The value of travel time variability with trip chains, flexible scheduling and correlated travel times

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Abstract

This paper analyzes the effects of flexibility in activity scheduling and correlation of travel times across trips made during a day on the value of mean travel time (VMTT) and day-to-day travel time variability (VTTV). As a first step the paper derives formulas for the VMTT and VTTV in a general single-trip scheduling model of which current popular models are special cases. Extending the analysis to a multi-trip scheduling model, the paper then shows that the VMTT and VTTV on a trip in general depend on the realized travel times on preceding trips, first through the arrival time to the preceding activity and second through the information provided about subsequent travel times. Analytical formulas for the VMTT and VTTV are obtained for a special case with linear marginal cost functions, and the variability of the VMTT and VTTV across days is characterized. The paper shows that with flexible scheduling there are typically a cost associated with a positive correlation of travel times, arising from persistent deviations from typical travel demand or supply on a given day. However, there is also a strict benefit in the dependence since it allows for a more efficient scheduling of later trips. This has implications for the value of reducing the intra-day correlation of travel times through for example faster restoration after incidents.

KEYWORDS: travel time variability, reliability, cost, correlation, trip chain, activity

1 Introduction

In recent years it has been increasingly recognized that there are costs associated with not only the typical (mean) travel time, but also with the unpredictable variation of travel time around the typical value, arising from fluctuations in travel demand, weather conditions, road works etc. (FWHA, 2008). Day-to-day travel time variability forces travelers and transporters to balance the risk of arriving late, including missed appointments, production chain disruption costs etc., against the cost of precautions, such as adding buffers of time and resources. Many modern technologies for traffic and transport management, such as congestion tolls and intelligent traffic information systems, have among their main goals to reduce the variability of travel times.

Considerable attention has been given to assessing the value of travel time variability, VTTV, (or conversely, reliability) for personal travel. One methodological approach, known as the "mean-variance" approach, is to include some measure of the variability, such as the standard deviation or an interpercentile range, directly in individuals' utility functions (e.g., Noland and Polak, 2002; Small et al., 2005; Börjesson et al., 2011). While empirically valid, the method does not provide a theoretical foundation for the cost (Fosgerau and Karlström, 2010). Another approach, therefore, is to derive the VTTV from an underlying model of individuals' travel preferences. The prevailing method is to use a scheduling model for trip departure times, in which costs may arise from early/late departures and/or early/late arrivals. The value of variability is then extracted assuming that the traveler chooses the departure time to minimize the expected cost under the stochastic travel time.

Until now, the VTTV has been assessed considering the scheduling of a single trip in isolation. The most widely used model is the piecewise constant marginal cost model proposed by Vickrey (1969) and made popular by Small (1982). In this model, the travel cost rises proportionally to the earliness of departure and to the lateness or earliness of arrival in relation to a preferred arrival time (sometimes with an additional discrete lateness penalty). From this model, the VTTV can be studied theoretically (e.g., Noland and Small, 1995; Bates et al., 2001) and empirically (e.g., Bates et al., 2001; Asensio and Matas, 2008; Börjesson and Eliasson, 2011). Fosgerau and Karlström (2010) show that the VTTV is independent of the mean and the spread of the travel time distribution but dependent on the shape of its right tail.

Fosgerau and Engelson (2011) use the scheduling model of Vickrey (1973), in which the marginal costs of departing earlier or arriving later are linear functions of time. They show that the VTTV, with the variability measured by the standard deviation, is proportional to the standard deviation, which is an attractive feature for network routing and traffic assignment models. Empirical estimation results are reported by Börjesson et al. (2011), who find that the model fits the data better than the Vickrey (1969) model. They also find that, although it has been shown that the mean-variance and scheduling approaches are equivalent under expected utility theory (Fosgerau and Karlström, 2010; Fosgerau and Engelson, 2011), empirical results suggest a more complex relationship.

The models of Vickrey (1969, 1973) can be seen as two special cases of a general single-trip scheduling model in which a pair of marginal cost functions determine the cost associated with a certain departure time and arrival time, respectively. This model was proposed by Tseng and Verhoef (2008) who also estimate a flexible, piecewise constant version of the model for the morning commute. The first contribution of this paper is to provide formulas for the VTTV as well as the value of mean travel time, VMTT, for the general single-trip scheduling model. The mean and standard deviation of the travel time are allowed to depend on the departure time. The general formulas provide insights into the properties of the VMTT and VTTV that specific functional forms cannot. Furthermore, the formulas for any special case can be directly obtained from the general formulas, whether closed form expressions for the optimal departure time exist (such as for the Vickrey (1969, 1973) models) or numerical calculations are necessary. We illustrate this fact by deriving the previously known VMTT and VTTV for the Vickrey (1969, 1973) models (Fosgerau and Karlström, 2010; Fosgerau and Engelson, 2011). The general formulas also provide the starting point for the next part of this paper, which is an extension of the VTTV analysis to a multi-trip setting.

Single-trip scheduling models provide many essential insights about the factors behind the cost of travel time variability as well as feasible frameworks for empirical estimation. However, the models only consider a single trip in isolation, assuming that lateness and earliness costs are independent of previous and subsequent trips and activities. Single-trip models do not capture the fact that the trip is typically a link in a daily chain of activities and trips where the scheduling of many activities is flexible, so that the utility derived from them is not completely determined by the time of day they are undertaken. This suggests that earliness and lateness costs on a given trip may be determined not only by the departure and arrival times on the trip itself but also by the arrival time to the preceding activity and the departure time on the subsequent trip. It also implies that the subsequent schedule can be adjusted in response to the arrival time of the trip in order to minimize the costs. The effects of these interdependencies on the VMTT and VTTV on different trips have so far not been investigated.

Furthermore, single-trip scheduling models cannot capture the impact on the VMTT and VTTV when travel times are correlated (or, more generally, not statistically independent) across trips. Poor weather conditions, unusally high travel demand or an incident on a particular day, for example, can cause travel times to be unusually long on several or all links in the trip chain, which can have increasingly severe effects on the schedule as the day progresses. This implies that the VMTT and VTTV will vary not only between trips but also with the realizations of travel times on preceding trips, that is, across days. This fact has not received much notice previously but has important empirical implications for the collection, calculation and interpretation of travel time values.

Hence, the second and most important contribution of this paper is to generalize the single-trip modeling framework for the analysis of the VMTT and VTTV into a multi-trip model, taking the possibility of flexible activity scheduling and dependent travel times into account. This represents an extension of the simple but analytically tractable single-trip models into the field of activity-based travel demand modeling (Axhausen and Gärling, 1992; Bowman and Ben-Akiva, 2001; Arentze and Timmermans, 2009). Furthermore, the stochasticity of travel times brings the analysis into the framework of multi-stage stochastic programming (e.g., Ruszczyński and Shapiro, 2003). The model can be seen as a generalization of the two-trip model of Jenelius et al. (2011) (based in turn on work by Ettema and Timmermans (2003)), which was used to assess the value of travel time and the costs of delays in a setting of deterministic travel times. We derive formulas for the VMTT and VTTV on any trip and show that they depend on the travel times on preceding trips in two distinct ways. First, the scheduling flexibility of activities means that the optimal timing of a trip depends on the arrival time to the preceding activity, which, taken across the distribution of all preceding travel times, is a stochastic variable. Second, the preceding travel time realizations themselves provide information about the subsequent travel times, which also determines the optimal timing of a trip. We define the unconditional VMTT and VTTV as the expected values across all realizations of preceding travel times and show that the model simplifies to the single-trip model when activity scheduling is fixed and travel times are statistically independent.

Finally we analyze a special case of the multi-trip model with two trips and linear marginal cost functions in line with the Vickrey (1973) single-trip model. For this special case we obtain instructive closed-form expressions for the VMTT and VTTV on each trip, and we also study the variability of the values across days. We find that when the scheduling of the intermediate activity is flexible, there is typically a cost associated with positive correlation of travel times, meaning that the costs of "bad days" more than offset the benefits of "good days". However, there is also a strict benefit in the dependence since it allows a more efficient scheduling of trip 2 given the experienced travel time on trip 1.

The paper is organized as follows. Section 2 presents the general single-trip model and derives optimality conditions and formulas for the VMTT and VTTV. In Section 3 we present the general multi-trip model with flexible scheduling and derive the corresponding optimality conditions and formulas for the VMTT and VTTV on any trip. Section 5 analyzes the two-trips, linear special case, and Section 6 concludes.

2 A general single-trip model

In this section we consider a daily schedule that consists of two activities and an intermediate trip. We fix two times $t = t_s$ and $t = t_e$ that mark the start and end of the period of interest, respectively. The schedule can then be uniquely represented by the departure time from activity 1, denoted d. The utility gained from spending another unit of time on activity $i \in \{1, 2\}$ at time t is expressed in the form of a deterministic marginal utility function $u_i(t)$. We assume that there is a point in time, normalized to $t = 0$, before which the traveler prefers to be at activity 1 and after which she prefers to be at activity 2. That is, $u_1(t) > u_2(t)$ for all $t \in [t_s, 0)$ and $u_2(t) > u_1(t)$ for all $t \in (0, t_e]$. The marginal utility derived from traveling, denoted ν, is assumed to be constant.[1]

The time required to travel from activity 1 to 2 is stochastic and depends in general on the departure time d, $T(d) > 0$. Specifically, we assume that $T(d) = \tau(d)+\sigma(d)X$, where X is the standardized travel time with mean zero, standard deviation 1 and cumulative distribution function Φ for all d.[2] We assume that the mean $E[T(d)] = \tau(d) > 0$ and the standard deviation $\sigma(d) > 0$ are smooth functions of d, and that Φ has convex support so that the inverse Φ^{-1} exists.

Absolute levels of utility are unobservable, and operational models must be based on differences in utility relative to some baseline level. The models remain equivalent regardless of this choice, however. Throughout this paper we use the marginal utility of travel, ν, as the reference level. Assuming that utility is money metric, we define $c_i(t) \equiv u_i(t)-\nu$, $i \in \{1, 2\}$, as the marginal cost of travelling rather than taking part in activity i at time t. Furthermore, following Tseng and Verhoef (2008) it is convenient to consider the cost relative to the utility derived from an optimal schedule when travel

[1] The assumption that ν is constant is not strictly necessary here but conforms to the multi-trip model in Section 3, where it is useful.

[2] The invariance of the standardized travel time distribution over time is found empirically to be a good approximation by Fosgerau and Fukuda (2010).

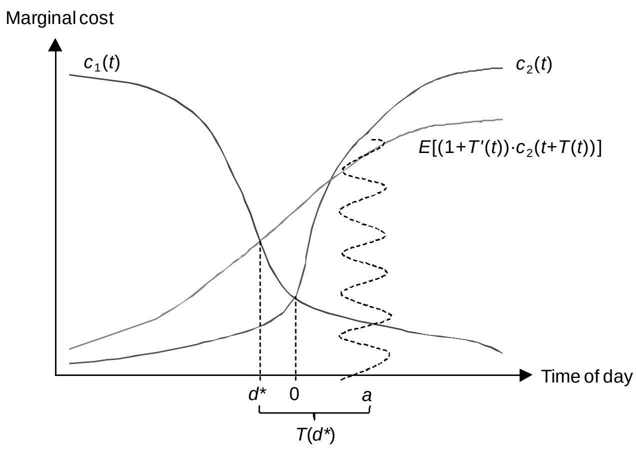

Figure 1: Illustration of the single-trip model and the optimal departure time. The curved, dashed line at time a indicates that the arrival time is stochastic.

is instantaneous, i.e., when $T \equiv 0$. The cost associated with the schedule d is then

$$C(d) = \int_d^0 c_1(t)dt + \int_0^{d+T(d)} c_2(t)dt. \tag{1}$$

Since travel time is stochastic, so is the cost $C(d)$, and travelers are assumed to minimize *expected* cost, $\min_d E[C(d)]$. For feasibility we require that $d \geq 0$ and that $d + T(d) \leq t_s$ almost surely.

