A refinement of amplitude homology and a generalization of discrete Morse theory

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Abstract

Let $n \geq 2$. A chain complex with boundary map δ has the property that $\delta^2 = 0$. Kapranov introduced the concept of an n-complex, in which we instead have that $\delta^n = 0$. Kapranov also generalized the concept of homology to n-complexes, introducing $n-1$ generalized homology groups, where the kth group is defined as the quotient $\ker \delta^k / \mathrm{im}\, \delta^{n-k}$. One goal of the present paper is to introduce $n-1$ new groups that we refer to as train groups. The train groups are closely related to generalized homology groups; there exists a filtration of each generalized homology group such that each quotient arising from the filtration is a direct sum of train groups. In particular, one may view the train groups as refinements of the generalized homology groups. Another goal is to generalize the algebraic version of Forman's discrete Morse theory to n-complexes. We will then make use of the theory of generalized homotopies introduced by Kapranov and further developed by Kassel and Wambst and by Dubois-Violette.

1 Introduction

Throughout this paper, R is a commutative ring with unity. Let C be an R-module, and let $\delta : C \to C$ be an endomorphism; we write $\mathcal{C} = (C, \delta)$. Assuming $C = \bigoplus_{i \in \mathbb{Z}} C_i$, $\delta(C_i) \subseteq C_{i-1}$, and $\delta^2 = 0$, recall that the homology in degree i of the chain complex

$$\mathcal{C} : \cdots \xrightarrow{\delta} C_{i+1} \xrightarrow{\delta} C_i \xrightarrow{\delta} C_{i-1} \xrightarrow{\delta} C_{i-2} \xrightarrow{\delta} \cdots$$

is defined as the quotient group

$$H_i(\mathcal{C}) = \frac{C_i \cap \ker \delta}{C_i \cap \mathrm{im}\, \delta}.$$

Replacing the assumption that $\delta^2 = 0$ with the assumption that $\delta^n = 0$ for a given fixed $n \geq 2$, Kapranov [10] introduced a sequence of generalized homology

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groups by

$$H_i^{(k)}(\mathcal{C}) = \frac{C_i \cap \ker \delta^k}{C_i \cap \operatorname{im} \delta^{n-k}}$$

for $1 \leq k \leq n-1$. A special case of this construction appeared already in the work of Mayer [13, 14] in the context of simplicial homology. Following Cibils, Solotar and Wisbauer [2], we refer to Kapranov's generalized homology groups as *amplitude homology groups*. Note that $H_i^{(k)}(\mathcal{C})$ depends on n, but we will always fix n whenever we discuss this group. In situations where the grading is unimportant, we will consider the full group $H^{(k)}(\mathcal{C}) = \ker \delta^k / \operatorname{im} \delta^{n-k}$. The pair (C, δ) is often referred to as an *n-complex*.

The goal of this paper is twofold.

- The first goal is to introduce a refinement of Kapranov's amplitude homology groups; we will refer to the refined groups as *train groups*. In the case of a finite-dimensional vector space, the dimension of the ith train group is equal to the number of i-dimensional summands of the cyclic decomposition of δ. In particular, the concept is well-studied in this particular case.
- The second goal is to extend Forman's discrete Morse theory [6] to n-complexes. The idea of discrete Morse theory is to simplify homology computations by transforming a chain complex into a "smaller" chain complex with isomorphic homology. Using the extended version, one may transform an n-complex in a similar way such that the amplitude homology groups and the train groups remain unmodified up to isomorphism.

1.1 Train groups

To explain the first goal in more detail, we first assume that C is a finite-dimensional vector space. It is a straightforward exercise in linear algebra that we may decompose C into a direct sum of subspaces $A_1, \ldots, A_r$, where each A_j has a basis $e_1, \ldots, e_{\rho_j}$ such that $\delta(e_i) = e_{i-1}$ for $2 \leq i \leq \rho_j$, and $\delta(e_1) = 0$. For $1 \leq k \leq n$, let q_k be the number of subspaces A_j in the decomposition such that the dimension ρ_j equals k. The numbers q_k are important invariants of $\mathcal{C}$. For $1 \leq k \leq n-1$, let r_k be the dimension of $H^{(k)}(\mathcal{C})$. Dubois-Violette [3, Prop. 2] observed that

