends heavily on the underlying model and the objective function over which the optimization occurs. The differences between these models and objective functions can be subtle but consequential, which makes it difficult to derive general results from a specific cross-layer model. In light of the difficulties involved in trying to glean insight from two layers of coding, we have proposed in [8] a two-layer abstraction to model heterogeneous networks and showed that the end-to-end coded transmission with a delay constraint can be characterized by a throughput-reliability function. The network reduction tools we have proposed in [8] and further developed in [9] can provide tight upper and lower bounds for the throughput-reliability function of directed acyclic networks with arbitrary topology. With these tools developed in [8], [9], we investigate how to negotiate cross-layer design trade-offs with practically motivated constraints, including an end-to-end delay constraint and limited physical-layer configurations. We formulate the cross-layer design as an optimization problem with goodput or the link/network capacity as the objective function and relax the integrality constraint. We show that for single-hop transmissions there exists a globally optimal operating point for the relaxed problem. For multiple-hop transmissions, the goodput function is a concave function of the generation size, and is component-wise concave with respect to the physical layer data rate over each individual link.

The rest of the paper is organized as follows. In Sec. II we describe the system model and formulate the cross-layer design as an optimization problem. We present the derivations for concavity and component-wise concavity in Sec. III, and illustrate the benefit of cross-layer design in Sec. IV via a

three-node relay network.

II. System Model

A source node intends to transmit some messages over a wireless network to a remote destination node, and the transmission should be completed within a delay constraint τ. Since there is no end-to-end feedback, appropriate channel and network coding has to be implemented such that the probability of success stays above a predefined threshold. The two-layer model proposed in [8] employs FEC on bit level to combat distortions on the physical layer (PHY) and RLNC on packet level to recover from packet erasures on the network layer (NET). RLNC re-encoding is performed at intermediate nodes and we assume that any packet that contains bit errors after FEC decoding will be detected and erased.

On the NET layer, each packet has l bits with payload of ηl bits and overhead $(1-\eta)l$ bits, where $0 < \eta < 1$ is the percentage of payload. For each generation of source messages, we first split them evenly into k segments, each of ηl bits (appending zeros when necessary), and then pass these uncoded segments through RLNC encoding (generating payload) and packetization process (adding overhead) to generate packets of size l bits. We let n be the number of RLNC coded packets generated by the source.

On the PHY layer, for link i, each codeword contains m_i symbols over $\mathbb{F}_{z_i}$. The RLNC coded or re-encoded packets from the NET are grouped into frames, each of $m_i r_i$ bits, where r_i (bits per symbol) is the PHY data rate for link i. The frame-to-codeword mapping $\mathbb{F}_2^{m_i r_i} \rightarrow \mathbb{F}_{z_i}^{m_i}$ accommodates physical layer operations such as channel coding, modulation, and media access. At the PHY, we only focus on the codeword length m_i and data rate r_i in the following analysis, since the number of coded RLNC packets transmitted over link i is a function of m_i and r_i given the delay constraint.

The overall system operation can be summarized as

$$(\mathbb{F}_2^{\eta l})^k \rightarrow (\mathbb{F}_2^{\eta l})^n \rightarrow (\mathbb{F}_2^{l})^n \rightarrow \left\{ (\mathbb{F}_2^{m_i r_i})^{\frac{ln}{m_i r_i}} \rightarrow (\mathbb{F}_{z_i}^{m_i})^{\frac{ln}{m_i r_i}} \right\}. \tag{1}$$

Since a packet may be dropped due to decoding errors at the PHY or erasures at the NET, we denote the overall packet loss probability on link i by $\xi(m_i, r_i, \mathcal{H}_i)$ to highlight its dependence on the codeword length, data rate, and the link quality (includes both PHY and NET).