2.1 Single-trip model: optimality conditions

A marginal change in the departure time affects the time spent in activity 1, the time spent traveling and the time spent in activity 2, and the necessary optimality condition requires the net effect on the expected cost be zero. More precisely, the first-order condition for the optimal departure time d^* is

$$0 = \frac{\partial E[C(d)]}{\partial d} = \frac{\partial}{\partial d}\left(\int_d^0 c_1(t)dt + E\left[\int_0^{d+T(d)} c_2(t)dt\right]\right),$$

that is,

$$c_1(d^*) = E[(1 + T'(d^*)) \cdot c_2(d^* + T(d^*))], \tag{2}$$

where $T'(d) = \tau'(d) + \sigma'(d)X$. As illustrated in Figure 1, the optimality condition can be interpreted as finding the intersection of the functions $c_1(t)$ and $E[(1 + T'(t)) \cdot c_2(t+T(t))]$. An important special case is when travel time is independent of departure time, that is, $T = \tau + \sigma X$ for all d. In this case the optimality condition simplifies to

$$c_1(d^*) = E[c_2(d^* + T)]. \tag{3}$$

2.2 Single-trip model: value of mean travel time and variability

Let us define the marginal value of mean travel time (VMTT) as the marginal expected cost of a uniform increase in the mean travel time given that the departure time is chosen optimally. Without loss of generality, we write the mean travel time as $\tau(d) = \tau_0 + \tau_{\mathrm{x}}(d)$, where $\tau_0 > 0$, and consider changes in the fixed component τ_0. Due to the limited time available in a day, a change of τ_0 affects the amounts of time spent in the two activities. Since the departure time is chosen optimally, however, the net marginal effect on expected cost of the changed departure time is zero. According to the envelope theorem (e.g., Mas-Colell et al., 1995), it is sufficient to consider the partial effect of the changed travel time on activity 2. The value of mean travel time is given by

$$\begin{aligned} \mathrm{VMTT} \equiv \frac{dE[C(d^*)]}{d\tau_0} &= \frac{\partial}{\partial \tau_0} E\left[\int_0^{d^*+\tau_0+\tau_{\mathrm{x}}(d^*)+\sigma(d^*)X} c_2(t)dt\right] \\ &= E[c_2(d^* + T(d^*))]. \end{aligned} \tag{4}$$

It can be seen that the VMTT is independent of the travel time distribution if $c_2(t)$ is constant. If travel time is independent of departure time, optimality condition (3) implies $\mathrm{VMTT} = c_1(d^*)$. In this case the VMTT is independent of the travel time distribution if $c_1(t)$ is constant.

Similarly to the VMTT, we can define the marginal value of travel time variability (VTTV) as the marginal expected cost of a uniform increase in the travel time standard deviation given that the departure time is chosen optimally. We write the standard deviation of travel time as $\sigma(d) = \sigma_0 + \sigma_{\mathrm{x}}(d)$, where $\sigma_0 > 0$, and consider changes in the fixed component σ_0. The value of travel time variability is

$$\begin{aligned} \mathrm{VTTV} \equiv \frac{dE[C(d^*)]}{d\sigma_0} &= \frac{\partial}{\partial \sigma_0} E\left[\int_0^{d^*+\tau(d^*)+[\sigma_0+\sigma_{\mathrm{x}}(d^*)]X} c_2(t)dt\right] \\ &= E[X \cdot c_2(d^* + T(d^*))], \end{aligned} \tag{5}$$

where again the envelope theorem was applied. Note that the absolute level of $c_2(t)$ does not affect the VTTV since $E[X] = 0$. Another measure of travel time variability that is sometimes used is the variance. It follows as a corollary to (5) that the value of travel time variance is $dE[C(d^*)]/d\sigma_0^2 = 1/(2\sigma_0)\mathrm{VTTV} = 1/(2\sigma_0)E[X \cdot c_2(d^* + T(d^*))]$.

Generally, the optimality condition (2) can be written as

$$c_1(d^*) = (1 + \tau'(d^*)) \cdot \mathrm{VMTT} + \sigma'(d^*) \cdot \mathrm{VTTV}. \tag{6}$$

This shows that the traveler trades the cost of departing from activity 1 against a linear combination of the costs of mean travel time and travel time uncertainty.

Formulas (4) and (5) provide a general and direct way to calculate the VMTT and VTTV for any single-trip scheduling model, whether closed form expressions exist for the optimal departure time or numerical calculations are necessary. Using these formulas avoids an explicit derivation of the expected cost function and its value at the optimal departure time. In the following we apply them to two important special cases for which analytical results are known.

2.3 Special case: the step model

This model was proposed by Vickrey (1969) and recently analyzed by Fosgerau and Karlström (2010). Here we show that it represents a special case of the general model above. Marginal costs relative to the utility of travel are given by

$$\begin{aligned} c_1(t) &= c_1, \\ c_2(t) &= c_2^{\mathrm{E}} + (c_2^{\mathrm{L}} - c_2^{\mathrm{E}})H(t), \quad t \in [t_s, t_e], \end{aligned}$$

where $c_2^{\mathrm{E}} < c_1 < c_2^{\mathrm{L}}$ and $H(t)$ is the Heaviside step function defined as $H(t) = 0$ for $t \leq 0$ and $H(t) = 1$ for $t > 0$. In the Appendix we show that this model formulation is equivalent to the parameterizations used by Small (1982) and Fosgerau and Karlström (2010), respectively. In particular, the well-known (α, β, γ) parameterization of Small (1982) for the cost of travel time, schedule delay early and schedule delay late, respectively, is related to our $(c_1, c_2^{\mathrm{E}}, c_2^{\mathrm{L}})$ parameterization as $(\alpha, \beta, \gamma) = (c_1, c_1 - c_2^{\mathrm{E}}, c_2^{\mathrm{L}} - c_1)$.

We here consider the case where travel time is independent of the departure time. The general first-order condition (3) becomes $c_1 = E[c_2^{\mathrm{E}} + (c_2^{\mathrm{L}} - c_2^{\mathrm{E}})H(d^* + T)]$. Noting that $E[H(d^* + T)] = \Pr(d^* + T \geq 0) = 1 - \Phi((-d^* - \tau)/\sigma)$, we can solve for the optimal departure time $d^* = -\tau - \sigma\Phi^{-1}((c_2^{\mathrm{L}} - c_1)/(c_2^{\mathrm{L}} - c_2^{\mathrm{E}}))$.

The general theory and the fact that $c_1(t)$ is constant immediately gives $\mathrm{VMTT} = c_1$, which is independent of the travel time distribution. According to the general formula (5) for the VTTV we then have

$$\begin{aligned} \mathrm{VTTV} &= E[X \cdot [c_2^{\mathrm{E}} + (c_2^{\mathrm{L}} - c_2^{\mathrm{E}})H(d^* + T)]] \\ &= (c_2^{\mathrm{L}} - c_2^{\mathrm{E}}) \int_{\frac{c_2^{\mathrm{L}} - c_1}{c_2^{\mathrm{L}} - c_2^{\mathrm{E}}}}^{1} \Phi^{-1}(p)dp, \end{aligned}$$

independent of τ and σ, as previously found by Fosgerau and Karlström (2010).

2.4 Special case: the slope model

This model was proposed by Vickrey (1973) and was recently analyzed by Fosgerau and Engelson (2011). Marginal costs are given by

$$\begin{aligned} c_1(t) &= \beta_0 + \beta_1 t, \\ c_2(t) &= \gamma_0 + \gamma_1 t, \quad t \in [t_s, t_e], \end{aligned}$$

where $\beta_1 < \gamma_1$. The normalization of time so that $c_1(0) = c_2(0)$ implies $\beta_0 = \gamma_0$.

Assuming that travel time is independent of departure time, the general first-order condition (3) requires that $\gamma_0 + \beta_1 d^* = E[\gamma_0 + \gamma_1(d^* + T)]$. We can solve for the optimal departure time $d^* = -\gamma_1/(\gamma_1 - \beta_1) \cdot \tau$ and obtain $\mathrm{VMTT} = \gamma_0 + \beta_1 d^* = \gamma_0 - \beta_1\gamma_1/(\gamma_1 - \beta_1) \cdot \tau$. The VMTT thus increases proportionally to the mean travel time, unless $\gamma_1 = 0$ or $\beta_1 = 0$ (constant marginal cost at the destination or the origin), in which cases it is constant, or $\gamma_1 < 0$, in which case it decreases. Using formula (5) we further obtain

$$\mathrm{VTTV} = E[X \cdot (\gamma_0 + \gamma_1(d^* + T))] = \gamma_1\sigma,$$

proportional to σ while independent of τ and any other characteristics of the travel time distribution. In other words, the value of travel time variance, $1/(2\sigma)\mathrm{VTTV} = \gamma_1/2$ is constant, as found by Fosgerau and Engelson (2011).

3 A general multi-trip model with flexible scheduling and dependent travel times

In this section we extend the single-trip model in Section 2 and consider a schedule that consists of $n + 1$ activities and a chain of $n \geq 1$ intermediate trips. We show how the single-trip formulas for the VMTT and the VTTV are generalized in the multi-trip case, capturing the dependencies between activities and trips. The model is an extension of the three activities, two trips model of Jenelius et al. (2011) (which in turn is based on a formulation of Ettema and Timmermans (2003)), who showed that partially flexible scheduling implies that the value of deterministic travel time (VTT) varies between different trips. The introduction of stochastic travel times brings the model into the framework of multi-stage stochastic programming (e.g., Ruszczyński and Shapiro, 2003).

We assume that the number and order of activities and trips are already determined and fixed; that is, we do not consider reordering, replacing or cancelling of activities. The activities are then numbered $i = 1, \ldots, n + 1$ in chronological order. To fixate ideas, we assume that the time period covered by the schedule represents a single day. As before, the utility derived from taking part in an activity is assumed to be independent of other activities. Further, the scheduling of a day is independent of preceding and subsequent days. This represents the boundary constraints of the problem and is not a critical assumption, but it facilitates the analysis (for discussions on the representation of day-to-day dependencies in activity-based models, see, e.g., Karlström (2005); Arentze et al. (2010)).

For the first and last activities the utility gained from spending another unit of time at time t is represented by a marginal utility function $u_i(t)$ as in the single-trip model. For intermediate activities $i \in \{2, \ldots, n\}$ we assume that the marginal utility at time t depends on a linear combination of the time of day t and the duration $t - a_i$ since the arrival at a_i, i.e., $u_i(t - \xi_i a_i)$, where $\xi_i \in [0, 1]$ is a parameter expressing the scheduling flexibility of the activity (Ettema and Timmermans, 2003; Jenelius et al., 2011). Note that $\xi_i = 0$ means that marginal utility depends only on time of day, while $\xi_i = 1$ means that marginal utility depends only on time since arrival, i.e., activity duration. Flexibility here thus refers to the degree to which the utility of taking part in an activity is independent of the time of day and is not associated with any particular assumption about the shape of the marginal utility function. Importantly, the marginal utility of travel, ν, is assumed to be constant across all trips. With a marginal utility of travel varying with time or with the duration of individual trips a traveler would trade off travel time on different trips, which would complicate the analysis.

To ensure that each trip is made within the day, we assume that for each pair of consecutive activities i and $i + 1$ there is some point in time, which may depend on the preceding arrival time and preceding travel times, before which the traveler prefers to be at activity i and after which she prefers to be at activity $i + 1$. The requirement can be satisfied for example if each activity $i \in \{2, \ldots, n + 1\}$ has an intial warm-up period with monotonically increasing marginal utility and each activity $i \in \{1, \ldots, n\}$ has an ultimate cool-down period with monotonically decreasing marginal utility, such that the traveler can always depart during the cool-down period of activity i and arrive during the start-up period of activity $i + 1$.

