Optimal Symbol-by-Symbol Costa Precoding for a Relay-Aided Downlink Channel


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JINFENG DU, ERIK G. LARSSON, MING XIAO, MIKAEL SKOGLUND

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School of Electrical Engineering and the ACCESS Linnaeus Center,
Royal Institute of Technology (KTH), SE-100 44 Stockholm, Sweden

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Abstract—In this article, we consider practical approaches to Costa precoding (also known as dirty paper coding). Specifically, we propose a symbol-by-symbol scheme for cancellation of interference known at the transmitter in a relay-aided downlink channel. For finite-alphabet signaling and interference, we derive the optimal (in terms of maximum mutual information) modulator under a given power constraint. A sub-optimal modulator is also proposed by formulating an optimization problem that maximizes the minimum distance of the signal constellation, and this non-convex optimization problem is approximately solved by semidefinite relaxation. For the case of binary signaling with binary interference, we obtain a closed-form solution for the sub-optimal modulator, which only suffers little performance degradation compared to the optimal modulator in the region of interest. For more general signal constellations and more general interference distributions, we propose an optimized Tomlinson-Harashima precoder (THP), which uniformly outperforms conventional THP with heuristic parameters. Bit-level simulation shows that the optimal and sub-optimal modulators can achieve significant gains over the THP benchmark as well as over non-Costa reference schemes, especially when the power of the interference is larger than the power of the noise.

Index Terms—Costa precoding, dirty paper coding, interference, modulation, relay channel.

I. INTRODUCTION

F rom information theory, it is known since [1] that the achievable rate of a communication channel remains unchanged if the receiver observes the transmitted signal in the presence of additive interference and white Gaussian noise, provided that the transmitter knows the interference non-causally. The resulting precoding method is known as “Costa precoding” or “dirty paper coding” (DPC) after the title of [1]. The problem of designing a DPC transmitter is important because the scenario with known interference arises in many contexts, notably, in precoding for intersymbol interference channels and for the downlink multuser MIMO wireless channel [2]–[5]. In [4] DPC has been shown to be capacity achieving in non-degraded MIMO broadcast channels. DPC can also be applied in a cooperative two-transmitter two-receiver wireless network [6], in relay-aided broadcast channels [7], and in relay interference channels with a cognitive source [8]. Essentially, an information theoretic strategy for achieving capacity is known; it is precisely the achievability proof in [1] and works as follows: First quantize the interference into a number of bins and then, depending on what bin the interference falls into, choose an appropriate code to encode the message at the transmitter. This approach has been used with success in [9], [10], for example, where sophisticated coding schemes were proposed based on superposition coding [9], lattices and trellis shaping [10]. Trellis and convolutional precoding was used in [11] where the trellis shaping was developed taking into account the knowledge of a noncausal interference sequence.

In this work we study practical DPC schemes in the context of a relay-aided downlink channel. Consider a communication network where the base station transmits information symbols \( \omega_1 \) and \( \omega_2 \) to user 1 and user 2, respectively, with the aid of a half-duplex relay (a relay that cannot transmit and receive simultaneously). As illustrated in Figure 1, the relay is dedicated to assist user 1 (the weaker/more distant user) whose direct link with the source fails. The base station transmits \( x_1 \) (signal for \( \omega_1 \)) during time slot \( t_1 \) and \( x_2 \) (signal for \( \omega_2 \)) during \( t_2 \). The relay listens to the base station during \( t_1 \) and transmits \( z = f(y_r) \) during \( t_2 \), where \( y_r \) is the received signal at the relay during \( t_1 \) and \( f(\cdot) \) is a relay mapping function. The relaying signal \( z \), which is useful for user 1, appears as interference for user 2. Assuming that the relaying function
f(⋅) is known at the base station and that the source-relay link is good, the “interference” z will be known non-causally at the base station with high probability, effectively resulting in the Costa problem.

The goal of our work is to obtain an understanding for what one can achieve in small (or a single) dimensions of signals and at low complexity, rather than to achieve the channel one can achieve in small (or a single) dimensions of signals.

The well known Tomlinson-Harashima precoder (THP) [12, 13], originally proposed for channels with inter-symbol interference, is a symbol-by-symbol DPC approach and therefore it serves as a good benchmark. The achievable rate for THP has been investigated in [14] and a scaled THP has been invented in [15]. THP with partial channel knowledge has been studied in [16]. Essentially THP (and its variations) subtracts the interference z from the information-bearing symbol and then performs a modulo operation to avoid a power boost. Another reason for introducing THP is that it already has wide applications. For instance, THP has been proposed as a building block for transmitter precoding for the downlink multiuser MIMO channel [17], [18]. Another symbol-by-symbol DPC scheme proposed in [19] minimizes the uncoded symbol error probability by joint design of the modulator and the demodulator. It is omitted in this paper due to the difficulty to evaluate its performance in terms of mutual information.

In our conference paper [20], we have presented the optimal modulation for binary signaling with binary interference based on an exhaustive search over 12 possible mappings, which typically outperforms THP even when the parameters of THP are optimally chosen. Based on these preliminary findings, we propose here a mapping set size reduction method that map the information symbol \( \omega_2 \in \mathbb{Z} \) and an interference symbol \( z \in \mathbb{R} \) (known to the base station but not to user 2) onto an output symbol \( x_2 \in \mathbb{R} \). Thereby, our focus is on symbol-by-symbol modulation rather than on coding. To get a better understanding of how our proposed scheme performs compared to the theoretical limit, we will use the mutual information between the transmitted \( \omega_2 \) and the received signal at user 2 as the criterion for design.

The goal of our work is to obtain an understanding for what one can achieve in small (or a single) dimensions of signals and at low complexity, rather than to achieve the channel one can achieve in small (or a single) dimensions of signals. Our proposed DPC schemes are evaluated in terms of mutual information, coded bit-error-rate (BER), as well as energy efficiency, and compared to two non-DPC approaches, namely orthogonal transmission and receiver centric interference cancellation.

The rest of this paper is organized as follows. The system model and design criteria are introduced in Sec. II, where a brief overview of THP is also presented. The optimal modulator and the sub-optimal modulator are discussed in Sec. III, and THP with optimized parameters for Gaussian interference is presented in Sec. IV. Two non-DPC schemes are discussed in Sec. V as a reference. Simulation results are presented in Sec. VI and conclusions are drawn in Sec. VII.