The design problem we consider involves picking the best code from a limited number of options on each layer, which is a relevant problem for many wireless systems, where transceivers use channel-state information to select a transmission mode from a small number of choices. We primarily consider goodput as the design objective,

$$\Gamma = k\eta\ell\,(1 - P_e)\,, \tag{2}$$

where P_e is the probability of failure (transmission fails to complete without error within the delay constraint).

The next important question concerns which parameters we are free to choose. To simplify notations and to highlight the key design parameters, in the following we assume that network-level parameters such as packet size l and the ratio of payload η are fixed across the network, and that all links have the same codeword length m. One advantage of fixing m is that the number of packets transmitted over link i is determined by r_i, so that we avoid having to optimize over both n and m. For a single link, this leaves us to solve an optimization problem over two parameters: r and k. For a large network whose topology can be represented by a directed acyclic graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, for every link $i \in \mathcal{E} \triangleq \{1, 2, \ldots, |\mathcal{E}|\}$, the pair of parameters (n_i, ξ_i) which represent the number of packets transmitted over link i and its associated packet erasure probability, are functions of the data rate r_i. Therefore we denote the end-to-end probability of failure by $P_e(k, \vec{n}(\vec{r}), \vec{\xi}(\vec{r}))$ and the goodput expression (2) can be written as

$$\Gamma(k, \vec{r}) = k\eta\ell\left(1 - P_e(k, \vec{n}(\vec{r}), \vec{\xi}(\vec{r}))\right). \tag{3}$$

Here we use vectors

$$\vec{r} = \langle r_1, \ldots r_{|\mathcal{E}|}\rangle,\ \vec{n} = \langle n_1, \ldots n_{|\mathcal{E}|}\rangle,\ \vec{\xi} = \langle \xi_1, \ldots \xi_{|\mathcal{E}|}\rangle,$$

to represent the data rates, number of transmitted packets, and the packet erasure probability, respectively, over all links.

III. Optimization

The optimization problem imposes natural integrality constraints on k and n, so we relax these constraints for all of the following arguments. In this relaxed problem, we will show that Γ is a concave function of k and is component-wise concave with respect to each element in $\vec{r}$. Therefore a good convex solver can be used the find solutions.

To prove the concavity of Γ in k, we need two steps: we first show that $P_e(k, \vec{n}, \vec{\xi})$ is convex in k under some conditions and then prove the concavity of Γ in k by proving its second derivative is non-positive due to the property we have shown in the first step. This result is presented in Sec. III-A.

To prove the component-wise concavity of Γ in $r_i, \forall i \in \mathcal{E}$, we only need to show that the end-to-end probability of error $P_e(k, \vec{n}, \vec{\xi})$ is component-wise convex with respect to r_i due to (3). The proof of the convexity of $P_e(k, \vec{n}, \vec{\xi})$ in the PHY-layer rate $r_i, \forall i$ requires more examination than convexity in k for two reasons. Firstly, both the number of transmitted packets $n_i(r_i)$ and the overall packet erasure probability $\xi_i(r_i)$ are functions of the PHY data rate r_i. Secondly, the interaction among links and the network topology also have influence on the end-to-end probability of error P_e. We will first prove the concavity of Γ in r (i.e., convexity of P_e in r) for one single link in isolation in Sec. III-B and establish the component-wise concavity in large networks in Sec. III-C.

A. Concavity of Γ in k

Since RLNC works over an erasure channel with ξ, the capacity of which is $(1-\xi)$, we will only consider rates less than the capacity for sake of good performance. With n fixed, this means that k is defined over the interval $[0, n(1-\xi)]$. Since the packet reception at the destination node can be modeled as a random arrival process of successfully received innovated

packets (or, the degree-of-freedom, DoF), denoted by D, we can relate P_e to the probability distribution of D as follows

$$P_e(k, \vec{n}, \vec{\xi}) = Pr\{D < k\}.$$

Note that the mean value $E[D]$, which is fully determined by the network topology $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and $(n_i, \xi_i, \forall i \in \mathcal{E})$, converges to the network capacity (measured by packets per unit time) asymptotically as the delay constraint goes to infinity.