Trip i is between activity i and $i + 1$. The travel time on each trip is stochastic and depends on the departure time d_i. We require that all trips be possible to make within the day for any outcome of the travel times. Therefore, we assume that for all departure times the travel times are bounded from above by some finite number. Anologously to the single-trip model, we assume that travel time can be decomposed as $T_i(d_i, X_i) = \tau_i(d_i) + \sigma_i(d_i)X_i > 0$, where X_i is the standardized travel time with mean zero and standard deviation 1. The standardized travel times on different trips are not independent in general. We assume that the mean $E[T_i(d_i, X_i)] = \tau_i(d_i) > 0$ and the standard deviation $\sigma_i(d_i) > 0$ of each trip are smooth functions of d_i.

The dependence of travel times means that the realized travel times up to a certain time of day provide information about the travel times on subsequent trips. We refer to a vector of departure times $\mathbf{d} = (d_1, \ldots, d_n)$ as a *schedule* for short. The traveler chooses the schedule $\mathbf{d}$ sequentially in order to maximize expected utility, $\max_{\mathbf{d}} E[U(\mathbf{d})]$. Specifically, the departure time of a trip $i = 2, \ldots, n$ is chosen once the traveler has arrived to the preceding activity, having observed the arrival time a_i and the realized standardized travel times $(x_1, \ldots, x_{i-1})$ (i.e., an outcome of $(X_1, \ldots, X_{i-1})$) on the preceding trips. It is convenient to also introduce the quantity x_0 that is observed prior to the departure of the first trip. This may represent prior information about the travel times during the day based on experiences from preceding days or other available information such as weekday, time of year etc. The analysis can either be performed conditional on this realization or across all possible realizations. The process can be expressed as (c.f. Ruszczyński and Shapiro, 2003):

$$\begin{aligned}&\text{observation } (x_0) \to \text{decision } (d_1) \to \text{observation } (x_1) \to \cdots \\ &\qquad \to \text{observation } (x_{n-1}) \to \text{decision } (d_n)\,.\end{aligned}$$

For feasibility we require that $d_1 \geq t_s$, $d_i \geq d_{i-1} + T_{i-1}(d_{i-1}, x_{i-1})$ for $i \in \{2, \ldots, n\}$ and any x_{i-1}, and $d_n + T_n(d_n, X_n) \leq t_e$ almost surely.

As in the single-trip model we use the marginal utility of travel, ν, as the reference level. However, we find it more convenient here to phrase the problem in terms of the utility U directly rather then the cost C relative to the situation with instantaneous travel. Note that a change dU in utility is equivalent to a change $-dC$ in cost. With $c_i(t) \equiv u_i(t) - \nu$, $i \in \{1, \ldots, n+1\}$, the utility can then be written

$$\begin{aligned}U(\mathbf{d}) = &\int_{t_s}^{d_1} c_1(t)dt + \sum_{i=2}^{n} \int_{[1-\xi_i][d_{i-1}+T_{i-1}(d_{i-1},X_{i-1})]}^{d_i - \xi_i[d_{i-1}+T_{i-1}(d_{i-1},X_{i-1})]} c_i(t)dt \\ &+ \int_{d_n+T_n(d_n,X_n)}^{t_e} c_{n+1}(t)dt + \nu[t_e - t_s].\end{aligned} \tag{7}$$

Note that the term $\nu[t_e - t_s]$ is constant with respect to the schedule and can be dropped without affecting the model.

3.1 Multi-trip model: optimality conditions

In this analysis we assume that the traveler will spend a positive amount of time in each activity for any outcome of the travel times. That is, we focus on interior optima $d_1^* > t_s$ for $i = 1$ and $d_i^* > a_i^* \equiv d_{i-1}^* + T_i(d_{i-1}^*, x_{i-1})$ for $i \in \{2, \ldots, n\}$ and any possible x_{i-1}. A boundary solution $d_i^* = a_i^*$ corresonds to departing on the next trip immediately after arriving, which we consider to be a situation of limited practical interest. Other adjustments, not covered by this model, seem more reasonable in such severe situations, for example cancelling some of the subsequent activities.

We use the general notation $\mathbf{v}_{i,j}$ to denote the vector $(v_i, v_{i+1}, \ldots, v_j)$. Using backward induction, we first consider the last trip, indexed n. The optimal departure time d_n^* of trip n does not depend on the departure times of the preceding trips directly, but assuming that the scheduling of the preceding activity is at least partially flexible ($\xi_n > 0$), it depends on the arrival time a_n to the preceding activity. Further, it depends in general on the realized standardized travel times on all preceding trips, $\mathbf{x}_{0,n-1}$, since these observations resolve some uncertainty about X_n through the joint travel time distribution.

For a given preceding arrival time a_n, a marginal change in the departure time d_n affects the time spent in activity n, the time spent traveling on trip n and the time spent in activity $n+1$, and the optimality condition requires that the net change in expected utility be zero. More precisely, the first-order condition for the optimal departure time $d_n^* = d_n^*(a_n, \mathbf{x}_{0,n-1})$ of trip n given preceding arrival time a_n and travel time realizations $\mathbf{x}_{0,n-1}$ is

$$\begin{aligned} 0 &= \frac{\partial E[U(\mathbf{d}) \,|\, \mathbf{x}_{0,n-1}]}{\partial d_n} \\ &= \frac{\partial}{\partial d_n}\left(\int_{(1-\xi_n)a_n}^{d_n-\xi_n a_n} c_n(t)dt + E\left[\int_{d_n+T_n(d_n,X_n)}^{t_e} c_{n+1}(t)dt \,\middle|\, \mathbf{x}_{0,n-1}\right]\right), \end{aligned}$$

that is, with $T_i'(d_n, X_i) = \tau_i'(d_i) + \sigma_i'(d_i)X_i$,

$$c_n(d_n^* - \xi_n a_n) = E[(1 + T_n'(d_n^*, X_n)) \cdot c_{n+1}(d_n^* + T_n(d_n^*, X_n)) \,|\, \mathbf{x}_{0,n-1}]. \tag{8}$$

Consider now an arbitrary trip $i \in \{2, n-1\}$. Assume that the departure times on all trips subsequent to i are chosen optimally given the arrival time to the immediately preceding activity and the realized travel times on all preceding trips. In other words, each subsequent departure time is given by $d_j^* = d_j^*(a_j, \mathbf{x}_{0,j-1})$, $j \in \{i+1, \ldots, n\}$. Since the travel times on trip i and all subsequent trips are known only to belong to the conditional joint distribution $\mathbf{X}_{i,n} \,|\, \mathbf{x}_{0,i-1}$ at the time of the departure on trip i, the optimal departure times d_j^* on the subsequent trips are stochastic variables *ex ante*.

Given preceding arrival time a_i, a marginal change in the departure d_i affects the time spent in activity i, the time spent traveling on trip i and the times spent in all subsequent activities $i+1, \ldots, n+1$, and the optimality condition requires that the net change in expected utility be zero. However, since the subsequent schedule is chosen optimally by assumption, the net change in expected utility due to these departure time changes will be zero. Hence, by the envelope theorem it is sufficient to consider only the partial effect of a change in d_i on expected utility, that is, the effect on the time spent in activity i, the time spent traveling on trip i and the times spent in the immediately subsequent activity $i+1$. The necessary optimality condition for the departure time $d_i^* = d_i^*(a_i, \mathbf{x}_{0,i-1})$ on trip i given preceding arrival time a_i and travel time realizations $\mathbf{x}_{0,i-1}$ is then

$$\begin{aligned} 0 &= \frac{\partial E[U(\mathbf{d}_{1,i}, \mathbf{d}_{i+1,n}^*) \,|\, \mathbf{x}_{0,i-1}]}{\partial d_i} \\ &= \frac{\partial}{\partial d_i}\left(\int_{(1-\xi_i)a_i}^{d_i-\xi_i a_i} c_i(t)dt + E\left[\int_{(1-\xi_{i+1})(d_i+T_i(d_i,X_i))}^{d_{i+1}^*-\xi_{i+1}(d_i+T_i(d_i,X_i))} c_{i+1}(t)dt \,\middle|\, \mathbf{x}_{0,i-1}\right]\right), \end{aligned}$$

that is,

$$\begin{aligned} c_i(d_i^* - \xi_i a_i) = E[(1 + T_i'(d_i^*, X_i)) \cdot ([1-\xi_{i+1}]c_{i+1}([1-\xi_{i+1}][d_i^* + T_i(d_i^*, X_i)]) \\ + \xi_{i+1}c_{i+1}(d_{i+1}^* - \xi_{i+1}[d_i^* + T_i(d_i^*, X_i)])) \,|\, \mathbf{x}_{0,i-1}], \\ i \in \{2, \ldots, n-1\}. \end{aligned}$$

(9)

For trip 1, finally, there is no preceding arrival time that could influence the departure time choice, but we express the first-order condition for the optimal departure time d_1^* conditional on the invariant observation x_0,

$$
\begin{aligned}
0 &= \frac{\partial E[U(d_1, \mathbf{d}^*_{2,n})]}{\partial d_1} \\
&= \frac{\partial}{\partial d_1}\left(\int_{t_s}^{d_1} c_1(t)dt + E\left[\int_{(1-\xi_2)(d_1+T_1(d_1,X_1))}^{d_2^*-\xi_2(d_1+T_1(d_1,X_1))} c_2(t)dt \,\middle|\, x_0\right]\right),
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
c_1(d_1^*) = E[(1+T_1'(d_1^*, X_1)) \cdot ([1-\xi_2]c_2([1-\xi_2][d_1^* + T_1(d_1^*, X_1)]) \\
+ \xi_2 c_2(d_2^* - \xi_2[d_1^* + T_1(d_1^*, X_1)])) \,|\, x_0].
\end{aligned}
\tag{10}
$$

We can express the optimality conditions in a more unified form. The concept of the *backward optimal* marginal utility function was introduced by Jenelius et al. (2011) in the case of deterministic travel times. Here we generalize this concept to a stochastic multi-trip setting and introduce the backward optimal marginal cost function for activity $i \in \{2, \ldots, n+1\}$, $\tilde{c}_i(a_i, \mathbf{x}_{0,i-1})$, as the marginal cost due to a later arrival time to activity i, given that the subsequent schedule is chosen optimally conditional on the preceding realized standardized travel times $\mathbf{x}_{0,i-1}$. For the last activity with fixed scheduling we simply have

$$
\tilde{c}_{n+1}(a_{n+1}, \mathbf{x}) \equiv c_{n+1}(a_{n+1}), \tag{11}
$$

and for activity $i \in \{2, \ldots, n\}$ it is given by

$$
\begin{aligned}
\tilde{c}_i(a_i, \mathbf{x}_{0,i-1}) &\equiv -\frac{\partial}{\partial a_i}\int_{(1-\xi_i)a_i}^{d_i^*-\xi_i a_i} c_i(t)dt \\
&= (1-\xi_i)c_i((1-\xi_i)a_i) + \xi_i c_i(d_i^*(a_i, \mathbf{x}_{0,i-1}) - \xi_i a_i), \\
i &\in \{2, \ldots, n\}.
\end{aligned}
\tag{12}
$$

When the scheduling of activity i is fixed, $\xi_i = 0$, we have $\tilde{c}_i(a_i, \mathbf{x}_{0,i-1}) = c_i(a_i)$ independent of preceding travel times, and when the scheduling is completely flexible, $\xi_i = 1$, we have $\tilde{c}_i(a_i, \mathbf{x}_{0,i-1}) = c_i(d_i^*(a_i, \mathbf{x}_{0,i-1}) - a_i)$. Formula (11) can be seen as a special case of (12) by defining $\xi_{n+1} \equiv 0$ and $d_{n+1}^* = t_e$, reflecting that the scheduling and ending time of the last activity are fixed.