**Notations:** \( X \) denotes a matrix and \( x \) denotes a vector. \( (\cdot)^T \) indicates matrix/vector transpose and \( \text{Tr}(\cdot) \) means the trace of a matrix. \( N! \) denotes the factorial of the integer \( N \). \( E[\cdot] \) stands for the expected value of a random variable and \( P(\cdot) \) denotes the probability of a discrete-valued random variable. \( p_y(t) \) indicates the value of the probability density function (pdf) of a continuous-valued variable \( y \) at the position where \( y = t \). The random variable and its realization will not be explicitly distinguished unless necessary.

**II. SYSTEM MODEL AND TOMLINSON-HARASHIMA PRECODER**

From now on, we consider a discrete, one-dimensional Gaussian channel, and all quantities are real-valued and scalar. As shown in Figure 1, the base station transmits \( x(\omega_1) \) during time slot \( t_1 \). The relay receives \( y_r = x(\omega_1) + n_r \) during \( t_1 \), where \( n_r \) is noise, and generates the relaying signal \( z = y_r \) dedicated for user 1. During time slot \( t_2 \), the base station transmits \( x_2 \) to user 2 and the relay transmits \( z_1 \) to user 1 through the same channel. Therefore the received signal at user 2 in \( t_2 \) can be written as

\[
y = x_2 + z + n, \tag{1}
\]

where \( n \) is noise. The design of the optimal relay mapping \( f(\cdot) \) is interesting and challenging, as discussed in [21], [22]. For example, we can choose the memoryless relaying function proposed in [21] to maximize the generalized signal-to-noise power ratio (GSNR) at user 1, or utilize the constellation rearrangement proposed in [22] to maximize the rate for user 1 if its direct link with the source exists. The joint optimization of the relay function and the modulator in the base station is rather complicated. To simplify the analysis and highlight the insights gained in this paper, hereafter we assume a perfect source-relay link\(^2\) in Figure 1 with a deterministic relay mapping \( z = \sqrt{\frac{P_x}{P_r}} x_1 \). The DPC modulator in the base station that we envision maps an information symbol \( \omega_2 \) from an \( M \)-ary alphabet \( \{0, \ldots, M-1\} \), and the interfering relay symbol \( z \in \mathbb{R} \), onto a modulated symbol \( x_2 \in \mathbb{R} \), through the (nonlinear) modulator mapping as follows

\[
x_2 = X(\omega_2, z).
\]

User 2 does not know \( z \), but we shall assume that it knows the probability distribution of \( z \), say \( p_z(u) \). This assumption

\(^1\)Extension to inphase/quadrature (narrowband) modulation, or to other orthogonal multiplexing formats is immediate by treating each dimension independently.

\(^2\)Throughout we use “optimal” in the sense of maximum mutual information or minimum error probability. When not explicitly stated, we refer jointly to both these criteria.

\(^3\)For \( P_x \gg \sigma_r^2 \), \( x_1 \) can be almost perfectly known/estimated at the relay.
is weak if z is drawn from a stationary and ergodic process, because then the base station can provide information about \( p_2(u) \) to user 2. We assume that the noise is Gaussian: \( n \sim \mathcal{N}(0, \sigma^2) \) where \( \sigma^2 \) is known. Furthermore, we assume that the available average transmit power is fixed to a constant \( P_2 \). With the optimization criterion of the mutual information \( I(y; \omega_2) \), the problem is then to find the best possible mapping \( x_2 = X(\omega_2, z) \) that maximizes \( I(y; \omega_2) \), i.e.,

\[
X(\omega_2, z) = \arg \max_{x} \, E\left[ I(x; \omega_2) \right],
\]

where

\[
I(y; \omega_2) = H(\omega_2) - H(\omega_2|y)
\]

\[
= \sum_{\omega_2=0}^{M-1} \int_{-\infty}^{\infty} p_y(y, \omega_2) \log P(\omega_2|y) dy - \sum_{\omega_2=0}^{M-1} P(\omega_2) \log P(\omega_2)
\]

\[
= \sum_{\omega_2=0}^{M-1} \left[ \int_{-\infty}^{\infty} p_y(y, \omega_2) \log \frac{p_y(y, \omega_2)}{p_b(y)} dy - P(\omega_2) \log P(\omega_2) \right]
\]

\[
= \sum_{\omega_2=0}^{M-1} P(\omega_2) \int_{-\infty}^{\infty} p_y(y|\omega_2) \log \frac{p_y(y|\omega_2)}{P(\omega_2)} dy.
\]

The last equality comes from the fact that \( \int_{-\infty}^{\infty} p_y(y|\omega_2) dy = 1, \forall \omega_2 \). In practice, \( I(y; \omega_2) \) can be easily computed by Monte-Carlo integration. Naturally \( p_y(y|\omega_2) \) and \( P(\omega_2) \) depend on both the specific modulator mapping \( X(\omega_2, z) \) and the distribution \( p_z(u) \).

### A. Tomlinson-Harashima precoding (THP)

THP is the best known available baseline for comparison and therefore we outline its principle here. THP first maps \( \omega_2 \) onto a constellation point by modulating it via \( x(\omega_2) \), and then subtracts the interference \( z \) from it. A modulo operation \( \text{mod}(\cdot, \Lambda) \) is then carried out so that the resulting transmitted signal falls into the region \([-\Lambda/2, \Lambda/2] \). Therefore we have

\[
x_2 = X(\omega_2, z) = \text{mod}(x(\omega_2) - z, \Lambda),
\]

\[
y = \text{mod}(x(\omega_2) - z, \Lambda) + z + n = x(\omega_2) + k\Lambda + n = x(\omega_2) + e,
\]

where \( k \) is an integer which depends both on \( \omega_2 \) and \( z \). Note that the equivalent noise term \( e = k\Lambda + n \) also depends on \( \omega_2 \). In papers dealing with THP, the following heuristic (and suboptimal) detector is usually used:

\[
\hat{\omega}_2_{\text{subop}} = \arg \min_{\omega_2} |\text{mod}(y, \Lambda) - x(\omega_2)|.
\]

To find the minimum error-probability receiver for THP, first note that

\[
p_y(y|\omega_2) = \sum_{k=-\infty}^{\infty} P(k|\omega_2) p_n(y - x(\omega_2) - k\Lambda),
\]

where the integer \( k \) is random with the following conditional distribution:

\[
P(k|\omega_2) = P(x(\omega_2) - z \in [(k+1/2)\Lambda, -(k-1/2)\Lambda] | \omega_2)
\]

\[
= P(x(\omega_2) + (k-1/2)\Lambda \leq z \leq x(\omega_2) + (k+1/2)\Lambda | \omega_2)
\]

\[
= F_z(x(\omega_2) + (k+1/2)\Lambda) - F_z(x(\omega_2) + (k-1/2)\Lambda).
\]