$$r_k = r_{n-k} = \sum_{p=1}^{n-1} \min\{p, n-p, k, n-k\} q_p. \tag{1}$$

We now introduce the refinement, which is valid for any endomorphism $\delta : C \to C$, not only nilpotent ones. For $k \geq 1$, we define a *k-train* to be a sequence $\tau_k(x) = (x, \delta(x), \delta^2(x), \ldots, \delta^{k-1}(x))$ such that $x \in \ker \delta^k$. We say that two k-trains $\tau_k(x)$ and $\tau_k(y)$ are *congruent* if $\delta^{k-1}(x-y) \in \operatorname{im} \delta^k$. Equivalently, there is an element $z \in \ker \delta^k \cap \operatorname{im} \delta$ such that the k-train $\tau_k(x-y-z)$ ends with the zero element. Hence $\tau_k(x)$ and $\tau_k(y)$ are congruent if and only if

$$x - y \in \ker \delta^{k-1} + \ker \delta^k \cap \operatorname{im} \delta \Longleftrightarrow \delta^{k-1}(x-y) \in \ker \delta \cap \operatorname{im} \delta^k.$$

We define $T^{(k)}(\mathcal{C})$ to be the R-module of congruence classes of k-trains. We refer to $T^{(k)}(\mathcal{C})$ as the kth *train group* of $\mathcal{C}$. By the above discussion, we have that

$$T^{(k)}(\mathcal{C}) \cong \frac{\ker \delta^k}{\ker \delta^{k-1} + \ker \delta^k \cap \operatorname{im} \delta} \cong \frac{\ker \delta \cap \operatorname{im} \delta^{k-1}}{\ker \delta \cap \operatorname{im} \delta^k}.$$

In particular, the train groups generalize the concept of homology.

To see the connection to Jordan decompositions in the case that C is a vector space, note that the dimension q_k in (1) equals $\dim T^{(k)}(\mathcal{C})$. Namely, for any A_j, the basis element e_i belongs to $\ker \delta^{k-1}$ whenever $i < k$ and lies outside $\ker \delta^k$ whenever $i > k$. Since e_k belongs to $\ker \delta^k \cap \operatorname{im} \delta$ if and only if $k < \rho_j$, the claim follows. From the perspective of Jordan decompositions, the train groups thereby capture all essential information about δ.

1.2 Amplitude homology groups versus train groups

From the viewpoint of homological algebra, trains groups are less natural than Kapranov's amplitude homology groups. What makes the groups $H_i^{(k)}(\mathcal{C})$ particularly attractive is that they admit an interpretation as homology groups. Specifically, the groups $H_*^{(k)}(\mathcal{C})$ and $H_*^{(n-k)}(\mathcal{C})$ appear as the homology of the chain complexes

$$\cdots \xrightarrow{\delta^k} C_{i+n-k} \xrightarrow{\delta^{n-k}} C_i \xrightarrow{\delta^k} C_{i-k} \xrightarrow{\delta^{n-k}} C_{i-n} \xrightarrow{\delta^k} \cdots$$

for $1 \le i \le n$. In particular, much of the existing homology theory is straightforward to adapt to $H_*^{(k)}(\mathcal{C})$; see Kapranov [10] and Dubois-Violette [3]. While the groups $T^{(k)}(\mathcal{C})$ benefit from their natural interpretation in the language of Jordan decompositions, they are still problematic in that they might be difficult to analyze using tools from homological algebra; there seems to be no straightforward representation of $T^{(k)}(\mathcal{C})$ as a quotient of a kernel by an image.

One reason for studying train groups is that they often provide more specific and detailed information about $\mathcal{C}$ than the amplitude homology groups, as illustrated by the formula (1). In Section 2.4, we will establish a decomposition result for the amplitude homology groups. When C is a vector space, (1) yields an isomorphism

$$H^{(k)}(\mathcal{C}) \cong \bigoplus_{p=1}^{n-1} \left(T^{(p)}(\mathcal{C})\right)^{t_p},$$

where $t_p = \min\{p, n-p, k, n-k\}$. For a general R-module C, the relationship between amplitude homology groups and train groups is given by $n-1$ short exact sequences, and we obtain an isomorphism as above whenever the exact sequences all split. See Theorem 2.4 for details.

1.3 Generalized discrete Morse theory

Discrete Morse theory [6], which is more a method than a theory, is a very useful technique for computing the homology (or cohomology) of a combinatorially defined chain (or cochain) complex such as a simplicial chain complex. The core of method is roughly to strip off pieces from the complex such that remaining part has the same homology as the original complex. We refer the reader to the references for more details [6, 1, 8, 9, 12, 15, 16].