For notational clarity, let $a_{i+1}(d_i, x_i) = d_i + T_i(d_i, x_i)$ denote the arrival time to activity $i+1$ as a function of the departure time d_i and the standardized travel time x_i. Then $a'_{i+1}(d_i, x_i) = 1 + T_i'(d_i, x_i)$ denotes the derivative of a_{i+1} with respect to d_i. Further, let X_i^{c} be the stochastic standardized travel time on trip i conditional on the preceding standardized travel times $\mathbf{x}_{0,i-1}$, that is, $X_i^{\mathrm{c}} \equiv X_i \,|\, \mathbf{x}_{0,i-1}$. In order to express the optimality conditions for each trip in one general equation, we define $\xi_1 \equiv 0$ and $a_1 \equiv t_s$, reflecting that the scheduling and starting time of the first activity are fixed. The necessary optimality conditions (8)–(10) can now be written as

$$
\begin{aligned}
c_i(d_i^* - \xi_i a_i) = E[a'_{i+1}(d_i^*, X_i^{\mathrm{c}}) \cdot \tilde{c}_{i+1}(a_{i+1}(d_i^*, X_i^{\mathrm{c}}), (\mathbf{x}_{0,i-1}, X_i^{\mathrm{c}}))], \\
i \in \{1, \ldots, n\}.
\end{aligned}
\tag{13}
$$

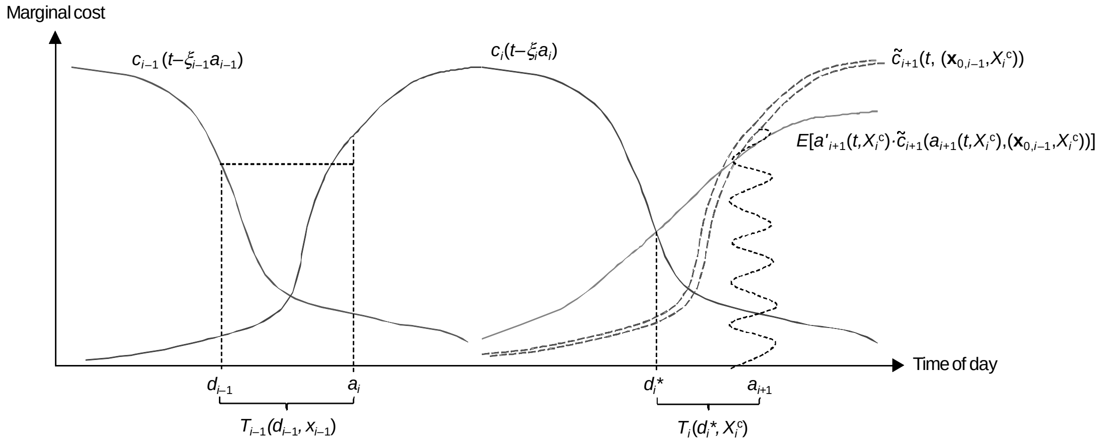

Figure 2: Illustration of a trip i in the multiple-trip model and the optimal departure time d_i^*. The double, dashed lines for $\tilde{c}_{i+1}(t, (\mathbf{x}_{0,i-1}, X_i^c))$ indicates that the backward optimal marginal cost function is stochastic. The curved, dashed line at time a_{i+1} indicates that the arrival time is stochastic.

For each trip i the optimality condition states that the marginal gain from departing later from the origin activity i (left-hand side) must equal the expected marginal loss from arriving later to the destination activity $i+1$ given that the subsequent schedule is chosen optimally, considering that the arrival time is affected by the departure time and is stochastic at the time of departure (right-hand side). The optimality condition can be interpreted as finding the intersection of the functions $c_i(t)$ and $E[a'_{i+1}(t, X_i^c) \cdot \tilde{c}_{i+1}(a_{i+1}(t, X_i^c), (\mathbf{x}_{0,i-1}, X_i^c))]$, as illustrated in Figure 2. An important special case is when travel time is independent of departure time, that is, $T_i = \tau_i + \sigma_i X_i$ for all d_i. In this case the optimality condition simplifies to

$$c_i(d_i^* - \xi_i a_i) = E[\tilde{c}_{i+1}(a_{i+1}(d_i^*, X_i^c), (\mathbf{x}_{0,i-1}, X_i^c))], \quad i \in \{1, \ldots, n\}. \tag{14}$$

We may compare Eqs. (13)–(14) with the corresponding Eqs. (2)–(3) for the single-trip model. The optimality condition for the multi-trip model extends that of the single-trip model in two distinct ways. First, the joint distribution of travel times means that the optimal scheduling of a trip takes the realized travel times on preceding trips into account. Second, the scheduling flexibility of activities means that the optimal timing of a trip depends on the arrival time to the preceding activity. Together, this means that the backward optimal marginal cost function itself is stochastic, which is in contrast to the fixed marginal cost function $c_2(t)$ in the single-trip model. If travel times are statistically independent (i.e., $X_i^c = X_i$ for all trips), and if the scheduling of the activities is completely fixed (i.e., $\xi_i = 0$ for all trips), the optimality condition (13) becomes

$$c_i(d_i^*) = E[a'_{i+1}(d_i^*, X_i) \cdot c_{i+1}(a_{i+1}(d_i^*, X_i))], \quad i \in \{1, \ldots, n\}, \tag{15}$$

which is the same condition as Eq. (2) for the single-trip model. In this case the multi-trip model thus separates into n single-trip model instances.

3.2 Multi-trip model: value of mean travel time and variability

Now using forward induction, it can be seen that the arrival time to any activity i and subsequent departure time in an optimal schedule are completely determined by the preceding realized travel times $\mathbf{x}_{0,i-1}$ and can be denoted $a_i^*(\mathbf{x}_{0,i-1})$ and $d_i^*(\mathbf{x}_{0,i-1})$, respectively. The arrival time $a_{i+1}^*(\mathbf{x}_{0,i})$ is related to the preceding departure time as $a_{i+1}^*(\mathbf{x}_{0,i}) = d_i^*(\mathbf{x}_{0,i-1}) + T_i(d_i^*(\mathbf{x}_{0,i-1}), x_i)$.

As the proper generalization from the single-trip model, we define the value of mean travel time (VMTT) on a trip as the marginal expected cost of a uniform increase in the mean travel time given that all departure times, on both preceding and subsequent trips, are chosen optimally. We write the mean travel time on trip $i \in \{1, \ldots, n\}$ as $\tau_i(d_i) = \tau_i^0 + \tau_i^{\mathrm{x}}(d_i)$, with $\tau_i^0 > 0$, and consider changes in the fixed component τ_i^0. The VMTT on trip i will vary depending on the realized travel times on the preceding trips and, if we consider x_0 to be stochastic, depending on the circumstances and the information available before the first trip. We may first define the *conditional* VMTT for a given travel time realization $\mathbf{x}_{0,i-1}$, i.e., on a given day. By the envelope theorem, we have

$$\begin{aligned}
\mathrm{VMTT}_i(\mathbf{x}_{0,i-1}) &\equiv -\frac{dE[U(\mathbf{d}^*) \mid \mathbf{x}_{0,i-1}]}{d\tau_i^0} \\
&= -\frac{\partial}{\partial \tau_i^0} E\left[\int_{[1-\xi_{i+1}][d_i^*+\tau_i^0+\tau_i^{\mathrm{x}}(d_i^*)+\sigma_i(d_i^*)X_i]}^{d_{i+1}^*-\xi_{i+1}[d_i^*+\tau_i^0+\tau_i^{\mathrm{x}}(d_i^*)+\sigma_i(d_i^*)X_i]} c_{i+1}(t)dt \,\middle|\, \mathbf{x}_{0,i-1} \right] \\
&= E[\tilde{c}_{i+1}(a_{i+1}^*(\mathbf{x}_{0,i-1}, X_i^{\mathrm{c}}), (\mathbf{x}_{0,i-1}, X_i^{\mathrm{c}}))], \\
i \in \{1, \ldots, n\}.
\end{aligned} \tag{16}$$

Taking the expectation across the whole distribution of $\mathbf{X}_{0,i-1}$, i.e., across all possible days, the *unconditional* VMTT is

$$\begin{aligned}
\mathrm{VMTT}_i &\equiv E[\mathrm{VMTT}_i(\mathbf{X}_{0,i-1})] \\
&= E[\tilde{c}_{i+1}(a_{i+1}^*(\mathbf{X}_{0,i}), \mathbf{X}_{0,i})], \quad i \in \{1, \ldots, n\},
\end{aligned} \tag{17}$$

where the general identity $E[E[Y \mid X]] = E[Y]$ for two stochastic variables X and Y was used.

Similarly to the VMTT, we define the value of travel time variability (VTTV) on a trip as the marginal expected cost of a uniform increase in the travel time standard deviation given that the schedule is chosen optimally. We write the standard deviation of travel time on trip $i \in \{1, \ldots, n\}$ as $\sigma_i(d_i) = \sigma_i^0 + \sigma_i^{\mathrm{x}}(d_i)$, with $\sigma_i^0 > 0$, and consider changes in the fixed component σ_i^0.

Just as the VMTT, the VTTV on trip i will vary depending on the realized travel times on the preceding trips, and we first consider the conditional VTTV for a given travel time realization $\mathbf{x}_{0,i-1}$. By the envelope theorem, we have

$$\begin{aligned}
\mathrm{VTTV}_i(\mathbf{x}_{0,i-1}) &\equiv -\frac{dE[U(\mathbf{d}^*) \mid \mathbf{x}_{0,i-1}]}{d\sigma_i^0} \\
&= -\frac{\partial}{\partial \sigma_i^0} E\left[\int_{[1-\xi_{i+1}][d_i^*+\tau_i(d_i^*)+(\sigma_i^0+\sigma_i^{\mathrm{x}}(d_i^*))X_i]}^{d_{i+1}^*-\xi_{i+1}[d_i^*+\tau_i(d_i^*)+(\sigma_i^0+\sigma_i^{\mathrm{x}}(d_i^*))X_i]} c_{i+1}(t)dt \,\middle|\, \mathbf{x}_{0,i-1} \right] \\
&= E[X_i^{\mathrm{c}} \cdot \tilde{c}_{i+1}(a_{i+1}^*(\mathbf{x}_{0,i-1}, X_i^{\mathrm{c}}), (\mathbf{x}_{0,i-1}, X_i^{\mathrm{c}}))], \\
i \in \{1, \ldots, n\}.
\end{aligned}$$

(18)

Taken across the whole distribution of $\mathbf{X}_{0,i-1}$, the unconditional VTTV is

$$\begin{aligned}\mathrm{VTTV}_i &\equiv E[\mathrm{VTTV}_i(\mathbf{X}_{0,i-1})]\\ &= E[X_i \cdot \tilde{c}_{i+1}(a^*_{i+1}(\mathbf{X}_{0,i}), \mathbf{X}_{0,i})], \quad i \in \{1, \ldots, n\}.\end{aligned} \tag{19}$$

The optimality condition (13) can now be expressed as

$$\begin{aligned}c_i(d^*_i(\mathbf{x}_{0,i-1}) - \xi_i a^*_i(\mathbf{x}_{0,i-1})) = [1 + \tau'_i(d^*_i(\mathbf{x}_{0,i-1}))] \cdot \mathrm{VMTT}_i(\mathbf{x}_{0,i-1})&\\ + \sigma'_i(d^*_i(\mathbf{x}_{0,i-1})) \cdot \mathrm{VTTV}_i(\mathbf{x}_{0,i-1}),&\\ i \in \{1, \ldots, n\},&\end{aligned} \tag{20}$$

which is the generalization to the multi-trip setting of formula (6) for the single-trip model. Thus, the traveler trades the cost of departing from activity i against a linear combination of the costs of mean travel time and travel time variability, where all costs depend on the preceding realized travel times. In the important special case where travel time is independent of departure time, we have $\mathrm{VMTT}_i(\mathbf{x}_{0,i-1}) = c_i(d^*_i(\mathbf{x}_{0,i-1}) - \xi_i a^*_i(\mathbf{x}_{0,i-1}))$, and across all realizations of preceding travel times we have $\mathrm{VMTT}_i = E[c_i(d^*_i(\mathbf{X}_{0,i-1}) - \xi_i a^*_i(\mathbf{X}_{0,i-1}))]$, providing alternative formulas for the conditional and the unconditional VMTT that may be more useful depending on the specific model.