In the following, we will first prove that $P_e(k, \vec{n}, \vec{\xi})$ is convex in k under certain conditions as described in Proposition 1, and then prove the concavity of Γ in k in Proposition 2.

Proposition 1: For a probability mass function (PMF) on D that is non-decreasing in D over the interval $[0, E[D]]$, $P_e(k, \vec{n}, \vec{\xi}) = Pr\{D < k\}$ is convex in k over the support of D within the interval $[0, E[D]]$.

Proof: Let dP_D be the Radon-Nikodym derivative of the cumulated distribution function (CDF) of D, and let $k_1 \leq k \leq k_2 \leq E[D]$, where $k_1,\ k,\ k_2 \in \text{support}\{D\}$ and,

$$k = \theta k_1 + \bar{\theta} k_2, \quad \theta \in [0, 1], \quad \bar{\theta} = 1 - \theta. \tag{4}$$

Because the PMF of D is non-decreasing in this interval, the average value over the range of k_1 to k will be lower than the average value over the range of k to k_2:

$$\frac{1}{k - k_1} \int_{k_1}^{k-1} dP_D \leq \frac{1}{k_2 - k} \int_{k}^{k_2 - 1} dP_D. \tag{5}$$

Multiplying both sides by $\frac{(k-k_1)(k_2-k)}{k_2-k_1}$, we obtain

$$\frac{k_2 - k}{k_2 - k_1} \int_{k_1}^{k-1} dP_D \leq \frac{k - k_1}{k_2 - k_1} \int_{k}^{k_2 - 1} dP_D. \tag{6}$$

Additionally, we may manipulate (4) to obtain

$$\theta = \frac{k_2 - k}{k_2 - k_1} \quad \text{and} \quad \bar{\theta} = \frac{k - k_1}{k_2 - k_1}. \tag{7}$$

Applying these identities to (6), we obtain

$$\begin{aligned}
\theta \int_{k_1}^{k-1} dP_D &\leq \bar{\theta} \int_{k}^{k_2 - 1} dP_D, \\
(\theta + \bar{\theta}) \int_{k_1}^{k-1} dP_D &\leq \bar{\theta} \int_{k_1}^{k_2 - 1} dP_D, \\
\int_0^{k-1} dP_D - \int_0^{k_1 - 1} dP_D &\leq \bar{\theta} \left(\int_0^{k_2 - 1} dP_D - \int_0^{k_1 - 1} dP_D \right), \\
Pr\{D<k\} - Pr\{D<k_1\} &\leq \bar{\theta} Pr\{D < k_2\} - \bar{\theta} Pr\{D < k_1\}, \\
Pr\{D<k\} &\leq \theta Pr\{D < k_1\} + \bar{\theta} Pr\{D < k_2\}.
\end{aligned}$$

The last statement satisfies the definition of convexity. ■

Note that the statement "over the support of D" in Proposition 1 is to include the cases where strict monotonicity below $E[D]$ does not hold over all integers, for example, when multiple packets are grouped into one frame and the erasure occurs on a frame-by-frame basis. In this case, restricting a maximization problem to the subset of integers that are evenly distributed over $[0, E[D]]$ is justifiable: packets arrive at the destination in a group-by-group basis.

Remark 1: When arguing monotonicity of the PMF, using the *Central Limit Theorem* we can approximate the PMF by the Q-function which itself decreases monotonically in its argument. For a single link, the monotonicity of the binomial distribution can be used directly: the PMF of $B(n, \xi)$ is monotonically increasing for all $k < (n+1)\xi$ and monotonically decreasing for all $k > (n+1)\xi$. For the special case where $(n+1)\xi$ is an integer, we have

$$P_r(k = (n+1)\xi) = P_r(k = (n+1)\xi - 1).$$

Since

$$(n+1)\xi - 1 < E(X) = n\xi < (n+1)\xi,$$

we can always say that the PMF is monotonically increasing for all $k < E[X]$ regardless the integrity of $(n+1)\xi$.