In (4), \( F_z(t) = \int_{-\infty}^{t} f_z(u) du \) is the cumulative distribution function of \( z \). The maximum a posteriori (MAP) receiver finds the most likely \( \omega_2 \) when \( y \) is received:

\[
\hat{\omega}_2_{\text{MAP}} = \arg \max_{\omega_2} P(\omega_2|y) = \arg \max_{\omega_2} p_y(y|\omega_2)
\]

\[
= \arg \max_{\omega_2} \sum_{k=-\infty}^{\infty} P(k|\omega_2) \exp \left( \frac{(y - x(\omega_2) - k\Lambda)^2}{2\sigma^2} \right),
\]

where the second equality comes from the assumption of equally probable \( \omega_2 \). In practice the sum in (5) can be truncated to a few terms since \( P(k|\omega_2) \) decreases rapidly (exponentially if \( z \) is Gaussian) as \( |k| \) increases. The difference in performance between the two receivers, however, is usually small except for “unlucky” choices of the mapping \( x(\omega_2) \) and \( \Lambda \), i.e., when \( P(k \neq k_0|\omega_2) \) is significant where \( k_0 \) satisfies \( \text{mod}(y, \Lambda) = y - k_0\Lambda \).

### III. DESIGN OF THE OPTIMUM MODULATOR

In this section we first find the optimal mapping modulator for binary signaling with binary interference and then generalize it to higher order modulations. A suboptimal modulator by maximizing the minimum distance among constellation points is also proposed by formulating an optimization problem.

#### A. Optimal mapping for binary signaling with binary interference

For discrete, binary random variables \( \omega_2 \) and \( z \) (over \( \mathbb{Z} \) and \( \mathbb{R} \), respectively), we assume that

\[
P(\omega_2=0) = P(\omega_2=1) = 1/2, \ P(z=-\beta) = P(z=\beta) = 1/2.
\]

That is, the input alphabet is binary (\( \omega_2 = 0, 1 \)) and the interference comes from a scaled BPSK constellation \( z = \pm \beta \). Also, \( \omega_2 \) and \( z \) are independent and all combinations of \( (\omega_2, z) \) are equally likely. Therefore the mapping \( X(\omega_2, z) \) can be explicitly written as

\[
X(\omega_2 = 0, z = -\beta) \triangleq s_0, \quad X(\omega_2 = 0, z = \beta) \triangleq s_1,
\]

\[
X(\omega_2 = 1, z = -\beta) \triangleq s_2, \quad X(\omega_2 = 1, z = \beta) \triangleq s_3.
\]

By symmetry \((\omega_2, z)\) and have symmetric probability densities),

we must have \( x \in \{-a, -b, a, b\} \) for some positive constants \( a, b \). The problem is then to find suitable \((a, b)\) to map \( s_0, s_1, s_3 \) onto the set \( \{-a, -b, a, b\} \) such that \( I(y; \omega_2) \) stated in (3) is maximized. Note that

\[
p_y(y|\omega_2) = \sum_{z=\pm \beta} p_{y,z}(y, z|\omega_2) = \sum_{z=\pm \beta} p_y(y|\omega_2, z) P(z),
\]

where

\[
p_y(y|\omega_2, z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(y - X(\omega_2, z))^2}{2\sigma^2} \right).
\]

There are \( 4! = 24 \) permutations of the elements in \( \{-a, -b, a, b\} \), of which 12 are redundant \((a, b \text{ and } b, a \text{ are ordered})\). The set of all possible mappings \((s_0, s_1, s_2, s_3)\) to be considered are:

- (I) \((a, -a, b, -b)\);
- (II) \((a, -b, b, -a)\);
- (III) \((-a, -b, a, b)\);
- (IV) \((-a, b, -a, -b)\);
- (V) \((-a, -a, b, -b)\);
- (VI) \((-a, b, b, a)\);
- (VII) \((-a, a, -b, b)\);
- (VIII) \((-a, b, -a, b)\);
- (IX) \((a, b, -a, b)\);
- (X) \((a, a, -b, b)\);
- (XI) \((a, a, -b, b)\);
- (XII) \((a, -a, -b, b)\).
The mapping $X(\omega_2, z)$ is a deterministic function that assigns one of the values $\{-a, -b, b, a\}$ to $x_2$ for each possible pair $(\omega_2, z)$. Since the variables $\omega_2$ and $z$ are independent and equiprobable (see (6)), it follows that all four possibilities for $x_2$, viz. $x_2\in\{-a, -b, b, a\}$ are equally likely. Thus the power constraint translates into $E[x_2^2] = (a^2 + b^2)/2 \leq P_t$.

A straightforward approach, as stated in our preliminary work [20], is to perform an exhaustive search over a fine grid which contains all $(a, b)$ that satisfy this constraint. And for each $(a, b)$ we examine all the 12 mappings to identify the optimal modulation which generates the highest mutual information. This optimization process can be carried out offline and the result can be stored in a look-up table (indexed by $P_t/\sigma^2$ and $\beta^2/\sigma^2$) with resolution as required.

The minimum error-probability receiver for the optimal (maximum mutual information) modulator has a rather simple form. To write it out explicitly, noting (8) that
\[
\omega_{2\text{MAP}} = \text{argmax}_{\omega_2} \sum_{z=\pm \beta} \exp\left(-\frac{(y - z - X(\omega_2, z))^2}{2\sigma^2}\right).
\]
When the assumption of a perfect source-relay link does not hold, i.e., when $z$ is not perfectly known at the relay, the conditional probability $p_y(y|\omega_2)$ must be adjusted to reflect the reliability of $z$. Given the transmit power $P_t$ and source-relay link noise power $\sigma_n^2$, the conditional probability (8) should be rewritten as
\[
p_y(y|\omega_2) = (1 - P_e) \sum_{z=\pm \beta} p_y(y|\omega_2, z)P(z) + P_e \sum_{z=\pm \beta} p_y(y|\omega_2, -z)P(z),
\]
where $P_e=Q(\sqrt{P_t/\sigma^2})$ is the error probability of detecting the BPSK modulated $\omega_1$ (hence $z$).

### B. Extension to higher order modulation

Despite the fact that the optimization can be done off-line, it is not directly feasible to extend the exhaustive search method proposed in Section III-A to higher-order modulation since the number of possible mappings increases explosively with the order of the modulation. For $M$-PAM with $N$-PAM interference, in total we have $MN$ combinations for $(\omega_2, z)$ and therefore the same number of possible $X(\omega_2, z)$ values. Their amplitudes are symmetric in the real field $\mathbb{R}$ around the origin and hence at most half of them, i.e. $MN/2$, are free to choose under the power constraint. Besides, there are in total $(MN)!$ permutations of the set of $MN$ parameters. Since $MN/2$ of these parameters have no ordering constraint, the number of all possible mappings is $(MN/2)!$. And then for each of these mappings, we still have to do an exhaustive grid search along $MN/2$ dimensions to find the optimal modulator for a particular combination of $P_t/\sigma^2$ and $P_e/\sigma^2$.