Formulas (17) and (19) may be compared with their counterparts (4) and (5) for the single-trip special case. Just as for the optimality conditions discussed above, there are two distinct features that influences the unconditional VMTT and VTTV that are not present in the single-trip model. First, the scheduling flexibility of activities means that the optimal timing of a trip depends on the arrival time to the preceding activity, which, taken across the distribution of all preceding travel times, is a stochastic variable. Second, the preceding travel time realizations themselves provide information about the subsequent travel times, which also determines the optimal timing of a trip. It follows that with statistically independent travel times ($X^c_i = X_i$ for all i) and fixed scheduling ($\xi_i = 0$ for all i) of all activities, the model separates into n single-trip model instances, and we obtain $\mathrm{VMTT}_i = E[c_{i+1}(d^*_i + T_i(d^*_i, X_i))]$ and $\mathrm{VTTV}_i = E[X_i \cdot c_{i+1}(d^*_i + T_i(d^*_i, X_i))]$ equivalent to formulas (4)–(5).

An alternative interpretation of the single-trip model is that it focuses on one specific trip in a chain of multiple trips and on a particular outcome of the preceding travel times. From this point of view, the scheduling flexibility of preceding and subsequent activities have already been incorporated in the specification of the marginal utility functions, and the conditioning on previous travel times has already been incorporated in the travel time function. Thus, $c_1(t)$ and $c_2(t)$ in the single-trip model are identical to $c_i(t - \xi_i a^*_i(\mathbf{x}_{0,i-1}))$ and $\tilde{c}_{i+1}(t, (\mathbf{x}_{0,i-1}, X^c_i))$, and the standardized travel time X is identical to the conditional standardized travel time $X^c_i = X_i \mid \mathbf{x}_{0,i-1}$, for some trip i in the multi-trip model. By ignoring the preceding trips and surrounding activities, the single-trip model fails to capture the variation of the conditional VMTT and VTTV across different days.

3.3 Multi-trip model: Value of travel time dependence

Suppose that the stochastic part of the travel time on trip i can be decomposed into one component that represents the dependence on the preceding trips, and one component that is independent of preceding trips. That is,

$$T_i(d_i) = \tau_i(d_i) + [1 + \hat{\sigma}^{\mathrm{x}}_i(d_i)][\sigma_{0,i-1} f_i(\mathbf{X}_{0,i-1}) + \sigma_{i,i} Y_i], \quad i = 1, \ldots, n,$$

where $f_i(\mathbf{X}_{0,i-1})$ represents the dependence on preceding trips, independent of Y_i, $E[f_i(\mathbf{X}_{0,i-1})] = E[Y_i] = 0$ and $\mathrm{Var}[f_i(\mathbf{X}_{0,i-1})] = \mathrm{Var}[Y_i] = 1$. This is consistent with the formulation $T_i(d_i, X_i) = \tau_i(d_i) + \sigma_i(d_i)X_i$ since we can define $X_i = [\sigma_{0,i-1}f_i(\mathbf{X}_{0,i-1}) + \sigma_{i,i}Y_i]/\sigma_i^{\mathrm{n}}$, where $\sigma_i^{\mathrm{n}} = (\sigma_{0,i-1}^2 + \sigma_{i,i}^2)^{1/2}$, and $\sigma_i(d_i) = \sigma_i^{\mathrm{n}}[1 + \hat{\sigma}_i^{\mathrm{x}}(d_i)]$.

Now we can consider the value of reducing the travel time dependence, VTTD, as the value of reducing the variability $\sigma_{0,i-1}$. Following the same approach as for the VMTT and the VTTV above, we have the value conditional on the realizations of preceding travel times,

$$\begin{aligned}\mathrm{VTTD}_i(\mathbf{x}_{0,i-1}) &\equiv -\frac{dE[U(\mathbf{d}^*)\,|\,\mathbf{x}_{0,i-1}]}{d\sigma_{0,i-1}} \\ &= [1 + \hat{\sigma}_i^{\mathrm{x}}(d_i^*(\mathbf{x}_{0,i-1}))]f_i(\mathbf{x}_{0,i-1}) \cdot E[\tilde{c}_{i+1}(a_{i+1}^*(\mathbf{x}_{0,i-1}, X_i^{\mathrm{c}}), (\mathbf{x}_{0,i-1}, X_i^{\mathrm{c}}))]\end{aligned} \tag{21}$$

and the unconditional value across realizations,

$$\begin{aligned}\mathrm{VTTD}_i &\equiv E[\mathrm{VTTD}_i(\mathbf{X}_{0,i-1})] \\ &= E[[1 + \hat{\sigma}_i^{\mathrm{x}}(d_i^*(\mathbf{X}_{0,i-1}))]f_i(\mathbf{X}_{0,i-1}) \cdot \tilde{c}_{i+1}(a_{i+1}^*(\mathbf{X}_{0,i}), \mathbf{X}_{0,i})], \\ i &\in \{1, \dots, n\}.\end{aligned} \tag{22}$$

4 Special case: Two trip step model

We now consider a special case of the general model in Section 3, which consists of $n = 2$ trips and 3 activities with piecewise constant marginal cost functions. This specification is an extension of the single-trip model proposed by Vickrey (1969) studied in Section 2.3. Marginal costs are given by

$$\begin{aligned}c_1(t) &= c_1, \\ c_2(t) &= c_2^1 + (c_2^2 - c_2^1)H(t - (1-\xi)t_1^*) - (c_2^2 - c_2^3)H(t - (t_2^* - \xi t_1^*)), \\ c_3(t) &= c_1, \quad t \in [t_s, t_e],\end{aligned}$$

where $c_2^1 < c_1$, $c_2^3 < c_1$ and $c_1 < c_2^{\mathrm{L}}$. The only activity with flexible scheduling is activity 2, which means that we can drop the activity index from the scheduling flexibility parameter and simply denote it as ξ. We assume that the traveler always arrives to activity 2 before the second, negative step in marginal cost, i.e., that $(1 - \xi)a_2 < t_2^* - \xi t_1^*$ for all feasible arrival times a_2. Further, the traveler always departs from activity 2 after the first, positive step in marginal cost, i.e., $d_2 - \xi a_2 > (1-\xi)t_1^*$ for all feasible departure times d_2 and arrival times a_2. We assume here that travel times are independent of departure times, i.e., that $T_i = \tau_i + \sigma_i X_i$ for $i = 1, 2$.

4.1 Two trip step model: Optimality conditions

In the following the optimality conditions and the travel time values are derived conditional on the realization of x_0 prior to the first trip, which is left out of the equations. For activity 3, the backward optimal marginal cost function defined in (11) is here simply $\tilde{c}_3(a_3, (x_1, x_2)) = c_1$. The optimal departure time on trip 2 is at the transition from the high cost c_2^2 to the low cost c_2^3, which occurs at $d_2^* = d_2^*(a_2) = t_2^* + \xi(a_2 - t_1^*)$, independent of the realized travel time on trip 1.

Consider then trip 1. The scheduling of the destination activity is flexible, and the backward optimal marginal cost function defined in (12), with the optimal departure time of trip 2 from (25) inserted, is

$$\begin{aligned}\tilde{c}_2(a_2) &= c_2^1 + (1-\xi)(c_2^2 - c_2^1)H([1-\xi][a_2 - t_1^*]) \\ &\quad + \xi(c_2^2 - c_2^1)H(d_2^*(a_2) - \xi a_2 - [1-\xi]t_1^*) \\ &= (1-\xi)c_2^1 + \xi c_2^2 + (1-\xi)(c_2^2 - c_2^1)H(a_2 - t_1^*).\end{aligned}$$

As can be seen by comparing with Section 2.3, this is the destination marginal cost function of a single-trip step model with cost parameters $c_2^{\mathrm{E}} = (1-\xi)c_2^1 + \xi c_2^2$ and $c_2^{\mathrm{L}} = c_2^2$. In the case that $(1-\xi)c_2^1 + \xi c_2^2 < c_1$, which is the standard assumption of the single trip model, the general necessary optimality condition (14) applied to the timing of trip 1 gives

$$c_1 = E\left[(1-\xi)c_2^1 + \xi c_2^2 + (1-\xi)(c_2^2 - c_2^1)H(d_1^* + T_1 - t_1^*)\right],$$

and we can readily find the optimal departure time $d_1^* = t_1^* - \tau - \sigma_1\Phi_1^{-1}([c_2^2 - c_1]/[(1-\xi)(c_2^2 - c_2^1)])$, where Φ_1 is the cumulative distribution function of X_1.

In the case that $(1-\xi)c_2^1 + \xi c_2^2 \geq c_1$ and $\xi < 1$, which means that early arrival is at least as attractive as activity 2, it is optimal to spend as little time as possible at activity 1, that is, the optimal departure time is $d_1^* = t_s$. In the extreme case $\xi = 1$, finally, there are no fixed time of day costs and all feasible departure times are optimal.

4.2 Two trip step model: travel time values

For trip 2, we trivially have $\mathrm{VMTT}_2 = c_1$ and $\mathrm{VTTV}_2 = 0$. For trip 1, the values differ between the three different cases identified above. In the traditional case that $(1-\xi)c_2^1 + \xi c_2^2 < c_1$, the theory for the single-trip step model gives $\mathrm{VMTT}_1 = c_1$ and

$$\mathrm{VTTV}_1 = (1-\xi)(c_2^2 - c_2^1)\int_{\frac{c_2^2 - c_1}{(1-\xi)(c_2^2 - c_2^1)}}^{1} \Phi_1^{-1}(p)dp.$$

In the case that $(1-\xi)c_2^1 + \xi c_2^2 \geq c_1$ and $\xi < 1$, we instead have

$$\begin{aligned}\mathrm{VMTT}_1 &= E[(1-\xi)c_2^1 + \xi c_2^2 + (1-\xi)(c_2^2 - c_2^1)H(t_s + T_1 - t_1^*)] \\ &= (1-\xi)c_2^1 + \xi c_2^2 + (1-\xi)(c_2^2 - c_2^1)\left(1 - \Phi_1\left(\frac{t_1^* - t_s - \tau_1}{\sigma_1}\right)\right),\end{aligned}$$

and

$$\begin{aligned}\mathrm{VTTV}_1 &= E[X_1 \cdot [(1-\xi)c_2^1 + \xi c_2^2 + (1-\xi)(c_2^2 - c_2^1)H(t_s + T_1 - t_1^*)]] \\ &= (1-\xi)(c_2^2 - c_2^1)\int_{\Phi_1\left(\frac{t_1^* - t_s - \tau_1}{\sigma_1}\right)}^{1} \Phi_1^{-1}(p)dp.\end{aligned}$$

Note that the mean and standard deviation of the travel time enter both VMTT_1 and VTTV_1 non-linearly in this case. In the extreme case $\xi = 1$, any additional travel time is taken from activity 1 or 3, and we have $\mathrm{VMTT}_1 = c_1$ and $\mathrm{VTTV}_1 = 0$.

5 Special case: Two trip slope model

We now consider another special case of the general model in Section 3 with linear marginal cost functions. This specification was previously studied in a setting of deterministic travel times by Jenelius et al. (2011), and is an extension of the single-trip model proposed by Vickrey (1973) and Fosgerau and Engelson (2011) studied in Section 2.4. As will be shown, the model allows for closed-form expressions for the VMTT and VTTV on both trips and a clear interpretation of the roles of scheduling flexibility and travel time correlation. The marginal cost functions are given by

$$
\begin{aligned}
&c_1(t) = \beta_0 + \beta_1 t,\\
&c_2(t) = \begin{cases} \gamma_0 + \gamma_1 t & t < \underline{t},\\ \delta_0 + \delta_1 t & t \geq \bar{t},\end{cases}\\
&c_3(t) = \epsilon_0 + \epsilon_1 t, \quad t \in [t_s, t_e],
\end{aligned}
$$

where $t_s < \underline{t} \leq \bar{t} < t_e$. We assume that the traveler always arrives during the first regime and always departs during the second regime of activity 2, respectively, i.e., that $(1-\xi)a_2 < \underline{t}$ and $d_2 - \xi a_2 > \bar{t}$ for all feasible arrival times a_2 and departure times d_2. The requirement that the traveler initially prefers the origin activity and ultimately the destination activity implies that $\beta_1 < \gamma_1$ and $\delta_1 < \epsilon_1$. We assume no particular sign for any of these parameters, although we will mainly discuss the case $\beta_1 < 0$, $\gamma_1 > 0$, $\delta_1 < 0$ and $\epsilon_1 > 0$, which may be the most common case in practice. With these parameter signs activities 1 and 2 end with a cool-down period with diminishing marginal utility, while activities 2 and 3 begin with a warm-up period with increasing marginal utility. As we will see, the traveler is then risk-averse on both trips.

