ON THE CHROMATIC NUMBER OF GENERALIZED STABLE KNESER GRAPHS

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ABSTRACT. For each integer triple (n, k, s) such that $k \geq 2$, $s \geq 2$, and $n \geq ks$, define a graph in the following manner. The vertex set consists of all k-subsets S of $\mathbb{Z}_n$ such that any two elements in S are on circular distance at least s. Two vertices form an edge if and only if they are disjoint. For the special case $s = 2$, we get Schrijver's stable Kneser graph. The general construction is due to Meunier, who conjectured that the chromatic number of the graph is $n - s(k - 1)$. By a famous result due to Schrijver, the conjecture is known to be true for $s = 2$. The main result of the present paper is that the conjecture is true for $s \geq 4$, provided n is sufficiently large in terms of s and k. The proof techniques do not apply to the case $s = 3$, which remains nearly completely open.

1. INTRODUCTION AND SUMMARY

Let n be an integer, and let $\mathbb{Z}_n$ denote the set of congruence classes of integers modulo n. For an integer x, we let $\overline{x}$ denote the corresponding congruence class modulo n; the modulus n will always be clear from context. For a nonempty subset $S \subseteq \mathbb{Z}_n$ and an integer x such that $\overline{x} \in S$, define $\sigma(x; S)$ to be the smallest $y > x$ such that $\overline{y} \in S$.

Let s be a positive real number. A set $A \subseteq S$ is *s-sparse* in $S \subseteq \mathbb{Z}_n$ if, for each a such that $\overline{a} \in A$ and each $i \geq 1$, we have that

$$\sigma^i(a; A) \geq \sigma^{\lfloor si \rfloor}(a; S). \tag{1}$$

If $S = \mathbb{Z}_n$, then (1) simplifies to

$$\sigma^i(a; A) - a \geq \lfloor si \rfloor . \tag{2}$$

Assuming that s is an integer, and viewing the elements of S as arranged in increasing order around a circle, the set A is s-sparse exactly when there are at least $s - 1$ elements in S between any two elements in A.

For any $S \subseteq \mathbb{Z}_n$ and any real number $s \geq 1$, let $\mathrm{SG}^s_{S,k}$ be the graph in which the vertices are all s-sparse k-subsets of S, and two vertices form an edge if and only if they are disjoint. We write $\mathrm{SG}^s_{n,k} = \mathrm{SG}^s_{\mathbb{Z}_n,k}$. Choosing $s = 1$, we obtain the Kneser graph $\mathrm{KG}_{n,k}$ [3, 4], whereas $s = 2$ yields Schrijver's stable Kneser graph [6].

For a loopless graph G and an integer $r \geq 2$, let $G^{(r)}$ be the r-uniform hypergraph on the same vertex set as G in which the edges are all cliques of size r in G. For an integer $c \geq 1$, a proper c-coloring of a hypergraph H is a function $h : V(H) \to \{1, \ldots, c\}$ such that no edge is monochromatic; $h(\tau)$ consists of at least two elements

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for each edge τ. Let $\chi(H)$ be the smallest c such that there exists a proper c-coloring of H. For a graph G, we write

$$\chi^{(r)}(G) = \chi(G^{(r)}).$$

Meunier suggested the following conjecture, which generalizes several well-known theorems and conjectures.

Conjecture 1.1 (Meunier [5]). *Let k, n, r, s be integers such that $k \geq 2$, $s \geq r \geq 2$, and $n \geq ks$. Then*

$$\chi^{(r)}(\mathrm{SG}^s_{n,k}) = \left\lceil \frac{n - s(k-1)}{r-1} \right\rceil.$$

The conjecture extends a similar conjecture due to Alon, Drewnowski, and Łuczak [1] for the case $r = s$. One may extend the conjecture further by letting s be any real number satisfying $s \geq 2$.

One direction of the conjecture is easy. Specifically, we have the following upper bound.

Proposition 1.2. *Let s be a real number, and let k, n, r be integers such that $k \geq 2$, $s \geq r \geq 2$, and $n \geq ks$. Then*

$$\chi^{(r)}(\mathrm{SG}^s_{n,k}) \leq \left\lceil \frac{n - s(k-1)}{r-1} \right\rceil.$$

Proof. Throughout this proof, identify $\mathbb{Z}_n$ with the set $\{1, \ldots, n\}$. To any vertex S in $\mathrm{SG}^s_{n,k}$, assign the color $\lceil (\min S)/(r-1) \rceil$. Note that any vertex with a given color x must have a nonempty intersection with the set

$$\{k : (x-1)(r-1) < k \leq x(r-1)\}.$$

Since this set has size $r-1$, there is no monochromatic r-clique in $\mathrm{SG}^s_{n,k}$. Choosing $i = k-1$ in (2), we get that no vertex in $\mathrm{SG}^s_{n,k}$ is contained in the set $\{i : n - s(k-1) + 1 \leq i \leq n\}$. As a consequence, we are done. $\square$

Meunier stated and proved Proposition 1.2 in the case that s is an integer; the proof is identical to the one above. Earlier, Alon, Drewnowski, and Łuczak obtained the same bound in the special case $r = s$.