For example, in the case of 4-PAM signaling with BPSK interference, there are in total 8!/4! = 1680 different mappings and we have to do an exhaustive grid search over 4 dimensions. Therefore for higher-order modulation, the number of candidate mappings can become prohibitively large and makes the off-line exhaustive search computationally impractical. In what follows we will present a method which can greatly reduce the number of mappings.

We start with the special case with binary signaling and binary interference, as stated in (6). By comparing all the mappings in (9), we come up with the following observations:

1) Two mappings are said to be equivalent if one can be obtained from the other by exchanging $X(0, z)$ and $X(1, z)$ for all $z$;

2) Mappings satisfying $X(0, z)X(1, z) > 0$ will result in smaller distance between $\omega = 0$ and $\omega = 1$ in the received signal constellations, and therefore should not be considered;

3) Mappings should satisfy $|X(0, z) - X(1, z)| = |X(0, -z) - X(1, -z)|$.

All the equivalent mappings defined by Observation 1 are identical in the sense that $a$ and $b$ are commutatable, and therefore we will group them together in a pair of parenthesis. For example, we group the following pairs of equivalent mappings together: (III, IX), (IV, X), (V, XI), and (VI, XII). By applying Observation 2, mappings I, II, VII, VIII are excluded. By applying Observation 3, mappings (IV, X) and (V, XI) are also excluded. Now we only have two groups left: (III, IX) and (VI, XII). We then search over a fine grid which contains all $(a, b)$ that satisfy the power constraint, and for each $(a, b)$ we only examine the above mentioned two mappings (one element from each group, say IX and XII) instead of twelve as in Section III-A.

For the general cases with uniformly distributed information symbols $\omega \in \{0, \ldots, M - 1\}$ and uniformly distributed interference $z \in \{z_0, \ldots, z_{N-1}\}$ with $N$-PAM modulation, by defining the modulation vector associated with $\omega$ as $X(\omega) \triangleq [X(\omega, z) \forall z = [X(\omega, z_0), \ldots, X(\omega, z_{N-1})]$, the following principles can be applied to reduce the number of mapping candidates:

1) Mappings $X_1(\omega, z)$ and $X_2(\omega, z)$ are equivalent if they have the same vector set4, i.e., \(\{X_1(\omega)|\forall \omega\} = \{X_2(\omega)|\forall \omega\}\), where $X_i(\omega) \triangleq [X_i(\omega, z)|\forall z|$, $i = 1, 2$;

2) For $\omega_i \neq \omega_j$, the elements in received signal constellation subset $\{X(\omega_i, z) + z|\forall z\}$ should be separated as far as possible away from any elements in $\{X(\omega_j, z) + z|\forall z\}$;

3) For each interference pair $(z, -z)$, the subsets $\{X(\omega, z) + z|\forall \omega\} \{X(\omega, -z) - z|\forall \omega\}$ should be equivalent in the sense that they are symmetric with respect to the origin.

The equivalent mappings defined by the first principle will be grouped together and all the mappings that do not follow the second and the third principles will be deemed “unfavorable” and therefore be dropped. For example, by applying the above principles, the number of mappings for 4-PAM signaling with BPSK interference can be reduced from 1680 down to 133.

### C. Maximized minimum distance based sub-optimal modulator

As stated in Section III-B, the off-line optimization can be greatly simplified by reducing the number of mappings.

4The assignment of each modulation vector in $\{X(0), \ldots, X(M - 1)\}$ to an information symbol $\omega \in \{0, \ldots, M - 1\}$ should not affect the achievable rate or symbol error probability.
However, since the optimization in (2) is non-convex, the complexity of a grid search will increase exponentially with the number of searching dimensions. Therefore, we propose here a low-complexity sub-optimal modulator based on the criterion of maximized minimum distances among the constellation points.

For uniformly distributed information symbols \( \omega \in \{0, \ldots, M - 1\} \) and uniformly distributed interference \( z \) with \( N \)-PAM modulation, the distance between received signal constellation points for \( \omega_i \neq \omega_j \) (omitting the noise term for simplicity) can be classified into two types:

\[
\begin{align*}
d_i &= |X(\omega_i, z_m) + z_m - (X(\omega_j, z_m) + z_m)| \\
&= |X(\omega_i, z_m) - X(\omega_j, z_m)|, \\
d_a &= |X(\omega_i, z_m) + z_m - (X(\omega_j, z_n) + z_n)| \\
&= |X(\omega_i, z_m) - X(\omega_j, z_n) + (z_m - z_n)|,
\end{align*}
\]

where \( z_m, z_n \) are interference symbols.

There are in total \( N_I = \frac{M(M - 1)N}{2} \) type I distances \( d_i \) and \( N_{II} = \frac{M(M - 1)(N - 1)}{2} \) type II distances \( d_a \). By reformulating the mapping \( \bar{X}(\omega, z) \) to a vector

\[
x = [X(0, z_0), \ldots, X(0, z_{N-1}), X(1, z_0), \ldots, X(M-1, z_{N-1})],
\]

and denoting \( x(n) \) as the \( n \)th element of \( x \), we can rewrite (10) as follows

\[
d_{ik}^2 = (x(i_k) - x(j_k))^2 = xA_kx^T, \tag{11}
\]

\[
d_{il}^2 = (x(i_l) - x(j_l) + \eta_l)^2 = xB_lx^T + 2xb_l^T + \eta_l^2, \tag{12}
\]

where \( b_l \) are \( 1 \times MN \) sparse vectors each with only two non-zero elements \( b_l(i_l) = \eta_l \) and \( b_l(j_l) = -\eta_l \), and \( A_k (B_l) \) are \( MN \times MN \) sparse symmetric matrices each with only four non-zero elements placed in their diagonal and anti-diagonal positions defined by \( i_k, j_k (i_l, j_l) \), i.e.,

\[
A_k \text{ or } B_l = \begin{bmatrix}
\vdots & \ddots & \ddots & \ddots \\
. & . & \eta_l & . \\
. & . & -\eta_l & . \\
\vdots & \ddots & \ddots & \ddots \\
. & . & . & .
\end{bmatrix}, \quad b_l^T = \begin{bmatrix}
\eta_l \\
. \\
. \\
-\eta_l \\
. \\
. \\
. \\
. \\
. \\
\end{bmatrix}.
\]

The sub-optimal modulator can therefore be formulated based on (11) as an inhomogeneous quadratically-constrained quadratic program (QCCQP) [23] problem,

\[
\max_{x \in \mathbb{R}^{MN}} \min_{l=1, \ldots, N_I} \\{ xA_kx^T, xB_lx^T + 2xb_l^T + \eta_l^2 \} \tag{13}
\]

subject to \( x \) \( xX \leq MN \cdot P_x \).