Proposition 2: Letting $\mathcal{C}$ be the minimum cut of the wireless network by modeling link i as a lossless packet channel with capacity $n_i(1 - \xi_i)$ on the NET layer, Γ is concave over $k \in (0, C_{min})$, where

$$C_{min} = \sum_{i \in \mathcal{C}} n_i (1 - \xi_i).$$

Proof: To simplify notations, we use an abbreviation $h(k) \triangleq P_e(k, \vec{n}, \vec{\xi})$. The first and second derivatives of Γ with respect to k are as follows (ignoring the constant factor ηl),

$$\frac{\partial \Gamma}{\partial k} = \frac{\partial}{\partial k} \left(k[1 - h(k)] \right) = 1 - h(k) - k h'(k), \tag{8}$$

$$\frac{\partial^2 \Gamma}{\partial k^2} = -2h'(k) - k h''(k) \leq 0. \tag{9}$$

The last inequality comes from the fact that $h'(k) \geq 0$ and $h''(k) \geq 0$, as $h(k) = P_e(k, \vec{n}, \vec{\xi})$ is monotonic and convex (by Proposition 1) in k. ■

B. Concavity of Γ in r, One Link

As the PHY-layer data rate r (bits per symbol) increases, the codeword decoding error p_e also increases, which will ultimately increase the overall packet erasure probability ξ seen at the NET. On the other hand, increasing r will increase the data rate of the link, which will also increase the number of transmitted packets n. Since the overall probability of failure $P_e(k, n, \xi)$ increases with ξ (more packet losses) but decreases with n (more redundancy added), it is unclear what effect an increase in r will have on $P_e(k, n, \xi)$ as we trade off rate for probability of decoding error on the PHY layer.

Let us first look at the nice properties of $n(r)$, $\xi(r)$, and $P_e(k, n, \xi)$ that are relevant for our purpose:

1) $n(r)$ is affine non-decreasing in r once we relax the integrality constraint;
2) $\xi(r)$ is convex non-decreasing in r, owing to the time-sharing principle;
3) $P_e(k, n, \xi)$ is convex non-increasing in n and convex non-decreasing in ξ, owing to the time-sharing principle.

We expect that many explicit formulations of $P_e(k, n, \xi)$ will be jointly convex in n and ξ, but this is hard to argue for general setups based on first principles. We will show the convexity of $P_e(k, n, \xi)$ in r in the following proposition.

Proposition 3: $P_e(k,n,\xi)$ is convex in $r \in [0, C_{\mathcal{H}})$ for large but finite delay constraints where $\xi \leq 1/2$ and $C_{\mathcal{H}}$ is the capacity of the channel $\mathcal{H}$ measured in bits per symbol time.

Proof: From the properties 1)–3) listed above, we obtain the following relations:

$$\begin{array}{llll} \frac{\partial P_e}{\partial n} \leq 0, & \frac{\partial P_e}{\partial \xi} \geq 0, & \frac{\partial n}{\partial r} > 0, & \frac{\partial \xi}{\partial r} \geq 0, \\ \frac{\partial^2 P_e}{\partial n^2} \geq 0, & \frac{\partial^2 P_e}{\partial \xi^2} \geq 0, & \frac{\partial^2 n}{\partial r^2} = 0, & \frac{\partial^2 \xi}{\partial r^2} \geq 0. \end{array} \tag{10}$$