We denote the optimal transition times between the activities when $T_1 = T_2 = 0$ as t_1^* and t_2^*, respectively. The first-order requirements $c_1(t_1^*) = (1-\xi)c_2((1-\xi)t_1^*) + \xi c_2(t_2^* - \xi t_1^*)$ and $c_2(t_2^* - \xi t_1^*) = c_3(t_2^*)$ allow us to express t_1^* and t_2^* in terms of the parameters of the marginal utility functions,

$$
t_1^* = \frac{\beta_0 - \tilde{\gamma}_0}{\tilde{\gamma}_1 - \beta_1}, \quad t_2^* = \frac{\hat{\delta}_0 - \epsilon_0}{\epsilon_1 - \hat{\delta}_1}, \tag{23}
$$

where we have introduced the useful auxiliary variables

$$
\begin{aligned}
\tilde{\gamma}_0 &\equiv (1-\xi)\gamma_0 + \xi\frac{\delta_0\epsilon_1 - \epsilon_0\delta_1}{\epsilon_1 - \delta_1},\\
\tilde{\gamma}_1 &\equiv (1-\xi)^2\gamma_1 - \xi^2\frac{\delta_1\epsilon_1}{\epsilon_1 - \delta_1},\\
\hat{\delta}_0 &\equiv \frac{\delta_0[(1-\xi)^2\gamma_1 - \beta_1] - \xi[\beta_0 - (1-\xi)\gamma_0]\delta_1}{(1-\xi)^2\gamma_1 - \xi^2\delta_1 - \beta_1},\\
\hat{\delta}_1 &\equiv \frac{[(1-\xi)^2\gamma_1 - \beta_1]\delta_1}{(1-\xi)^2\gamma_1 - \xi^2\delta_1 - \beta_1}.
\end{aligned} \tag{24}
$$

Note that with fixed scheduling, $\xi = 0$, we obtain $\tilde{\gamma}_0 = \gamma_0$, $\tilde{\gamma}_1 = \gamma_1$, $\hat{\delta}_0 = \delta_0$ and $\hat{\delta}_1 = \delta_1$. With completely flexible scheduling, $\xi = 1$, we obtain $\tilde{\gamma}_0 = (\delta_0\epsilon_1 - \epsilon_0\delta_1)/(\epsilon_1 - \delta_1)$, $\tilde{\gamma}_1 = \delta_1\epsilon_1/(\epsilon_1 - \delta_1)$, $\hat{\delta}_0 = (\beta_0\delta_1 + \delta_0\beta_1)/(\beta_1 + \delta_1)$ and $\hat{\delta}_1 = \beta_1\delta_1/(\beta_1 + \delta_1)$. With expected parameter signs $\beta_1 < 0 < \gamma_1$ and $\delta_1 < 0 < \epsilon_1$ we have $\tilde{\gamma}_1 > 0$ and $\hat{\delta}_1 < 0$ for all ξ.

5.1 Two trip slope model: optimality conditions

Since the scheduling of the last activity is fixed, the backward optimal marginal cost function defined in (11) is simply $\tilde{c}_3(a_3, (x_1, x_2)) = \epsilon_0 + \epsilon_1 a_3$. Applied to trip 2, the general necessary condition (14) for the departure times gives

$$\delta_0 + \delta_1(d_2^* - \xi a_2) = E[\epsilon_0 + \epsilon_1(d_2^* + \tau_2 + \sigma_2 X_2 \mid x_1)],$$

which allows us to explicitly solve for the optimal departure time $d_2^* = d_2^*(a_2, x_1)$ as a function of the arrival time a_2 and the realized standardized travel time x_1 on trip 1,

$$d_2^*(a_2, x_1) = \frac{\delta_0 - \epsilon_0}{\epsilon_1 - \delta_1} - \xi \frac{\delta_1}{\epsilon_1 - \delta_1} a_2 - \frac{\epsilon_1}{\epsilon_1 - \delta_1}(\tau_2 + \sigma_2 E[X_2 \mid x_1]). \tag{25}$$

Note how the departure time d_2^* depends linearly on the arrival time to the preceding activity (unless $\xi = 0$) and the expected travel time on trip 2 conditional on the realized travel time on trip 1. An increase in the travel time mean or standard deviation on trip 2 moves the departure time earlier while a later arrival time moves the departure time later, if $\delta_1 < 0 < \epsilon_1$ as expected.

Consider then trip 1. The scheduling of the destination activity is flexible, and the backward optimal marginal cost function defined in (12), with the optimal departure time of trip 2 from (25) inserted, is

$$\begin{aligned} \tilde{c}_2(a_2, x_1) &= (1 - \xi)(\gamma_0 + (1 - \xi)\gamma_1 a_2) + \xi(\delta_0 + \delta_1(d_2^*(a_2, x_1) - \xi a_2)) \\ &= \tilde{\gamma}_0 + \tilde{\gamma}_1 a_2 - \xi \frac{\delta_1 \epsilon_1}{\epsilon_1 - \delta_1}(\tau_2 + \sigma_2 E[X_2 \mid x_1]). \end{aligned}$$

We see that the backward optimal marginal cost function is linear in the arrival time a_2. The general necessary optimality condition (14) applied to the timing of trip 1 now gives

$$\begin{aligned} \beta_0 + \beta_1 d_1^* &= E\left[\tilde{\gamma}_0 + \tilde{\gamma}_1 \cdot (d_1^* + \tau_1 + \sigma_1 X_1) - \xi \frac{\delta_1 \epsilon_1}{\epsilon_1 - \delta_1}(\tau_2 + \sigma_2 E[X_2 \mid X_1])\right] \\ &= \tilde{\gamma}_0 + \tilde{\gamma}_1(d_1^* + \tau_1) - \xi \frac{\delta_1 \epsilon_1}{\epsilon_1 - \delta_1}\tau_2. \end{aligned}$$

We can now solve explicitly for the optimal departure time on trip 1 and the associated arrival time to activity 2,

$$\begin{aligned} d_1^* &= t_1^* - \frac{\tilde{\gamma}_1}{\tilde{\gamma}_1 - \beta_1}\tau_1 + \xi \frac{\delta_1 \epsilon_1}{(\tilde{\gamma}_1 - \beta_1)(\epsilon_1 - \delta_1)}\tau_2, \\ a_2^*(x_1) &= t_1^* - \frac{\beta_1}{\tilde{\gamma}_1 - \beta_1}\tau_1 + \xi \frac{\delta_1 \epsilon_1}{(\tilde{\gamma}_1 - \beta_1)(\epsilon_1 - \delta_1)}\tau_2 + \sigma_1 x_1. \end{aligned} \tag{26}$$

We see that the optimal departure time depends linearly on the mean travel times on both trip 1 and (with $\xi > 0$) trip 2. With expected parameter signs $\beta_1 < 0 < \gamma_1$ and $\delta_1 < 0 < \epsilon_1$, an increase in the mean travel time on any trip moves the departure time earlier. Insertion of (26) into (25) gives the overall optimal departure time on trip 2 and associated arrival time as

$$
\begin{aligned}
d_2^*(x_1) &= t_2^* + \xi \frac{\beta_1\delta_1}{(\tilde{\gamma}_1-\beta_1)(\epsilon_1-\delta_1)}\tau_1 - \frac{\epsilon_1}{\epsilon_1-\hat{\delta}_1}\tau_2 \\
&\quad - \xi\frac{\delta_1}{\epsilon_1-\delta_1}\sigma_1 x_1 - \frac{\epsilon_1}{\epsilon_1-\delta_1}\sigma_2 E[X_2 \mid x_1], \\
a_3^*(x_1,x_2) &= t_2^* + \xi \frac{\beta_1\delta_1}{(\tilde{\gamma}_1-\beta_1)(\epsilon_1-\delta_1)}\tau_1 - \frac{\hat{\delta}_1}{\epsilon_1-\hat{\delta}_1}\tau_2 \\
&\quad - \xi\frac{\delta_1}{\epsilon_1-\delta_1}\sigma_1 x_1 - \frac{\epsilon_1}{\epsilon_1-\delta_1}\sigma_2 E[X_2 \mid x_1] + \sigma_2 x_2,
\end{aligned}
\tag{27}
$$

which are linear in the travel time means as well as the standard deviations. With expected parameter signs, an increase in the mean travel time or standard deviation on trip 1 moves the departure time later, while a corresponding increase on trip 2 moves the departure time earlier.

5.2 Two trip slope model: value of mean travel time and variability

According to general formula (17) and the optimal arrival time from (26), the unconditional value of mean travel time on trip 1 is

$$
\begin{aligned}
\mathrm{VMTT}_1 &= E\left[\tilde{\gamma}_0 + \tilde{\gamma}_1 \cdot a_2^*(X_1) - \xi\frac{\delta_1\epsilon_1}{\epsilon_1-\delta_1}(\tau_2+\sigma_2 E[X_2 \mid X_1])\right] \\
&= \frac{\beta_0\tilde{\gamma}_1-\tilde{\gamma}_0\beta_1}{\tilde{\gamma}_1-\beta_1} - \frac{\beta_1\tilde{\gamma}_1}{\tilde{\gamma}_1-\beta_1}\tau_1 + \xi\frac{\beta_1\delta_1\epsilon_1}{(\tilde{\gamma}_1-\beta_1)(\epsilon_1-\delta_1)}\tau_2,
\end{aligned}
\tag{28}
$$

which is linearly increasing in the mean travel times of both trips but independent of the standard deviations and other characteristics of the joint travel time distribution. If the scheduling of activity 2 is completely fixed, i.e., $\xi = 0$, we have $\mathrm{VMTT}_1 = (\beta_0\gamma_1 - \gamma_0\beta_1 - \beta_1\gamma_1\tau_1)/(\gamma_1-\beta_1)$, independent of trip 2. As expected, this is identical to the value for the single-trip model of Vickrey (1973) considered in Section 2.4, given that the optimal transition time t_1^* is normalized to 0 (see also Fosgerau and Engelson, 2011). If the scheduling is completely flexible, $\xi = 1$, we have $\mathrm{VMTT}_1 = (\epsilon_0\beta_1\delta_1 - \delta_0\beta_1\epsilon_1 - \beta_0\delta_1\epsilon_1 + \beta_1\delta_1\epsilon_1(\tau_1+\tau_2))/(\beta_1\delta_1 - \delta_1\epsilon_1 - \beta_1\epsilon_1)$, which is symmetric in the two mean travel times.

Further, according to the general formula (17) and the optimal arrival time from (27), the unconditional value of mean travel time on trip 2 is

$$
\begin{aligned}
\mathrm{VMTT}_2 &= E[\epsilon_0 + \epsilon_1 \cdot a_3^*(X_1,X_2)] \\
&= \frac{\hat{\delta}_0\epsilon_1-\epsilon_0\hat{\delta}_1}{\epsilon_1-\hat{\delta}_1} + \xi\frac{\beta_1\delta_1\epsilon_1}{(\tilde{\gamma}_1-\beta_1)(\epsilon_1-\delta_1)}\tau_1 - \frac{\hat{\delta}_1\epsilon_1}{\epsilon_1-\hat{\delta}_1}\tau_2,
\end{aligned}
\tag{29}
$$

which, again, is linearly increasing in the mean travel times of both trips. If the scheduling of activity 2 is completely fixed, i.e., $\xi = 0$, we have $\mathrm{VMTT}_2 = (\delta_0\epsilon_1 - \epsilon_0\delta_1 - \delta_1\epsilon_1\tau_2)/(\epsilon_1-\delta_1)$, independent of trip 1. Again, this is equivalent to the single-trip model of Vickrey (1973) given that the optimal transition time t_2^* is normalized to 0. If the scheduling is completely flexible, $\xi = 1$, we have $\mathrm{VMTT}_2 = (\epsilon_0\beta_1\delta_1 - \delta_0\beta_1\epsilon_1 - \beta_0\delta_1\epsilon_1 + \beta_1\delta_1\epsilon_1(\tau_1+\tau_2))/(\beta_1\delta_1 - \delta_1\epsilon_1 - \beta_1\epsilon_1)$, which is identical to VMTT_1 for the same case.