By Proposition 1.2, to obtain Conjecture 1.1 for given values k, n, r, s, it suffices to prove that

$$\chi^{(r)}(\mathrm{SG}^s_{n,k}) \geq \left\lceil \frac{n - s(k-1)}{r-1} \right\rceil.$$

In the present paper, the focus is on ordinary graphs, and the main result reads as follows.

Theorem 1.3. *Let s be a real number, and let t be a positive integer such that $2 \leq t \leq s/2$. Define $d = \lfloor tn/s \rfloor$. Whenever $n \geq ks$, we have that*

$$\frac{\chi(\mathrm{SG}^s_{n,k})}{n} \geq \frac{\chi(\mathrm{SG}^t_{d,k})}{d}.$$

In particular, if $\chi(\mathrm{SG}^t_{d,k}) \geq d - t(k-1)$, then

$$\chi(\mathrm{SG}^s_{n,k}) \geq n - s(k-1) - \frac{tn/s - \lfloor tn/s \rfloor}{\lfloor tn/s \rfloor} s(k-1).$$

See Section 2 for the proof of Theorem 1.3.

It remains an open problem whether it is possible to extend Theorem 1.3 to the case $s/2 < t \leq s$.

Theorem 1.3 does not remain true in general if $t(k-1)$ is a non-integer; see Section 1.2. It is plausible that the theorem does remain true in the case that $t(k-1)$ is an integer, but our proof only works when t itself is an integer. The idea of the proof is to pick an (n/d)-sparse subset $S \subset \mathbb{Z}_n$ of size d and then show that t-sparse subsets of S are s-sparse in $\mathbb{Z}_n$. This is typically false for non-integers t.

Explaining the idea of the proof a bit more, we will consider the n sets $S_g = S-g$, where $g \in \mathbb{Z}_n$. Given an optimal proper coloring of $\mathrm{SG}^s_{n,k}$, we will deduce that each $\mathrm{SG}^t_{S_g,k}$ is a subgraph of $\mathrm{SG}^s_{n,k}$ requiring at least $\chi(\mathrm{SG}^t_{d,k})$ colors. Examining how the sets S_g intersect, we will then observe that each color is used on at most d of the subgraphs $\mathrm{SG}^s_{n,k}$. The conclusion will be that $n\chi(\mathrm{SG}^t_{d,k}) \leq d\chi(\mathrm{SG}^s_{n,k})$.

1.1. **Consequences of the main result.** To describe some of the consequences of Theorem 1.3, let us review what is known about Conjecture 1.1. Using topological methods, Lovász [4] showed that $\chi(\mathrm{KG}_{n,k}) = n-2(k-1)$ whenever $k \geq 2$ and $n \geq 2k$. Shortly after, Schrijver [6] strengthened this result, showing that the chromatic number remains the same for the subgraph $\mathrm{SG}^2_{n,k}$.

Theorem 1.4 (Schrijver [6])**.** *For $k \geq 2$ and $n \geq 2k$, we have that*

$$\chi(\mathrm{SG}^2_{n,k}) = n-2(k-1).$$

Thanks to Theorem 1.4, we may deduce the following from Theorem 1.3.

Corollary 1.5. *Let $q \geq 1$ and $k \geq 2$ be integers. Whenever $n \geq 2^q k$, we have that*

$$\chi(\mathrm{SG}^{2^q}_{n,k}) = n - 2^q(k-1).$$

Proof. Let $q \geq 2$, and assume by induction that $\chi(\mathrm{SG}^{2^{q-1}}_{d,k}) = d - 2^{q-1}(k-1)$ for $d \geq 2^{q-1}k$. Choosing $s = 2^q$ and $t = 2^{q-1}$ in Theorem 1.3, we get that

$$\chi(\mathrm{SG}^{2^q}_{n,k}) \geq n - 2^q(k-1) - \frac{n/2 - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} 2^q(k-1)$$

for $n \geq 2^q k$. Now,

$$\frac{n/2 - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} 2^q(k-1) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{2^q(k-1)}{n-1} & \text{if } n \text{ is odd,} \end{cases}$$

which is strictly less than 1 in both cases; $n \geq 2^q k$. By Proposition 1.2, we are done. □