The second derivative of P_e with respect to r are therefore:

$$\frac{\partial P_e}{\partial r} = \frac{\partial P_e}{\partial n}\frac{\partial n}{\partial r} + \frac{\partial P_e}{\partial \xi}\frac{\partial \xi}{\partial r}, \tag{11}$$

$$\begin{aligned} \frac{\partial^2 P_e}{\partial r^2} =& \frac{\partial^2 P_e}{\partial n^2}\left(\frac{\partial n}{\partial r}\right)^2 + \frac{\partial P_e}{\partial n}\frac{\partial^2 n}{\partial r^2} + \frac{\partial^2 P_e}{\partial \xi^2}\left(\frac{\partial \xi}{\partial r}\right)^2 \\ &+ \frac{\partial P_e}{\partial \xi}\frac{\partial^2 \xi}{\partial r^2} + 2\frac{\partial^2 P_e}{\partial n \partial \xi}\frac{\partial \xi}{\partial r}\frac{\partial n}{\partial r}. \end{aligned} \tag{12}$$

All of the terms in (12) are non-negative by the relations in (10) except the last one; the sign of $\frac{\partial^2 P_e}{\partial n \partial \xi}$ is yet unknown. In the following we will show that it is positive and therefore $\frac{\partial^2 P_e}{\partial r^2} \geq 0$, which is sufficient to prove our convexity argument.

For large delay constraint, the number of packets n transmitted through the link is large. We can approximate the probability of failure using the *Central Limit Theorem* (after relaxing the integrality constraint)

$$P_e(k,n,\xi) \simeq Q\left(\frac{n(1-\xi)-(k-1)}{\sqrt{n(1-\xi)\xi}}\right).$$

Since the Q-function decreases monotonically in its argument, we only need to prove $\frac{\partial^2 f}{\partial n \partial \xi} \leq 0$ for $f(n,\xi) = \frac{n(1-\xi)-(k-1)}{\sqrt{n(1-\xi)\xi}}$. We first take the partial derivative w.r.t. n, then ξ, obtaining

$$f(n,\xi) = \frac{n(1-\xi)-(k-1)}{\sqrt{n(1-\xi)\xi}} = \sqrt{n(\xi^{-1}-1)} - \frac{(k-1)}{\sqrt{n(1-\xi)\xi}}, \tag{13}$$

$$\frac{\partial f}{\partial n} = \sqrt{(\xi^{-1}-1)}n^{-1/2}/2 + \frac{(k-1)n^{-3/2}}{2\sqrt{(1-\xi)\xi}}, \tag{14}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial n \partial \xi} =& -\frac{(k-1)n^{-3/2}((1-\xi)\xi)^{-3/2}(1-2\xi)}{4} \\ &- \frac{n^{-1/2}\xi^{-2}}{4\sqrt{(\xi^{-1}-1)}} < 0, \ \forall \xi \in [0, 1/2). \end{aligned} \tag{15}$$

Therefore we can argue that $\frac{\partial^2 P_e}{\partial n \partial \xi} > 0, \forall \xi \in [0, 1/2)$, which leads to the convexity of $P_e(k, n(r), \xi(r))$ in $r \in [0, C_{\mathcal{H}})$ subject to the constraint that $\xi(r) < 1/2$. The constraint $0 < r < C_{\mathcal{H}}$ follows naturally from channel capacity. ■

C. Concavity of Γ in r_i, Multiple Links

Proposition 4: For any wireless network whose topology can be represented by a directed acyclic graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, Γ is concave in $r_i \in [0, C_{\mathcal{H}_i})$, $\forall i \in \mathcal{E}$.

Proof: Recall that for any choice of generation size k, the number of packets successfully passing through link i (i.e., the DoF), denoted by S_i, is characterized by the probability function

$$Pr\{S_i \geq k\} = \vec{\Lambda}_i[k] = 1 - P_{e,i}(k, n(r_i), \xi(r_i)), \tag{16}$$

where $P_{e,i}(k, n(r_i), \xi(r_i))$ denotes the equivalent probability of failure associated to link i should itself formulates a single-hop network. Because we have shown, in the proof of Proposition 3, that $P_{e,i}(k, n(r_i), \xi(r_i))$ is convex in r_i, then $\vec{\Lambda}_i[s]$ is concave in r_i for any choice of $s \leq k$.