Thus, with flexible scheduling there is a single unconditional value of mean travel time for both trips. This result is not true for general marginal cost functions $c_i(t)$.

Specifically, it stems from the fact that for both trips and any level of flexibility ξ, the VMTT in this linear marginal cost specification is independent of the travel time variability. This means that it is equal to the marginal value of travel time (VTT) for the case with deterministic travel times, which is derived by Jenelius et al. (2011). It is shown in that paper that the VTT for the two trips are equal when $\xi = 1$ and travel times are independent of departure times (this latter result holds not only for linear marginal costs but for general functions $c_i(t)$).

According to the general formula (19), the unconditional value of travel time variability on trip 1 is

$$\begin{aligned}\mathrm{VTTV}_1 &= E\left[X_1 \cdot \left(\tilde{\gamma}_0 + \tilde{\gamma}_1 \cdot a_2^*(X_1) - \xi \frac{\delta_1 \epsilon_1}{\epsilon_1 - \delta_1}(\tau_2 + \sigma_2 E[X_2 \,|\, X_1])\right)\right] \\ &= \tilde{\gamma}_1 \sigma_1 - \xi \frac{\delta_1 \epsilon_1}{\epsilon_1 - \delta_1} \sigma_2 \mathrm{Cov}[X_1, X_2],\end{aligned} \tag{30}$$

which is independent of the travel time means and linear in the travel time standard deviation of both trip 1 and (if $\xi > 0$) trip 2. The first term captures the direct impact of the variability of the arrival time to activity 2, incorporating the endogenous choice of departure time on trip 2 through the proportionality constant $\tilde{\gamma}_1$. The second term is an addition compared to the single-trip model in Section 2.4 and captures the amount that the realized travel time on trip 1 affects the expectation about the travel time on trip 2, and hence the departure time on the trip. This dependency enters the VTTV in the form of the covariance between the standardized travel times. We would typically expect the covariance to be positive if different from zero, which means that it contributes to a higher cost of travel time variability if $\delta_1 < 0 < \epsilon_1$ as expected. The term vanishes when $\xi = 0$.

Similarly, the unconditional value of travel time variability on trip 2 is, applying the general formula (19),

$$\begin{aligned}\mathrm{VTTV}_2 &= E[X_2 \cdot (\epsilon_0 + \epsilon_1 \cdot a_3^*(X_1, X_2))] \\ &= \epsilon_1 \sigma_2 - \xi \frac{\delta_1 \epsilon_1}{\epsilon_1 - \delta_1} \sigma_1 \mathrm{Cov}[X_1, X_2] - \frac{\epsilon_1^2}{\epsilon_1 - \delta_1} \sigma_2 \mathrm{Var}[E[X_2 \,|\, X_1]].\end{aligned} \tag{31}$$

Again, the first term captures the direct impact of the variability of the arrival time to activity 3. The correlation between the travel times affects the scheduling of trip 1 through the departure time on trip 2 and enters the cost symmetrically to trip 1, with σ_1 replacing σ_2 in the second term. The third, non-positive term captures the increased ability to predict the travel time on trip 2 having observed the travel time on trip 1. This effect enters the VTTV in the form of the variance of the expectation of X_2 conditional on X_1, $\mathrm{Var}[E[X_2 \,|\, X_1]]$, or loosely speaking how much the expectation about the travel time on trip 2 varies depending on the realized travel time on trip 1. With $\epsilon_1 \neq 0$ this represents a reduction of the cost that does not depend on the scheduling flexibility of activity 2.

If X_1 and X_2 are independent, then $\mathrm{Cov}[X_1, X_2] = 0$ and $\mathrm{Var}[E[X_2 \,|\, X_1]] = \mathrm{Var}[E[X_2]] = \mathrm{Var}[0] = 0$, so that all interaction terms in the VTTV disappear. We may also have $\mathrm{Cov}[X_1, X_2] = 0$ or $\mathrm{Var}[E[X_2 \,|\, X_1]] = 0$ even though X_1 and X_2 are not independent. Furthermore, $\mathrm{Var}[E[X_2 \,|\, X_1]] = 0$ implies $\mathrm{Cov}[X_1, X_2] = 0$, while the opposite is not true in general.[3] In practice, however, it seems likely that dependence of travel times would most often manifest itself as correlation.

[3]To see that $\mathrm{Var}[E[X_2 \,|\, X_1]] = 0$ implies $\mathrm{Cov}[X_1, X_2] = 0$, note first that it implies $E[X_2 \,|\, X_1] = m$ for some constant m. In fact, $m = 0$ since $0 = E[X_2] = E[E[X_2 \,|\, X_1]] = E[m] = m$. Note then that

5.3 Value of travel time dependence

Every second-order moment term in the minimum expected cost $E[C(d_1^*, d_2^*)]$ must contain either σ_1^2, σ_2^2 or $\sigma_1\sigma_2$. There are no higher-order moments than second-order moments. Thus, from (28)–(31) it can be seen that the minimum expected cost is

$$\begin{aligned} E[C(d_1^*, d_2^*)] = f(\tau_1, \tau_2) + \frac{\tilde{\gamma}_1}{2}\sigma_1^2 + \frac{\epsilon_1}{2}\sigma_2^2 - \xi\frac{\delta_1\epsilon_1}{\epsilon_1 - \delta_1}\sigma_1\sigma_2\mathrm{Cov}[X_1, X_2] \\ - \frac{\epsilon_1^2}{2(\epsilon_1 - \delta_1)}\sigma_2^2\mathrm{Var}[E[X_2 \mid X_1]], \end{aligned} \tag{32}$$

where $f(\tau_1, \tau_2)$ summarizes the first-order moment terms.

We see that the dependence of the two travel times affects the expected cost in two ways. First, there is (typically) a cost associated with positive correlation of travel times, captured by the fourth term. Obviously, the costs of "bad days" (long travel times on both trips) more than offsets the benefits of "good days" (short travel times on both trips). This negative impact only appears if the scheduling of activity 2 is at least partially flexible ($\xi > 0$). Second, there is a benefit in the dependence since it allows a more accurate expectation about the travel time on trip 2 given the experienced travel time on trip 1, as captured by the last term. Which effect is the largest is ambiguous in general.

This becomes just a special case of the multiple-trip model above: We now consider a simple model of the joint distribution of travel times in which the standardized travel time on trip 2 is a linear combination of the standardized travel time on trip 1 and an independent standardized component. More precisely,

$$\begin{aligned} T_1 &= \tau_1 + \sigma_1 X_1, \\ T_2 &= \tau_2 + \sigma_{21} X_1 + \sigma_{22} Y_2, \end{aligned} \tag{33}$$

with X_1 and Y_2 independent, $E[X_1] = E[Y_2] = 0$ and $\mathrm{Var}[X_1] = \mathrm{Var}[Y_2] = 1$. We then have $E[T_2] = \tau_2$ and $\sigma_2^2 \equiv \mathrm{Var}[T_2] = \sigma_{21}^2 + \sigma_{22}^2$, and we can define the standardized travel time $X_2 \equiv (T_2 - \tau_2)/\sigma_2$. The travel time correlation between the two trips is $\rho \equiv \mathrm{Cov}[X_1, X_2] = \sigma_{21}/\sigma_2$, and the variance of the conditional expected travel time on trip 2 is $\mathrm{Var}[E[X_2 \mid X_1]] = \sigma_{21}^2/\sigma_2^2 = \rho^2$.

Using this travel time model and differentiating (32), we can assess the value of travel time correlation,

$$\mathrm{VTTC} \equiv \frac{dE[C(d_1^*, d_2^*)]}{d\rho} = -\frac{\epsilon_1^2}{\epsilon_1 - \delta_1}\left(\xi\frac{\delta_1}{\epsilon_1} + \rho\frac{\sigma_2}{\sigma_1}\right)\sigma_1\sigma_2. \tag{34}$$

The VTTC is positive if and only if $\rho\sigma_2/\sigma_1 < -\xi\delta_1/\epsilon_1$. Whether correlation is a cost or a benefit thus depends on the cost of time lost at activity 2 relative to activity 3, $\xi\delta_1/\epsilon_1$, and the predicable part of the travel time variability on trip 2 relative to trip 1, $\rho\sigma_2/\sigma_1$. Note that a completely fixed schedule ($\xi = 0$) implies that travel time correlation is strictly beneficial, so that correlation can be costly only if scheduling is flexible.

$\mathrm{Cov}[X_1, X_2] = E[X_1X_2] = E[X_1 \cdot E[X_2 \mid X_1]] = E[X_1 \cdot 0] = 0$. To see that $\mathrm{Cov}[X_1, X_2] = 0$ does not imply $\mathrm{Var}[E[X_2 \mid X_1]] = 0$ in general, consider for example the case $X_1 \sim U(-1, 1)$ and $X_2 \sim U(0, 2)$ for $|X_1| \geq 1/2$ and $X_2 \sim U(-2, 0)$ for $|X_1| < 1/2$. Here we have $\mathrm{Cov}[X_1, X_2] = E[X_1] = E[X_2] = 0$, but $E[X_2 \mid X_1]$ is a non-degenerate stochastic variable taking values 1 and -1. In fact, $\mathrm{Var}[E[X_2 \mid X_1]] = 1$.

Marginal cost function	Intercept (Euro/h)	Slope (Euro/h^2)
Morning home: $c_1(t) = \beta_0 + \beta_1 t$	$\beta_0 = 143$	$\beta_1 = -18.1$
Morning work: $c_2(t) = \gamma_0 + \gamma_1 t, \quad t < \underline{t}$	$\gamma_0 = -322$	$\gamma_1 = 50.0$
Afternoon work: $c_2(t) = \delta_0 + \delta_1 t, \quad t \geq \bar{t}$	$\delta_0 = 383$	$\delta_1 = -25.0$
Evening home: $c_3(t) = \epsilon_0 + \epsilon_1 t$	$\epsilon_0 = -335$	$\epsilon_1 = 19.6$

Table 1: Parameter values for the linear marginal cost functions. The flexibility parameter for activity 2 is set to $\xi = 0.2$.

The quantity σ_{21} represents the variability of the travel time on trip 2 that is inherited from the variability on trip 1. It is of interest to know the value of reducing this inherited travel time variability (through, for example, more efficient restoration of the road network after an incident). The value of travel time dependence (VTTD) is

$$\mathrm{VTTD} \equiv \frac{dE[C(d_1^*, d_2^*)]}{d\sigma_{21}} = -\frac{\delta_1 \epsilon_1}{\epsilon_1 - \delta_1} \left(\xi \sigma_1 + \rho \sigma_2\right). \tag{35}$$

With expected parameter signs, this value is always positive.

5.4 Numerical example

In this section we study some properies of the model numerically for a realistic set of parameter values, although the results should be seen mainly as an illustration of the model to be followed by more rigorous estimation and more reliable results in future work. We are mainly interested in the impact of travel time correlation on the VMTT and VTTV on each trip.

The parameters are calibrated so that the linear marginal cost functions correspond as closely as possible to the marginal cost functions used in Jenelius et al. (2011) to model a day consisting of three activities: being at home in the morning, working during the day and being at home in the evening. In that paper, sigmoid logistic marginal cost functions are calibrated to match the empirical results presented in Tseng and Verhoef (2008); we refer to the former paper for a detailed description of the procedure. In this paper we calibrate the slope parameters β_1, γ_1, δ_1 and ϵ_1 to be equal to the slopes of the corresponding logistic functions of Jenelius et al. (2011) at their saturation (or "mid") points. The values are reported in Table 1 and fulfil our expectations about the parameter signs from the analysis above.