Corollary 1.6. *Let $s \geq 4$ be a real number, and let $q = \lfloor \log_2(s/2) \rfloor$. Whenever $n \geq ks$ and $k \geq 2$, we have that*

$$\chi(\mathrm{SG}^s_{n,k}) \geq n - s(k-1) - \frac{2^q n/s - \lfloor 2^q n/s \rfloor}{\lfloor 2^q n/s \rfloor} s(k-1).$$

In particular, for each real number $s \geq 4$ and each integer $k \geq 2$, we have that

$$\chi(\mathrm{SG}^s_{n,k}) = \lceil n - s(k-1) \rceil$$

whenever

$$n \geq \max\left\{ \frac{s}{2^q \epsilon}(s(k-1)+\epsilon), ks \right\},$$

where $\epsilon = \lfloor s(k-1)+1 \rfloor - s(k-1)$.

Proof. We obtain the first statement of the corollary by choosing $t = 2^q$ in Theorem 1.3 and applying Corollary 1.5 with $n' = \lfloor 2^q n/s \rfloor$ instead of n. Corollary 1.5 does apply, because

$$n' \geq \lfloor 2^q(ks)/s \rfloor = \lfloor 2^q k \rfloor = 2^q k.$$

For the last statement, note that $\chi(\mathrm{SG}^s_{n,k}) = \lceil n - s(k-1) \rceil$ as soon as

$$\frac{2^q n/s - \lfloor 2^q n/s \rfloor}{\lfloor 2^q n/s \rfloor} s(k-1) < \epsilon. \tag{3}$$

Since $\lfloor 2^q n/s \rfloor > 2^q n/s - 1$, we obtain that (3) is true as soon as

$$\frac{1}{2^q n/s - 1} s(k-1) \leq \epsilon \quad \Longleftrightarrow \quad n \geq \frac{s}{2^q \epsilon}(s(k-1) + \epsilon).$$

This concludes the proof. □

As a side remark, note that $\epsilon = 1$ whenever $s(k-1)$ is an integer.

The restriction $s \geq 4$ in Corollary 1.6 is because of the restriction $t \leq s/2$ in Theorem 1.3. For $2 < s < 4$, very little seems to be known. Using computer, Meunier [5] has established Conjecture 1.1 for $(r, s) = (2, 3)$ in the case that $n \leq 2k + 5$ and also in the case that $(n, k) = (14, 4)$.

1.2. **Some remarks.** Theorem 1.3 does not remain true in general if t is a non-integer. For example, assume that $t > 4$ is a rational number such that $t(k-1)$ is not an integer, and assume that $s(k-1)$ is an integer. Let n have the property that $d = tn/s$ is an integer. Choosing n, and hence d, large enough, Conjecture 1.1 is true for $\mathrm{SG}^t_{d,k}$; apply Corollary 1.6. We get that

$$\begin{aligned}
\frac{\chi(\mathrm{SG}^s_{n,k})}{n} - \frac{\chi(\mathrm{SG}^t_{d,k})}{d} &\leq \frac{n - s(k-1)}{n} - \frac{\lceil d - t(k-1) \rceil}{d} \\
&< \frac{n - s(k-1)}{n} - \frac{d - t(k-1)}{d} \\
&= \frac{(k-1)(nt - ds)}{dn} = 0;
\end{aligned}$$

the second inequality is strict, as we assumed that $t(k-1)$ is not an integer.

Conjecture 1.1 (extended to real s) implies Theorem 1.3 for any $s \geq t \geq 2$ such that $t(k-1)$ is an integer. Namely, assuming the conjecture is true for $\mathrm{SG}^s_{n,k}$, we get that

$$\frac{\chi(\mathrm{SG}^s_{n,k})}{n} - \frac{\chi(\mathrm{SG}^t_{d,k})}{d} \geq \frac{n - s(k-1)}{n} - \frac{d - t(k-1)}{d} = \frac{(k-1)(nt - ds)}{dn} \geq 0.$$

1.3. **Hypergraphs.** Let us also review the situation for hypergraphs. Throughout this section, all parameters are integers. Again using topological methods, Alon, Frankl, and Lovász [2] extended the result of Lovász [4] to Kneser hypergraphs, proving that

$$\chi^{(r)}(\mathrm{KG}_{n,k}) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil$$

whenever $k \geq 2$, $r \geq 2$, and $n \geq kr$. Regarding stable Kneser hypergraphs, Alon, Drewnowski, and Łuczak [1] managed to settle Conjecture 1.1 in the particular case that $r = s = 2^q$ for some integer q.