In Section 5, we demonstrate that a fairly straightforward adaptation of this technique applies to n-complexes. Dubois-Violette [3] introduced the concept of homotopies of n-complexes, and we provide a generalization of this concept that we refer to as pseudo-homotopies; see Section 4. Adapting the algebraic discrete Morse theory developed by Jöllenbeck and Welker [8] and Sköldberg [16] (and using some own ideas [9]), we use pseudo-homotopies to generalize discrete Morse theory to n-complexes. Roughly speaking, we show how to strip off pieces from C to get a δ-invariant submodule U such that the train groups (as well as the amplitude homology groups) associated to $\mathcal{C}$ are immediately deducible from those associated to (U, δ). When C is a vector space, the removed pieces are subspaces of the form A_j as described in Section 1.1.

In Section 7, we illustrate the technique with examples. Some of the examples make use of a "divide and conquer" technique, which we describe in Section 6.

1.4 Remarks

In the case that C is not a vector space, it is not entirely correct to refer to train groups as a refinement of amplitude homology groups. Indeed, two n-complexes might have non-isomorphic amplitude homology groups and isomorphic train groups. For example, let $\mathcal{C} = (\mathbb{Z}^6, \delta_1)$ and $\mathcal{E} = (\mathbb{Z}^6, \delta_2)$ be 3-complexes, where the matrices of δ_1 and δ_2 in the standard basis are

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively. Then

$$T^{(k)}(\mathcal{C}) \cong T^{(k)}(\mathcal{E}) \cong \mathbb{Z}/2\mathbb{Z}$$

for $k \in \{1, 2\}$, whereas

$$H^{(1)}(\mathcal{C}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \neq \mathbb{Z}/4\mathbb{Z} \cong H^{(1)}(\mathcal{E}).$$

2 Amplitude homology and train groups

Fix an integer $n \geq 2$ and a commutative ring R. Let G be either $\mathbb{Z}$ or $\mathbb{Z}_m$ for some $m \geq 1$, and let $C = \bigoplus_{i \in G} C_i$ be a G-graded R-module. Let $\delta : C \to C$ be an endomorphism satisfying $\delta^n = 0$ and $\delta(C_i) \subseteq C_{i-1}$ for all $i \in G$. The pair $\mathcal{C} = (C, \delta)$ is an *n-complex.*

2.1 Amplitude homology groups

The *kth amplitude homology group* of $\mathcal{C}$ in degree i is defined by

$$H_i^{(k)}(\mathcal{C}) = \frac{C_i \cap \ker \delta^k}{C_i \cap \operatorname{im} \delta^{n-k}}$$

for $1 \leq k \leq n-1$ and $i \in G$. For convenience, we extend the definition to include the cases $k = 0$ and $k = n$. Note that $H_i^{(0)}(\mathcal{C}) = H_i^{(n)}(\mathcal{C}) = 0$ for all $i \in G$. Unless the grading is of importance, we will typically speak of the full amplitude homology group

$$H^{(k)}(\mathcal{C}) = \frac{\ker \delta^k}{\operatorname{im} \delta^{n-k}} \cong \bigoplus_{i \in G} H_i^{(k)}(\mathcal{C}).$$

Write $H^{(k)} = H^{(k)}(\mathcal{C})$ and $H_i^{(k)} = H_i^{(k)}(\mathcal{C})$. Let us recall some concepts and results due to Kapranov [10] and Dubois-Violette and Kerner [4]. Kapranov noted that the inclusion map from $\ker \delta^k$ to $\ker \delta^{k+1}$ induces a homomorphism

$$u_i^{(k)} : H_i^{(k)} \to H_i^{(k+1)}$$

for each $i \in G$. Moreover, δ itself induces a map

$$d_i^{(k)} : H_i^{(k)} \to H_{i-1}^{(k-1)}$$

for each $i \in G$. The map $u_i^{(k)}$ is well-defined for $0 \leq k \leq n-1$, whereas the map $d_i^{(k)}$ is well-defined for $1 \leq k \leq n$. Note that $u_i^{(0)}$, $u_i^{(n-1)}$, $d_i^{(1)}$, and $d_i^{(n)}$ are zero maps. One easily checks that $u_{i-1}^{(k-1)} d_i^{(k)} = d_i^{(k+1)} u_i^{(k)}$ for $1 \leq k \leq n-1$ and $i \in G$.

For simplicity, we will typically write $u = u_i^{(k)}$ and $d = d_i^{(k)}$. In situations where k or i are of importance, their value will be clear from context. For $i \in G$ and $0 \leq r \leq k \leq n$, we have that d^r defines a map from $H_i^{(k)}$ to $H_{i-r}^{(k-r)}$, and u^r defines a map from $H_i^{(n-k)}$ to $H_i^{(n-k+r)}$. Dubois-Violette and Kerner [4] proved the following important result, which relates the different amplitude homology groups via the maps d and u.