The solution of (13) will yield constellations with large mutual information, since a constellation that offers a large constellation-constraint mutual information also has a large minimum distance. However, the exact solution of (13) is hard to find since the problem is non-convex. But after reformulation [23] and semi-definite relaxation (SDR) [24] approximation, we can introduce some new matrices

\[
A_k = \begin{bmatrix}
A_k & 0 \\
0 & 0
\end{bmatrix}, \quad B_l = \begin{bmatrix}
B_l & b_l^T \\
0 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix},
\]

and therefore obtain the following relaxed version of (13):

\[
\max_{x \in \mathbb{S}^{MN+1}} t \tag{14}
\]

subject to \( \text{Tr}(A_kx) \geq t, \quad k = 1, \ldots, N_k \),

\[
\text{Tr}(B_lx) + \eta_l^2 \geq t, \quad l = 1, \ldots, N_{II},
\]

\[
\text{Tr}(Cx) \leq MN \cdot P_x, \quad x \succeq 0,
\]

\[
X(MN + 1, MN + 1) = 1,
\]

where \( \mathbb{S}^{n} \) denotes the set of \( n \times n \) symmetric matrices. Since (14) is an instance of semi-definite programming [23], it can be solved in a numerically reliable and efficient fashion by convex optimization software, e.g. CVX [25]. However, the globally optimal solution \( X^* \) to (14) in general has rank greater than 1, and therefore is not a feasible solution to the original problem (13). We can extract from \( X^* \) a feasible (normally sub-optimal) solution \( x \) to (13) through randomization with provable approximation accuracy, see [24] and references therein for more details.

Note that (13) and (14) are actually a realization of the Principle 2) stated in Section III-B. Besides, Principle 3) can also be utilized to add extra \( MN \) linear constraints to (13). Then following the same procedure of reformulation and relaxation, we can formulate a new optimization problem similar to (14). Detailed discussions on relaxation, randomization, and approximation are omitted here due to space limitations.

For the special case of \( M = N = 2 \), by confining ourselves to the selected mappings IX and XII in (9), we can solve (13) analytically (see Appendix A for a detailed derivation), resulting in a closed-form solution for the modulation mapping \( \bar{X}(\omega, z) \) as follows:

\[
\begin{align*}
\text{XI}, \quad a &= \sqrt{P_x}, \quad b = \sqrt{P_x}, \quad \text{if } P_x \leq \beta^2; \\
\text{XII}, \quad a = \beta, \quad b = \sqrt{2P_x - \beta^2}, \quad \text{if } \beta^2 < P_x < 5\beta^2; \\
\text{IX}, \quad a = \sqrt{P_x - \beta^2 + \beta}, \quad b = \sqrt{P_x - \beta^2 - \beta}, \quad \text{if } P_x \geq 5\beta^2.
\end{align*}
\]

This sub-optimal modulation can be carried out on-line given the instantaneous channel conditions.

IV. OPTIMIZED THP FOR ARBITRARY SIGNAL AND INTERFERENCE

In Section III we have discussed the modulator design \( x = \bar{X}(\omega, z) \) given information symbols \( \omega \) from an \( M \)-ary alphabet and an interference signal \( z \) modulated with \( N \)-PAM. We provided the optimal nonlinear mapping based on an exhaustive grid search, and a sub-optimal mapping based on convex optimization and relaxation. For an interference signal with a more general distribution (say Gaussian), however, it appears impractical (at least without approximations) to design the Costa modulator based the methods proposed in Section III. The THP modulation, however, fits for arbitrary signal and interference constellations and therefore can be regarded as a good candidate for such scenarios. The advantage of staying within the framework of THP is twofold. First, there are only two parameters to optimize over, as shown later in this section. Second, THP with heuristic parameter choices (which is commonly used in the literature) is known to provide significant gains over no-interference-cancellation.

Let \( \alpha \) be half of the minimum distance between the uniformly distributed constellation points, i.e., \( x \in \{-\alpha, \alpha\} \) for
BPSK and \( x \in \{-3\alpha, -\alpha, \alpha, 3\alpha\} \) for 4-PAM modulation, and let \( \Lambda \) be the parameter for modulation operations. Then THP modulation with transmit power constraint \( P_x \) has two parameters \( \alpha, \Lambda \) to optimize over. THP with the heuristic parameter choice \( \Lambda = 2M\alpha \) for \( M \)-PAM modulation appears to be customary and is the choice described in Chapter 10 of [2]. This method, referred as heuristic THP hereafter, is rather simple and can be used for general signal constellations and interference distributions. The actual value of \( \Lambda \) (therefore the value of \( \alpha \)) is determined by the interference power \( P_x \) and the transmit power constraint \( P_x \).

For higher-order modulation with equiprobable information symbols, the resulting transmitted signal after modulo operations turns to be approximately uniformly distributed in the region of \([-\Lambda/2, \Lambda/2]\), resulting in a transmit power of \( \Lambda^2/12 \). Hence we have

\[
\Lambda = \sqrt{12P_x}, \quad \alpha = \Lambda/(2M). \tag{16}
\]

However, when the modulation order is small, such as for a BPSK modulated signal, this approximation turns out to be biased. For the case of binary signaling with binary interference, the exact value of \( \alpha \) (hence also \( \Lambda \)) can be determined as follows (see Appendix B for a detailed derivation):

\[
\begin{align*}
\Lambda &= 4\alpha; \\
\alpha &= \sqrt{P_x - \beta^2}, & \text{if } P_x \geq 2\beta^2; \\
\alpha &= \frac{2\beta + \sqrt{4P_x - \beta^2}}{2\beta + 1}, & \text{if } \frac{\beta^2}{5} \leq P_x < 2\beta^2; \\
\alpha &= \frac{2n\beta + \sqrt{4n^2 \beta^2 + 4n^2 + 1} - P_x - \beta^2}{4n^2 + 1}, & \text{if } \frac{\beta^2}{4n^2 + 1} \leq P_x < \frac{\beta^2}{4(n-1)^2 + 1},
\end{align*}
\tag{17}
\]

This heuristic parameter choice \( \Lambda = 2M\alpha \), however, appears “unlucky” in some specific situations. Therefore we propose to use optimized parameters \( \alpha, \Lambda \) for THP. This optimization can be accomplished via a similar procedure as described in Section III, i.e., performing a grid search over all \( \alpha, \Lambda \) which satisfy the power constraint \( P_x \). Unlike for the optimal modulator where the search dimension increases with the modulation order, the optimization problem here is always two-dimensional. The search over all possible mappings is not necessary either.