Theorem 1.7 (Alon, Drewnowski, and Łuczak [1]). *Let $q \geq 1$ and $k \geq 2$. For every $n \geq 2^q k$, we have that*

$$\chi^{(2^q)}(\mathrm{SG}^{2^q}_{n,k}) = \left\lceil \frac{n - 2^q(k-1)}{2^q - 1} \right\rceil.$$

Meunier [5] proved the following result, extending a similar result due to Alon, Drewnowski, and Łuczak [1].

Theorem 1.8 (Meunier [5]). *Suppose that the following hold for given parameters $r_1 \geq 2$ and $s_2 \geq r_2 \geq 2$.*

- *Conjecture 1.1 is true for $(r,s) = (r_1, r_1)$ and all n and k such that $n \geq kr_1$ and $k \geq 2$.*
- *Conjecture 1.1 is true for $(r,s) = (r_2, s_2)$ and all n and k such that $n \geq ks_2$ and $k \geq 2$.*

Then Conjecture 1.1 is true for $(r,s) = (r_1 r_2, r_1 s_2)$ and all n and k such that $n \geq ks$ and $k \geq 2$.

The following result is a consequence of Corollary 1.5, Theorem 1.7, and Theorem 1.8.

Corollary 1.9. *Let p and q be any positive integers such that $p \leq q$, and let $k \geq 2$. Then*

$$\chi^{(2^p)}(\mathrm{SG}^{2^q}_{n,k}) = \left\lceil \frac{n - 2^q(k-1)}{2^p - 1} \right\rceil.$$

for $n \geq 2^q k$.

Proof. The case $p = 1$ is Corollary 1.5. For $p \geq 2$, use Theorem 1.8 with $r_1 = 2^{p-1}$ and $(r_2, s_2) = (2, 2^{q-p+1})$; apply Theorem 1.7 and Corollary 1.5. □

2. Proof of Theorem 1.3

Let $s \geq 4$ be a real number, and let t be an integer such that $2 \leq t \leq s/2$. For a given integer n, write $d = \lfloor tn/s \rfloor$. Let

$$S = \left\{ \overline{\lfloor in/d \rfloor} : 0 \leq i \leq d-1 \right\} \subseteq \mathbb{Z}_n.$$

Lemma 2.1. *If a set $A \subseteq S$ is t-sparse in S, then A is s-sparse in $\mathbb{Z}_n$. In particular, $\mathrm{SG}^s_{n,k}$ contains $\mathrm{SG}^t_{S,k}$ as a subgraph.*

Proof. Assume that A is t-sparse in S. Consider an element $a_0 \in \mathbb{Z}$ such that $\overline{a_0} \in A$. For $m \geq 1$, write $a_m = \sigma^m(a_0; A)$. We want to prove that

$$a_m - a_0 \geq \lfloor ms \rfloor.$$

We have that $a_m = \sigma^\ell(a_0; S)$ for some $\ell > 0$. By assumption,

$$\ell \geq mt.$$

Now, $a_0 = \lfloor in/d \rfloor$ for some i, and $a_m = \lfloor jn/d \rfloor$ for some j, which means that $\ell = j - i$. We get that

$$a_m - a_0 = \left\lfloor \frac{jn}{d} \right\rfloor - \left\lfloor \frac{in}{d} \right\rfloor \geq \left\lfloor \frac{jn}{d} - \frac{in}{d} \right\rfloor = \left\lfloor \frac{\ell n}{d} \right\rfloor \geq \left\lfloor \frac{mtn}{tn/s} \right\rfloor = \lfloor ms \rfloor,$$

which concludes the proof. □

Let $S_g = S - g$ for $g \in \mathbb{Z}_n$. Define a graph $G(n,S)$ with vertex set $\mathbb{Z}_n$ and with an edge between g and h whenever $S_g \cap S_h = \emptyset$. While not useful in what follows, one may observe that $G(n,S) \cong \mathrm{SG}^{n/d}_{n,d}$; this is because S and its translates are the only (n/d)-sparse d-subsets of $\mathbb{Z}_n$.

A vertex set I in a graph G is *independent* if no two vertices in I are adjacent in G. The *independence number* of G is the greatest value a such that there is an independent set in G of size a.