To investigate the interaction among links in the network, let us first consider two links in tandem and in parallel. To make the dependency on r_i explicit, let us represent $\vec{\Lambda}_1[s]$ by the indexed function $f_s(r_1)$; similarly, we say for the second link that $\vec{\Lambda}_2[s] = g_s(r_2)$. Proving the convexity of $P_e(k, \vec{n}(\vec{r}), \vec{\xi}(\vec{r}))$ in either r_1 or r_2 for tandem links, we note that, given a choice of k, we have

$$\begin{aligned} 1 - P_e\left(k, \vec{n}(\vec{r}), \vec{\xi}(\vec{r})\right) &= f_k(r_1)g_k(r_2), \\ P_e\left(k, \vec{n}(\vec{r}), \vec{\xi}(\vec{r})\right) &= -f_k(r_1)g_k(r_2) + 1. \end{aligned} \tag{17}$$

In either variable we simply multiply a concave function by a negative coefficient and add a constant, producing a convex function.

For parallel links, the number of DoF we receive is given by $D = S_1 + S_2$. Then,

$$\begin{aligned} &Pr\{S_1 + S_2 \geq k\} \\ &= \sum_j Pr\{S_1 = j\}Pr\{S_2 \geq k-j\} \\ &= \sum_j \left[Pr\{S_1 \geq j\} - Pr\{S_1 \geq j+1\}\right] Pr\{S_2 \geq k-j\} \\ &= \sum_j \left[f_j(r_1) - f_{j+1}(r_1)\right] g_{k-j}(r_2). \end{aligned} \tag{18}$$

Because the CCDFs are monotonically decreasing, $f_j(r_1) - f_{j+1}(r_1) \geq 0$ for any choice of r_1, thus the sum in (18) is the sum of concave functions, and is itself concave. Then,

$$P_e\left(k, \vec{n}(\vec{r}), \vec{\xi}(\vec{r})\right) = 1 - Pr\{S_1 + S_2 \geq k\} \tag{19}$$

is convex in each r_i.

We derive the end-to-end probability of failure by applying the reduction operations for tandem and parallel links iteratively. For networks with topology that can't be directly reduced to a single link after applying the tandem and parallel link reduction methods, we have shown in [9] that for any directed acyclic graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ we can establish tight upper and lower bounds for the probability function P_e, in addition to several heuristic network reduction approaches developed there to reduce the network to a single link. As we go about removing links from the network, the property of convexity is carried over to the resulting probability of failure, so that finally we obtain $P_e(k, \vec{n}(\vec{r})$, which is component-wise convex in $r_i, \forall i \in \mathcal{E}$. Consequently, we have shown the component-wise concavity (w.r.t. every element in $\vec{r}$) of

$$\Gamma = \eta \ell k \left(1 - P_e(k, \vec{n}(\vec{r}), \vec{\xi}(\vec{r}))\right). \quad ■$$

IV. Optimization for Design

We have proved in Sec. III the convexity of the relaxed end-to-end error probability in r_i and k without defining explicit functions for error probability at either layer. It is important to note that we have not proved *joint convexity*. The supremum of the support of k (erasure channel capacity) is a function of $\vec{r}$, which makes joint convexity difficult to prove. Consequently, using the gradient or Newton's method to minimize P_e might only produce a local minimum. However, we can still use well-known tools for convex optimization in each variable to reduce the complexity of the design problem.