V. NON-DPC BENCHMARKS

We present here two non-DPC approaches as a reference to evaluate our DPC schemes.

A. Relay uses an orthogonal channel

The relay can use an orthogonal channel to help user 1 so that the relaying signal \( z \) will not interfere with the reception of \( x_2 \), under the same available resource (time, bandwidth, energy) constraints. We use time sharing between the relay and the base station over the same bandwidth to realize the orthogonal transmission. Let \( \rho \in [0, 1] \) be the time sharing coefficient for the transmission from the base station to user 2, and assume that the transmitted signal \( x_2 = \hat{X}(\omega_2) \) is uniform M-PAM modulated and subject to the power constraint \( E[x_2] = P_x/\rho \). Hence the total used energy \( \rho \cdot E[x_2] \) remains the same as for the other schemes. The mutual information conveyed through this channel is

\[
I_{\rho}(y; \omega_2) = \rho \cdot I(y; \omega_2), \tag{18}
\]

where \( I(y; \omega_2) \) is calculated according to (3), with

\[
p_y(y|\omega_2) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left(-\frac{(y - X(\omega_2))^2}{2(\sigma^2)}\right).
\]

Note that \( \rho \) affects the throughput of both user 1 and user 2. Therefore, to choose \( \rho \) requires total throughput and fairness considerations. We will choose \( \rho = 1/2 \) in our simulation for simplicity.

B. Interference cancellation at the receiver

One can also use no precoding at the base station but perform interference cancellation at user 2. When the interference is much stronger than the signal, user 2 can perform successive interference cancellation (SIC) [2]: It first decodes \( \omega_1 \) treating \( x_2 = X(\omega_2) \) as noise, and then subtracts the relaying signal \( z(\omega_1) \) from \( y \) and uses the remaining signal to decode \( \omega_2 \). But for moderate and weak interference, SIC will not work. We therefore propose here a new interference cancellation scheme which works for all cases by keeping user 2 receiving signals in both time slot \( t_1 \) and \( t_2 \). The received signals at user 2 during \( t_1 \) and \( t_2 \) can be written as

\[
y_1 = x_1(\omega_1) + n_1, \quad y = X(\omega_2) + z(\omega_1) + n, \tag{19}
\]

where \( n_1 \) is additive white Gaussian noise and \( x_1(\omega_1) \) is the signal for user 1 under average power constraint \( P_x \).

The mutual information between the transmitted information symbol \( \omega_2 \) and the received signal \( (y_1, y) \) can therefore be written as

\[
I(y_1, y; \omega_2) = \sum_{\omega_2} \int y_1 \int y \int p_{y_1,y}(y_1, y; \omega_2) P(\omega_2) \log \frac{p_{y_1,y}(y_1, y; \omega_2)}{p_y(y; \omega_2)} dy dy_1, \tag{20}
\]

where \( p_{y_1,y}(y_1, y; \omega_2) \) can be obtained from the Bayes rule

\[
p_{y_1,y}(y_1, y; \omega_2) = \sum_{\omega_1} p_{y_1,y}(y_1, y; \omega_1, \omega_2) P(\omega_1) = \sum_{\omega_1} p_{y_1}(y_1; \omega_1) p_y(y; \omega_1, \omega_2) P(\omega_1). \tag{21}
\]

The second equality in (21) comes from the fact that \( y_1, y \) are independent if \( \omega_1 \) (and therefore \( x_1 \) and \( z \)) is known. We further get from (19) that

\[
p_{y_1}(y_1; \omega_1) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left(-\frac{(y_1 - X(\omega_1))^2}{2\sigma^2}\right),
\]

\[
p_y(y; \omega_1, \omega_2) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left(-\frac{(y - X(\omega_2) - z(\omega_1))^2}{2\sigma^2}\right).
\]

The maximum a posteriori receiver given \( (y_1, y) \) is therefore

\[
\hat{\omega}_{2MAP} = \text{argmax}_{\omega_2} P(\omega_2|y_1, y) = \text{argmax}_{\omega_2} p_{y_1,y}(y_1, y; \omega_2), \tag{22}
\]

where the second equality comes from the assumption of equally probable \( \omega_2 \).

VI. NUMERICAL RESULTS

In this section we will evaluate our proposed DPC schemes in terms of mutual information, coded BER and energy efficiency. Bit-level Monte-Carlo simulation is used to obtain the results.
information. Figure 2 shows

\[ I(y; \omega_2) \]

for binary signaling with binary interference, as a function of the signal-to-noise ratio (SNR, \( P_z/\sigma^2 \)) at a fixed interference-to-noise ratio (INR, \( P_z/\sigma^2 \)) of 6 dB. The optimal modulator and the sub-optimal modulator are slightly worse compared to the no-interference case. THP with optimized parameters \((\alpha, \Lambda)\), which uniformly outperforms THP with the heuristic parameter choice \( \Lambda = 4\alpha \), experiences notable degradation in low to medium SNR regions but converges to the optimal modulator at high SNR. Interference-cancellation performs well in high SNR regions but suffers from the corrupted observation of the interference in low to medium SNR regions. By using orthogonal channels with time sharing \( \rho = 1/2 \), the performance is relatively good in low SNR regions where the benefits of excluding interference dominate (this is the power-limited regime). In medium and high SNR regions (i.e., in the bandwidth-limited regime), however, the penalty of shortening the transmission time (and therefore less channel use) becomes the bottleneck.

Note that although the mapping parameters for the optimal modulator and for the optimized THP might change with the resolution of the searching grid, the actual performance will only differ slightly. The performance degradation of the optimal modulator compared to the no-interference case varies with the INR, and the maximum loss (less than 1.5 dB in SNR) appears when the INR is about 0 dB, i.e., when \( P_z \approx \sigma^2 \), as demonstrated in Figure 4 of [20].