Proposition 2.1 *For each $0 \leq r \leq k \leq n$, we have the following exact hexagon:*

$$\begin{tikzcd} H^{(r)} \arrow[r, "u^{k-r}"] & H^{(k)} \arrow[r, "d^r"] & H^{(k-r)} \arrow[d, "u^{n-k}"] \\ H^{(n-k+r)} \arrow[u, "d^{n-k}"] & H^{(n-k)} \arrow[l, "u^r"'] & H^{(n-r)} \arrow[l, "d^{k-r}"'] \end{tikzcd} \tag{2}$$

With the degrees written out, this means that the sequences

$$H_i^{(a)} \xrightarrow{u^b} H_i^{(a+b)} \xrightarrow{d^a} H_{i-a}^{(b)}$$
$$H_{i+b}^{(n-a)} \xrightarrow{d^b} H_i^{(n-a-b)} \xrightarrow{u^a} H_i^{(n-b)}$$

are exact for any $a \geq 0$ and $b \geq 0$ satisfying $a + b \leq n$.

Disregarding the grading, note the perfect symmetry between d and u. Specifically, the hexagon (2) is preserved under the transformation given by replacing $H^{(p)}$ with $H^{(n-p)}$ for each p and swapping d and u.

2.2 Train groups

The *kth train group* of $\mathcal{C}$ in degree i is defined by

$$T_i^{(k)}(\mathcal{C}) = \frac{C_i \cap \ker \delta^k}{C_i \cap (\ker \delta^{k-1} + \ker \delta^k \cap \operatorname{im} \delta)}.$$

We write $T^{(k)}(\mathcal{C}) = \bigoplus_{i \in G} T_i^{(k)}(\mathcal{C})$.

While maybe not obvious from their definition, the train groups align with the symmetry between d and u. To see this, note that

$$\begin{aligned} \frac{\ker \delta^k}{\ker \delta^{k-1} + \ker \delta^k \cap \operatorname{im} \delta} &\cong \frac{\ker \delta^k / \operatorname{im} \delta^{n-k}}{(\ker \delta^{k-1} + \ker \delta^k \cap \operatorname{im} \delta)/\operatorname{im} \delta^{n-k}} \\ &\cong \frac{H^{(k)}(\mathcal{C})}{H^{(k)}(\mathcal{C}) \cap (\ker d^{k-1} + \operatorname{im} d)}. \end{aligned}$$

Most importantly, the train groups can be defined directly in terms of the amplitude homology. With $r = k - 1$ in Proposition 2.1, we obtain that $H^{(k)} \cap \ker d^{k-1} = H^{(k)} \cap \operatorname{im} u$; hence

$$T^{(k)}(\mathcal{C}) \cong \frac{H^{(k)}(\mathcal{C})}{H^{(k)}(\mathcal{C}) \cap (\operatorname{im} u + \operatorname{im} d)}.$$

2.3 Amplitude systems

By the discussion in the preceding section, we may define the train groups in terms of amplitude homology without any reference to the underlying n-complex. Moreover, it turns out that many results about amplitude homology and train groups only rely on Proposition 2.1, not on any additional properties of the n-complex. For this reason, we now provide an intrinsic treatment of amplitude homology groups.

Specifically, an *amplitude system* $\mathcal{H} = (H^{(*)}, d, u)$ is a sequence

$$(0 = H^{(0)}, H^{(1)}, \ldots, H^{(n-1)}, H^{(n)} = 0)$$

of R-modules called *amplitude groups* along with commuting maps

$$0 \underset{d}{\overset{u}{\rightleftarrows}} H^{(1)} \underset{d}{\overset{u}{\rightleftarrows}} H^{(2)} \underset{d}{\overset{u}{\rightleftarrows}} \cdots \underset{d}{\overset{u}{\rightleftarrows}} H^{(n-1)} \underset{d}{\overset{u}{\rightleftarrows}} 0$$

such that (2) is exact for each $0 \leq r < k \leq n$. Equivalently,

$$\begin{aligned} \ker d^r \cap H^{(k)} &= \operatorname{im} u^{k-r} \cap H^{(k)} \\ \ker u^r \cap H^{(n-k)} &= \operatorname{im} d^{k-r} \cap H^{(n-k)} \end{aligned}$$

for each $0 \leq r < k \leq n$. As before, we assume that each group $H^{(k)}$ admits a grading $H^{(k)} = \bigoplus_{i \in G} H_i^{(k)}$ and that $d(H_i^{(k)}) \subseteq H_{i-1}^{(k-1)}$ and $u(H_i^{(k)}) \subseteq H_i^{(k+1)}$. For convenience, we define $H_i^{(k)} = 0$ for $k < 0$ and $k > n$, and we extend u and d to all $H_i^{(k)}$ in the obvious manner.