Although Proposition 4 does not imply joint concavity in $\vec{r}$, we can show that finding the maximizing point in the polyhedron $\mathcal{P} = \{\vec{r}\colon\ r_i \in [0, C_{\mathcal{H}_i}),\ \forall i \in \mathcal{E}\}$ can be done by finding the maximizing value of each r_i separately. This is equivalent to finding the values of r_i that minimize P_e. For tandem links, we obtain the gradient from (17):

$$\nabla P_e = \left\langle \frac{\partial P_e}{\partial r_1}, \frac{\partial P_e}{\partial r_2} \right\rangle = -\left\langle f_k'(r_1) g_k(r_2),\ f_k(r_1) g_k'(r_2) \right\rangle . \tag{20}$$

Because $f(r_1)$ and $g(r_2)$ are nonnegative, any value

$$r_1^* = \underset{r_1 \in [0, \mathcal{H}_1)}{\arg\min}\ P_e(k, r_1, r_2)$$

is independent of r_2, as $f'(r_1^*) = 0$ or P_e will be monotonic in r_1, so that either $r_1 = 0$ or $r_1 = C_{\mathcal{H}_1}$ will achieve the minimum. The same argument holds for convexity in r_2. For parallel links, differentiating (18) gives the gradient

$$\begin{aligned} \nabla P_e = -\langle \sum_j \left[g_j(r_2) - g_{j+1}(r_2)\right] f_{k-j}'(r_1), \\ \sum_j \left[f_j(r_1) - f_{j+1}(r_1)\right] g_{k-j}'(r_2) \rangle. \end{aligned} \tag{21}$$

Again, since $g_j(r_2) - g_{j+1}(r_2)$ and $f_j(r_1) - f_{j+1}(r_1)$ must be nonnegative, we apply the same reasoning that we applied to (20) and see that for both operations, P_e can be minimized in r_1 and r_2 independently.

It is therefore possible to approach the optimal value $\vec{r}$ asymptotically using well-known tools for convex optimization. We refer to [10] for further discussions on the performance and convergence of the biconvex optimization algorithms. Since we have relaxed the integrality constraint, necessary operations have to be taken to restore the solution to the original problem from the optimal solution of the relaxed problem. A natural option is using rounding, which may or may not be the best optimization. The effects of (randomized) rounding will be discussed in the future work.

We illustrate in Fig. 1 the benefit of cross-layer design maximizing the goodput against optimizing end-to-end or individual DoFs via a three-node relay network, where Reed-Solomon codes are used at the PHY to combat distortions from BSC and RLNC is implemented at the NET. Assuming we have channel state information and we are able to adjust PHY/NET-layer settings, picking up proper values of r_i and k from the optimization problem that give us the maximum goodput will provide better throughput-reliability tradeoff than optimizing either the end-to-end or individual DoFs, which only focus on asymptotical average of throughput (closely related to capacity) rather than delay constrained behavior. For this small design example the gain is not large, but the insights are clear. Although the most impaired cut (min-cut) still has the most significant impact on end-to-end performance, second-order effects need to be considered if we are aiming to approach the optimal throughput-reliability tradeoff given the delay constraint.

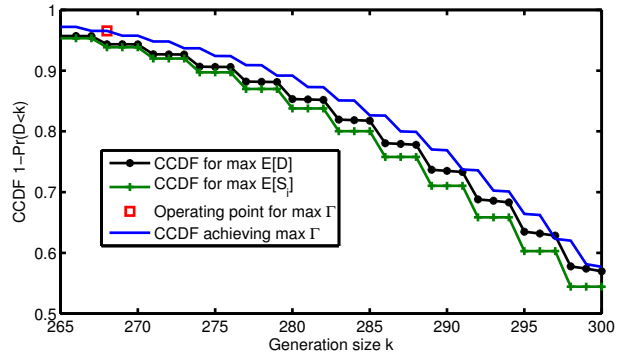

Fig. 1. The throughput-reliability curves of coded transmission over a three-node relay network where channels at the PHY are BSCs and NET channels are packet erasure links. Reed-Solomon codes of rate $\{1/8, \ldots, 7/8\}$ are possible at the PHY. The curves correspond to optimized design for maximizing the goodput Γ, the end-to-end DoF $E[D]$, and individual DoF $E[S_i]$. Note that the steps in the function are uneven because the two parallel links are delivering different number of packets in each codeword.

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