In Figure 3 we present the case for quaternary signaling (4-PAM) with binary interference, focusing on the performance of the optimal modulator and the sub-optimal modulator (14). The optimal modulator, in all SNR regions, suffers only a minor performance degradation compared to the no-interference case and achieves a significant gain (up to 3 dB at low SNR) over heuristic THP. The curve of the sub-optimal modulator is not smooth due to the fact that it is an approximate solution of (13) based on relaxation and randomization, as discussed in

\[ \log \left( \frac{P_2(y)}{P_1(y)} \right) = \log \left( \frac{P(\omega_2 = 1)}{P(\omega_2 = 0)} \right) \]

The decoding metrics were computed by evaluating

\[ \log P(\omega_2 | y) \]

for the interference cancellation scheme, \( \log P(\omega_2 | y_1, y) \) is used instead. More precisely, the log-likelihood ratios are used as soft input to the turbo decoder as follows:

\[ L(y) = \log \frac{P_2(y | \omega_2 = 1)}{P_1(y | \omega_2 = 0)} = \log \frac{P(\omega_2 = 1 | y)}{P(\omega_2 = 0 | y)} \]

Section III-C. The performance of optimized THP, not shown here to improve the readability of this figure, lies in between the curves of the optimal modulator and the heuristic THP.

\[ P_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]
the conclusion holds for heuristic THP (not shown here to simplify is not capacity achieving) as shown in Figure 2. The same schemes, we de

C. Energy efficiency

In order to measure the energy efficiency of the different schemes, we defined “equivalent SNR” $\gamma_X$ as the required SNR for scheme $X$ to achieve the same mutual information as in the no-interference case despite the presence of interference. That is, given an SNR $P_z/\sigma^2$ and an INR $P_z/\sigma^2$, there exists a constant $\gamma_X(P_z/\sigma^2)\sigma^2$ such that $I_X(\gamma_X) = I_{\text{no interf}}(P_z/\sigma^2)$, \hspace{1cm} (23)

where $I_{\text{no interf}}(P_z/\sigma^2)$ is the end-to-end mutual information with SNR $P_z/\sigma^2$ for the no-interference case, and $I_X(\gamma_X)$ is the mutual information for scheme $X$ at an SNR of $\gamma_X$.

In Figure 5, we compare the energy efficiencies of the interference-cancellation (left) and of the sub-optimal modulator (right) compared to the optimal modulator. A positive number indicates better energy efficiency.

$$L(y_1, y) = \log \frac{P_{y_1|y; y}|\omega_2 = 1}{P_{y_1|y; y}|\omega_2 = 0} = \log \frac{P(\omega_2 = 1|y, y)}{P(\omega_2 = 0|y, y)},$$

where $P(\omega_2=0) = P(\omega_2=1) = 1/2$ is assumed. As shown in Figure 4, for a strong interference scenario with INR=6 dB and for a required BER of $10^{-4}$, the optimal modulator (and the sub-optimal modulator) suffers only a 0.1 dB loss compared to the no-interference case, and shows a gain of 1.2 dB to the interference-cancellation scheme and 2.4 dB to the optimized THP. These gains approximately equal the difference in required SNR to achieve a mutual information of $\sim 0.33$ bits/channel use (or slightly larger, since the code is not capacity achieving) as shown in Figure 2. The same conclusion holds for heuristic THP (not shown here to simplify the figure).

D. Optimized THP with Gaussian interference

We next demonstrate partial interference cancellation at the transmitter via the optimized THP proposed in Section IV. For this purpose we consider transmission of binary (BPSK) and quaternary (4-PAM) symbols on a channel with Gaussian interference. Figure 6 shows the result for both heuristic and optimized THP (computation of the optimal modulator is not directly feasible for this case; cf. Section IV). It is clear that we can gain from optimizing the parameters of THP. For quaternary signaling, the gain is significant especially in low signal-to-interference ratio (SIR, $P_z/P_i$) regions where interference dominates. This indicates that THP is a fairly effective (yet strictly suboptimal) means for combating Gaussian interference known at the transmitter. The gain achieved by optimizing the parameters of THP is much smaller in the binary case, however.

VII. SUMMARY OF RESULTS AND CONCLUSIONS

In this paper we have studied DPC solutions for a relay-aided downlink channel that partially solve the Costa precoding problem using symbol-by-symbol processing. We started from the simplest scenario of binary signaling with binary interference, and derived the optimal modulator which maximizes the mutual information between the transmitter and

7We are more interested in interference dominated channels where DPC is most useful.
As stated in Section III-B, only two mappings IX and XII as stated in (9) need to be considered. We can find from mappings IX and XII that \( d_1 = d_4 \), and therefore only \( \{d_1, d_2, d_3\} \) are used to identify the modulator \( X(\omega_2, z) \).

When \( P_x \leq \beta^2 \), using mapping XII with \( a = b = \sqrt{P_x} \) will result in \( d_2 = d_3 > d_1 = 2\sqrt{P_x} \). The optimal detector which compares \( |y| \) with the threshold \( \beta \) will give almost the same performance as the no-interference case.

When \( P_x > \beta^2 \), we can calculate the maximized minimum distance for each mapping and then identify the larger one. Without loss of generality, we assume \( a, b \geq 0 \) in the following. For mapping IX we have \( d_{\text{min}} = \min\{a+b, 2(\beta-a), 2(\beta+b)\} \).

If \( \beta \geq a \), we have \( 2(\beta+b) \geq 2(a+b) > a+b \) which means \( d_{\text{min}} = \min\{a+b, 2(\beta-a)\} \). The maximum of \( d_{\text{min}} \) is achieved when \( a+b = 2(\beta-a) \). Combine this condition with the power constraint \( a^2 + b^2 = 2P_x \), we can formulate a new equation of \( a \) as \( 5a^2 - 6a\beta + 2\beta^2 - P_x = 0 \), which has two roots
\[
a = \frac{3\beta - \sqrt{5P_x - \beta^2}}{5}, \quad a = \frac{3\beta + \sqrt{5P_x - \beta^2}}{5}.
\]
The former root is valid (greater than 0) only if \( P_x \leq 2\beta^2 \) and the latter root conflicts with the precondition \( a \leq \beta \). Hence for \( \beta^2 < P_x \leq 2\beta^2 \) we have
\[
d^*_x = \frac{4\beta + 2\sqrt{5P_x - \beta^2}}{5}, \quad \text{with } a = \frac{3\beta - \sqrt{5P_x - \beta^2}}{5}, b = \frac{3\beta + \sqrt{5P_x - \beta^2}}{5}.
\]
If \( \beta < a \), we have \( d_{\text{min}} = \min\{a+b, 2(a-\beta), 2(\beta+b)\} \), whose maximum is achieved when \( a+b = 2(\beta-a) = 2(\beta+b) \), i.e. \( b = a-2\beta \). Combine this with the power constraint, we get
\[
a^2 - 2\beta a + 2\beta^2 - P_x = 0,
\]
which has a valid solution (only if \( P_x \geq 2\beta^2 \)) as follow
\[
a = \beta + \sqrt{P_x - \beta^2}, \quad b = \sqrt{P_x - \beta^2} - \beta.
\]

Hence for \( P_x \geq 2\beta^2 \) we have
\[
d^*_x = 2\sqrt{P_x - \beta^2}, \quad \text{with } a = \beta + \sqrt{P_x - \beta^2}, \quad b = \sqrt{P_x - \beta^2} - \beta.
\]

When mapping XII is used, \( d_{\text{min}} = \min\{a+b, 2(\beta+b-a)\} \). If \( 2\beta+b-a \leq 0 \), the maximum of \( d_{\text{min}} \) is achieved when \( a+b = a-b-2\beta \), i.e., \( b+\beta=0 \) which is impossible. If \( 2\beta+b-a > 0 \), the maximum of \( d_{\text{min}} \) is achieved when \( a+b = 2\beta+b-a \), i.e. \( a = \beta \). Hence \( b = \sqrt{2P_x - \beta^2} \) with \( d_{\text{min}} = \beta + \sqrt{2P_x - \beta^2} \).