As in Jenelius et al. (2011), the flexibility parameter is set to $\xi = 0.2$. The intercept parameters β_0, γ_0, δ_0 and ϵ_0 are calibrated so that the constant term of the VMTT on each trip (i.e., the VMTT in the limit $\tau_1 = \tau_2 = 0$) is -2.0 Euro/h, in line with Tseng and Verhoef (2008).[4] While it has no direct impact on the time values, they are further calibrated to give the optimal transition times $t_1^* = 8.0$ h and $t_2^* = 17.0$ h, which may represent a typical work day. The parameter values that uniquely correspond to these conditions can be found using formulas (23), (24), (28) and (29) and are shown in Table 1.

With the calibrated parameter values, the value of mean travel time for each trip is $\mathrm{VMTT}_1 = -2.0 + 11.6\tau_1 + 0.79\tau_2$ Euro/h and $\mathrm{VMTT}_2 = -2.0 + 0.79\tau_1 + 10.9\tau_2$ Euro/h. The fact that VMTT_1 depends more on τ_1 than on τ_2 and vice versa is due to the relatively low scheduling flexibility. Further, the value of travel time variability on

[4] The finding of Tseng and Verhoef (2008) that $\mathrm{VMTT} < 0$ for short trips may be due to estimation errors but may also capture a real positive value associated with making short trips.

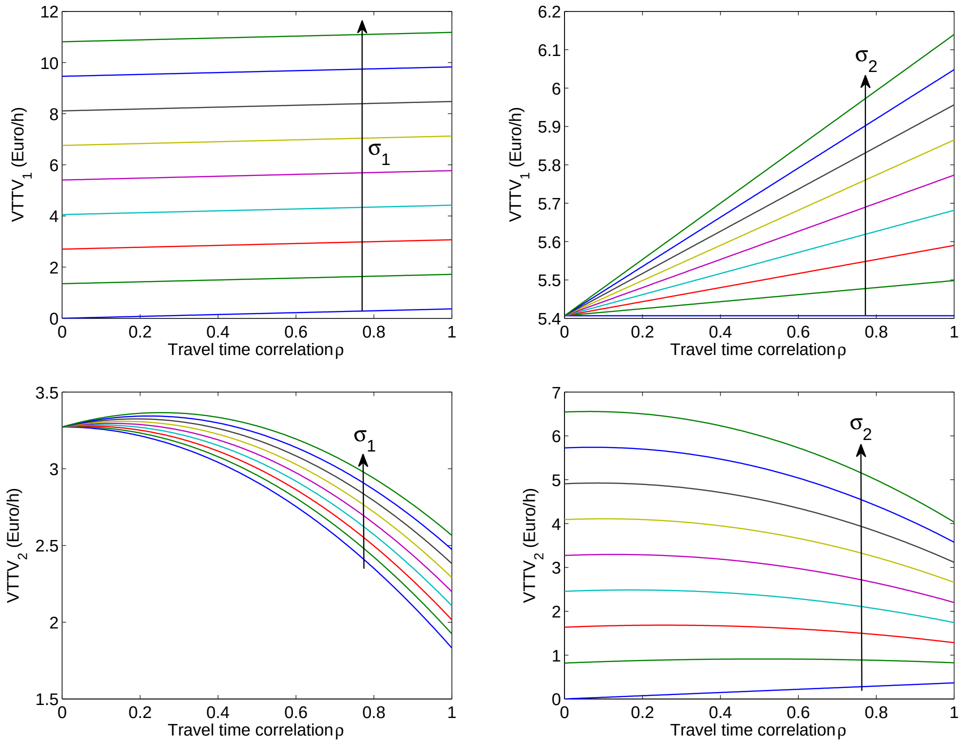

Figure 3: The unconditional VTTV on trip 1 (top) and trip 2 (bottom) as functions of the travel time correlation ρ for different values of the travel time standard deviation on trip 1 (left) and on trip 2 (right). To the left, σ_1 increases linearly from 0 to 20 minutes while σ_2 is constant at 10 minutes. To the right, symmetrically, σ_2 increases linearly from 0 to 20 minutes while σ_1 is constant at 10 minutes. Parameter values are given in Table 1.

each trip is $\mathrm{VTTV}_1 = 32.4\sigma_1 + 2.20\rho\sigma_2$ Euro/h and $\mathrm{VTTV}_2 = 2.20\rho\sigma_1 + (19.6 - 8.64\rho^2)\sigma_2$ Euro/h. Thus, while VTTV_1 increases linearly with the travel time correlation ρ, VTTV_2 follows a second-degree polynomial in ρ, obtaining the maximum value at $\rho = 0.13\sigma_1/\sigma_2$ if this is less than 1. For higher values of ρ, VTTV_2 decreases as the positive effect of correlation (accurate travel time prediction) overcomes the negative effect (the exacerbation of delay impacts). Figure 3 shows how the VTTV on each trip varies with the travel time correlation ρ for different values of σ_1 and σ_2, respectively, holding the other variable constant.

VTTC as function of ρ

While the purpose here is not to estimate the VMTT and VTTV, some comments regarding the practical relevance of the numerical results may be in place. Considering that there is a significant time between the two trips, we would expect the travel time correlation to be rather weak in practice. Further, the calculated VMTT are significantly lower than those found in most empirical studies (see, e.g., Abrantes and Wardman, 2011) for typical mean travel times. They are, however, in line with the results of Tseng and Verhoef (2008) against which the parameters were calibrated. It follows that the ratio VTTV/VMTT is considerably larger than what is commonly re-

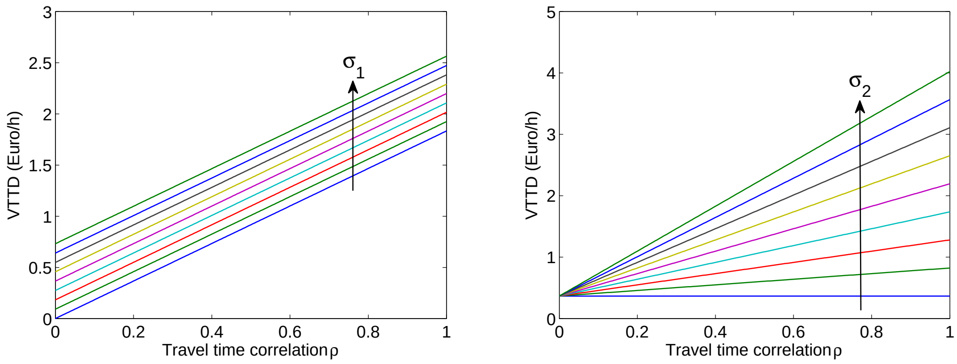

Figure 4: VTTD as a function of the travel time correlation ρ for different values of the travel time standard deviation on trip 1 (left) and on trip 2 (right). To the left, σ_1 increases linearly from 0 to 20 minutes while σ_2 is constant at 10 minutes. To the right, symmetrically, σ_2 increases linearly from 0 to 20 minutes while σ_1 is constant at 10 minutes.

ported (e.g., Noland and Polak, 2002).

6 Conclusion

The main contribution of this paper is to extend the analysis of the value of mean travel time (VMTT) and travel time variability (VTTV) from simple single-trip scheduling models to a more general multi-trip setting, incorporating the effects of flexibility in activity scheduling and dependence of travel times across trips. As a first step, we have presented a unifying single-trip model, of which current popular models are special cases, and derived formulas for the VMTT and VTTV. These are based directly on the marginal cost functions of the departure and arrival times, which avoids the need to explicitly calculate the expected cost function. The formulas serve as a platform from which specific models with desirable properties or empirical support can be obtained, both when closed form expressions exist and when numerical calculations are necessary.

The multi-trip model incorporates two features that are not captured by single-trip models: First, the scheduling of activities may be flexible, meaning that the utility derived from them depends not only on the time of day but also on the duration of activity participation. Second, travel times on different trips may be correlated, which is particularly relevant if they are close in time and space. The analysis shows that these features affect the VMTT and VTTV on a trip separately as well as in interaction with each other: Scheduling flexibility means that the VMTT and VTTV depend on the preceding arrival time and hence on the travel times on preceding trips, whether these are statistically dependent or not. Travel time dependence means that preceding realized travel times provide information about future travel times and hence affect scheduling and the VMTT and VTTV, whether scheduling is flexible or not.

We have considered a specific form of the multi-trip model with two trips and linear marginal cost functions for which closed-form expressions for the VMTT and VTTV are obtained. The formulas show that the dependence of the travel times affects the VTTV in two ways. First, if the scheduling of the intermediate activity is somewhat flexible, there are (with expected parameter signs) negative impacts on both trips associated with a positive correlation of travel times. Second, there is a benefit in the dependence for trip 2, since it allows a more accurate expectation about the travel time on trip 2 on a given day (i.e., given the realized travel time on trip 1), and hence a more efficient scheduling of the trip. We have also characterized how the VMTT and VTTV on trip 2 vary across days.

The results of this paper have implications for how we perceive, collect data on, calculate and analyze travel time values. The analysis shows that the VMTT and VTTV for an individual traveler vary not only between trips on a given day, depending on the purpose and the urgency of preceding and subsequent activities, but also across days, depending on the day-to-day variability of travel times. This highlights the contextual dependency of travel time values. In the process, our aim has been to demonstrate the viability of extending single trip scheduling models into activity-based multi-trip models as an approach to evaluating the costs of travel time variability. In order to apply multi-trip models in practice, the scheduling flexibility of activities and the correlation of trip travel times need to be estimated, which requires more information than single trip-models.

The only decision variables in the present model are the departure times on the trips. In future work it would be valuable to incorporate other choice dimensions such as trip cancelling and route, destination and mode changes. It would also be relevant to consider dependencies between activities (for example, the cancelling of an activity could make a subsequent activity less attractive) and between days (for example, an activity could be postponed to the following day, although with a larger cost of further postponement). Simply put, our directions for further development are much the same as within the field of activity-based modeling in general. Although the rapidly increasing number of decision variables would make the analysis (and estimation) increasingly challenging, the essential features would remain the same. In particular, the concept of the backward optimal marginal cost function, properly generalized, should still be relevant for calculating the VMTT and VTTV.

A The Vickrey (1969) model

This single-trip scheduling model was proposed by Vickrey (1969) and recently analyzed by Fosgerau and Karlström (2010). Here we show that the parameterizations used by these authors are equivalent to the parameterization used in Section 2.3 of this paper. The Small (1982) (α, β, γ) parameterization is

$$C(d) = \alpha T + \beta(d+T)^{-} + \gamma(d+T)^{+}. \tag{36}$$

Using the relationship $(d+T)^{-} = (d+T)^{+} - (d+T)$ and collecting terms gives the Fosgerau and Karlström (2010) parameterization

$$C(d) = -\lambda d + \omega T + \psi(d+T)^{+},$$

where $(\omega, \lambda, \psi) = (\alpha-\beta, \beta, \beta+\gamma)$, as Fosgerau and Karlström (2010) have previously noted. If, instead, we add and subtract the term αd in (36), use the relationship $d+T = (d+T)^{+} - (d+T)^{-}$ again and collect terms, we obtain a parameterization consistent with the general model in (1) which we denote $(c_1, c_2^{\mathrm{E}}, c_2^{\mathrm{L}})$,

$$\begin{aligned} C(d) &= -c_1 d - c_2^{\mathrm{E}}(d+T)^{-} + c_2^{\mathrm{L}}(d+T)^{+} \\ &= -c_1 d + [c_2^{\mathrm{E}} + (c_2^{\mathrm{L}} - c_2^{\mathrm{E}})H(d+T)](d+T). \end{aligned}$$

This parameterization is related to the (α, β, γ) model as $(c_1, c_2^{\mathrm{E}}, c_2^{\mathrm{L}}) = (\alpha, \alpha - \beta, \alpha + \gamma)$, and to the (ω, λ, ψ) model as $(c_1, c_2^{\mathrm{E}}, c_2^{\mathrm{L}}) = (\omega + \lambda, \omega, \omega + \psi)$.

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