Lemma 2.2. *The independence number of $G(n,S)$ is d.*

Proof. Let I be an independent set in $G(n,S)$. By cyclic symmetry, we may assume that $\overline{0} \in I$. Suppose that $\overline{p} \in I$. Then $S_{\overline{0}} \cap S_{\overline{p}} \neq \emptyset$, which is true if and only if there are i and j such that

$$\lfloor in/d \rfloor = \lfloor jn/d \rfloor - p \quad \Longleftrightarrow \quad p = \lfloor jn/d \rfloor - \lfloor in/d \rfloor.$$

This implies that

$$p = \lfloor (j-i)n/d \rfloor \quad \text{or} \quad p = \lfloor (j-i)n/d \rfloor + 1.$$

We conclude that

$$I \subseteq \{g, g + \overline{1} : g \in S\}.$$

Now, $n/d \geq s/t \geq 2$, which implies that $S_g \cap S_{g+\overline{1}} = \emptyset$. In particular, at most one of g and $g + \overline{1}$ belongs to I for each $g \in S$; hence $|I| \leq |S| = d$.

To see that the independence number is d, note that the set S forms an independent set in $G(n,S)$; S_g contains the element $\overline{0}$ for each $g \in S$. $\square$

Proof of Theorem 1.3. Consider a proper coloring of $\mathrm{SG}^s_{n,k}$ with $\chi(\mathrm{SG}^s_{n,k})$ colors. For each color i, let C_i denote the set of elements $g \in \mathbb{Z}_n$ such that some vertex of $\mathrm{SG}^t_{S_g,k}$ is given the color i. By Lemma 2.1, $\mathrm{SG}^t_{S_g,k}$ is contained in $\mathrm{SG}^s_{n,k}$, which implies that

$$\sum_i |C_i| \geq \sum_{g \in \mathbb{Z}_n} \chi(\mathrm{SG}^t_{S_g,k}) = n \cdot \chi(\mathrm{SG}^t_{S,k}). \tag{4}$$

Now, each C_i is an independent set in the graph $G(n,S)$, because the coloring is proper. By Lemma 2.2, we get that $|C_i| \leq d$, which yields that

$$\sum_i |C_i| \leq d \cdot \chi(\mathrm{SG}^s_{n,k}). \tag{5}$$

Combining (4) and (5), we obtain the theorem. $\square$

2.1. **Remark.** In the proof of Theorem 1.3, we considered the n subgraphs $\mathrm{SG}^t_{S_g,k}$ of the graph $\mathrm{SG}^s_{n,k}$. We defined a graph $G(n,S)$ and observed that

$$\sum_{g \in \mathbb{Z}_n} \chi(\mathrm{SG}^t_{S_g,k}) \leq \alpha(G(n,S)) \cdot \chi(\mathrm{SG}^s_{n,k}),$$

where $\alpha(G(n,S))$ denotes the independence number of $G(n,S)$. This is a special case of a more general fact, which we state for completeness.

Let $H = (V,E)$ be a graph, and let $V_1, \ldots, V_n$ be subsets of V. Define a graph G with vertex set $\{1, \ldots, n\}$ and with an edge between i and j whenever the complete bipartite graph with blocks V_i and V_j is a subgraph of H. Then

$$\sum_{i=1}^{n} \chi(H[V_i]) \leq \alpha(G) \cdot \chi(H),$$

where $H[V_i]$ is the induced subgraph of H on the vertex set V_i.

To see this, consider an optimal coloring of H. For each color i, let C_i be the set of indices j such that some vertex in V_j is given the color i. As in the proof of Theorem 1.3, we deduce that

$$\sum_i |C_i| \geq n \cdot \chi(H[V_i]).$$

Moreover, each C_i is an independent set in G. Namely, if a and b are adjacent in G, then all $x \in V_a$ are adjacent to all $y \in V_b$. We conclude that

$$\sum_i |C_i| \leq \alpha(G) \cdot \chi(H).$$

References

[1] N. Alon, L. Drewnowski, and T Łuczak, Stable Kneser hypergraphs and ideals in ℕ with the Nikodým property, Proc. Amer. Math. Soc. **137** (2009), 467–471.

[2] N. Alon, P. Frankl, and L. Lovasz, The chromatic number of Kneser hypergraphs, Trans. Amer. Math. Soc. **298** (1986), 359–370.

[3] M. Kneser, Aufgabe 360, *Jahresber. Deutsch. Math.-Verein.* **58** (1955), 2. Abteilung, 27.

[4] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, *J. Combinatorial Theory* **25** (1978), 319–324.

[5] F. Meunier, The chromatic number of almost stable Kneser hypergraphs, *J. Combin. Theory Ser. A*, Accepted.

[6] A. Schrijver, Vertex-critical subgraphs of Kneser graphs, *Nieuw Arch. Wiskd. (3)* **26** (1978), no. 3, 454–461.

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