Compared with \( d^*_x \) in (24) and (25), \( d^*_x \) has greater value for \( \beta^2 < P_x \leq 5\beta^2 \) and therefore mapping XII will be selected in this region and mapping IX will be selected when \( P_x \geq 5\beta^2 \). Together with the finding for \( P_x \leq 2\beta^2 \), one can easily conclude the results shown in (15).

**APPENDIX B**

**PARAMETERS FOR HEURISTIC THP**

For binary signaling \( w \in \{-\alpha, \alpha\} \) with binary interference \( z \in \{-\beta, \beta\} \), there are four different combinations/values \((\beta+\alpha, -\beta-\alpha, -\beta+\alpha, -\beta-\alpha)\) subject to modulo operation with \( \Lambda = 4\alpha \) to ensure \( |x| \leq \Lambda/2 \). By the assumption of equiprobable signals and interference as stated in (6), only

**APPENDIX A**

**DERIVATION OF THE SUB-OPTIMAL MODULATOR**

For \( M=2 \) and \( N=2 \), there are in total four different symbol distances among received signal constellation points, namely,
\[
d_1 = |X(0, -\beta) - X(1, -\beta)|, \quad d_4 = |X(0, \beta) - X(1, \beta)|,
\]
\[
d_2 = |X(0, -\beta) - X(1, \beta) - 2\beta|, \quad d_3 = |X(0, \beta) - X(1, -\beta) + 2\beta|.
\]
2 out of these 4 values are of interest due to their amplitude symmetry. Without loss of generality, we just select $\beta + \alpha$ and $\beta - \alpha$.

When $\beta \leq \alpha$, all the values satisfy the requirement $|x| \leq \Lambda/2$ and therefore we have

$$P_x = E[x^2] = \alpha^2 + \beta^2 \geq 2\beta^2.$$ 

Hence for $P_x \geq 2\beta^2$ we have $\alpha = \sqrt{P_x - \beta^2}$. When $\alpha < \beta \leq 3\alpha$, we have

$$2\alpha < \beta + \alpha \leq 4\alpha; \quad 0 < \beta - \alpha \leq 2\alpha.$$ 

After modulo operation the resulting $x$ has average power

$$P_x = E[x^2] = 5\alpha^2 - 4\beta\alpha + \beta^2 < 2\alpha^2,$$

which has only one feasible solution $\alpha = \frac{2\alpha^2 + \sqrt{5\alpha^4 - \beta^2}}{\beta}$ for

$$\beta^2/5 \leq P_x < 2\beta^2.$$ 

When $(2n - 1)\alpha < \beta \leq (2n + 1)\alpha$, for $n = 2, 3, \ldots$, we have

$$2n\alpha < \beta + \alpha \leq (2n+2)\alpha, \quad (2n-2)\alpha < \beta - \alpha \leq 2n\alpha.$$ 

For even $n$ even, the modulo operation will subtract $2n\alpha$ from the above two values and result in $\beta - (2n-1)\alpha$ and $\beta - (2n+1)\alpha$ respectively; for odd $n$, the modulo operation will subtract $(2n+2)\alpha$ and therefore result in $\beta - (2n+1)\alpha$ and $\beta - (2n-1)\alpha$. Hence we conclude that

$$P_x = E[x^2] = (4n^2 + 1)\alpha^2 - 4n\beta\alpha + \beta^2 < 2\alpha^2.$$ 

Similarly, by solving the above equation for $\alpha$ we can get

$$\alpha = \frac{2\alpha^2 + \sqrt{(4n^2+1)\alpha^4 - \beta^2}}{\beta} \quad \text{when} \quad \beta^2 \leq \beta^2 < \frac{4\alpha^2}{4(n^2+1)+1}.$$ 

Another solution is infeasible and thus dropped. Summarize all the above derivation we have proved (17).

### References


Ming Xiao (S’2002–M’2007) was born in the SiChuan Province, P. R. China, on May 22nd, 1975. He received Bachelor and Master degrees in Engineering from the University of Electronic Science and Technology of China, ChengDu in 1997 and 2002, respectively. He received Ph.D. degree from Chalmers University of technology, Sweden in November 2007. From 1997 to 1999, he worked as a network and software assistant engineer in ChinaTelecom. From 2000 to 2002, he also held a position in the SiChuan communications administration. From November 2007 to now, he has been in ACCESS Linnaeus center, school of electrical engineering, Royal Institute of Technology, Sweden, where he is currently an assistant professor. He received the “Chinese Government Award for Outstanding Self-Financed Students Studying Abroad” in 2007. He received a “Hans Werthén Grant” from the Royal Swedish Academy of Engineering Science (IVA) in March 2006, and Ericsson’s Research Foundation in 2010.

Mikael Skoglund (S’93–M’97–SM’04) received the Ph.D. degree in 1997 from Chalmers University of Technology, Sweden. In 1997, he joined the Royal Institute of Technology (KTH), Stockholm, Sweden, where he was appointed to the Chair in Communication Theory in 2003. At KTH, he heads the Communication Theory Lab and he is the Assistant Dean for Electrical Engineering.

Dr. Skoglund’s research interests are in the theoretical aspects of wireless communications. He has worked on problems in source–channel coding, coding and transmission for wireless communications, Shannon theory and statistical signal processing. He has authored some 220 scientific papers, including papers that have received awards, invited conference presentations, and papers ranking as highly cited according to the ISI Essential Science Indicators. He has also consulted for industry, and he holds six patents.

Dr. Skoglund has served on numerous technical program committees for IEEE conferences. During 2003–08 he was an associate editor with the IEEE TRANSACTIONS ON COMMUNICATIONS and he is presently on the editorial board for IEEE TRANSACTIONS ON INFORMATION THEORY.