Any n-complex $\mathcal{C} = (C, \delta)$ gives rise to an amplitude system as discussed in Section 2.1. We refer to this system as the amplitude system *induced* by $\mathcal{C}$. For a given amplitude system $\mathcal{H}$, one may ask whether there is some n-complex inducing $\mathcal{H}$. In Section 2.6, we will see that this is not always the case.

For $i \in G$ and $1 \leq k \leq n-1$, define

$$T_i^{(k)}(\mathcal{H}) = \frac{H_i^{(k)}}{H_i^{(k)} \cap (\operatorname{im} u + \operatorname{im} d)};$$

as before, we refer to this group as the kth train group of degree i. Our first result shows that we may identify $T_i^{(k)}(\mathcal{H})$ with a subgroup of $H_{i-k+1}^{(n-k)}$. In the next section, we will prove a much more general result.

Theorem 2.2 *For $1 \leq k \leq n-1$ and $i \in G$, the map $\psi_i^{(k)} : H_i^{(k)} \to H_{i-k+1}^{(n-k)}$ defined by $\psi_i^{(k)} = d^{k-1}u^{n-k-1}$ induces an isomorphism between $T_i^{(k)}(\mathcal{H})$ and $\operatorname{im} \psi_i^{(k)} \subseteq H_{i-k+1}^{(n-k)}$.*

Proof. First, note that if $x = d(a) + u(b) \in H_i^{(k)}$, then

$$\psi_i^{(k)}(x) = d^k u^{n-k-1}(a) + u^{n-k} d^{k-1}(b) = 0,$$

which implies that $\psi_i^{(k)}$ indeed induces a map $\hat{\psi}_i^{(k)} : T_i^{(k)} \to H_{i-k+1}^{(n-k)}$. It remains to show that this induced map is a monomorphism. For this, consider an element $x \in H_i^{(k)}$ such that $\psi_i^{(k)}(x) = 0$. This means that

$$d^{k-1}(x) \in H_{i-k+1}^{(1)} \cap \ker u^{n-k-1} = H_{i-k+1}^{(1)} \cap \operatorname{im} d^k;$$

hence $d^{k-1}(x) = d^k(y)$ for some $y \in H_{i+1}^{(k+1)}$. Since $d^{k-1}(x - d(y)) = 0$, we have that

$$x - d(y) \in H_i^{(k)} \cap \ker d^{k-1} = H_i^{(k)} \cap \operatorname{im} u,$$

which implies that $x \in \operatorname{im} d + \operatorname{im} u$ as desired. $\square$

Corollary 2.3 *For $1 \leq k \leq n-1$ and $i \in G$, we have that*

$$T_i^{(k)} \cong H_{i-k+1}^{(n-k)} \cap \ker d \cap \ker u.$$

Proof. Since

$$\begin{aligned} & H^{(n-k)} \cap \operatorname{im} d^{k-1}u^{n-k-1} = H^{(n-k)} \cap d^{k-1}(\ker d^k) \\ = \quad & H^{(n-k)} \cap \ker d \cap \operatorname{im} d^{k-1} = H^{(n-k)} \cap \ker d \cap \ker u, \end{aligned}$$

the isomorphism is a consequence of Theorem 2.2. $\square$

2.4 Decomposing amplitude groups

The main goal of this section is to generalize Theorem 2.2. More precisely, we show that each graded amplitude group $H_i^{(k)}$ admits a filtration

$$0 = H_i^{(k,n-1)} \subseteq H_i^{(k,n-2)} \subseteq \cdots \subseteq H_i^{(k,2)} \subseteq H_i^{(k,1)} \subseteq H_i^{(k,0)} = H_i^{(k)} \tag{3}$$

such that each quotient $H^{(k,m)}/H^{(k,m+1)}$ decomposes into a direct sum of train groups. This result provides a refinement of Dubois-Violette's formula (1).

For $i \in G$, $1 \le k \le n-1$, and $0 \le m \le n-1$, define

$$H_i^{(k,m)} = H_i^{(k)} \cap \left(\operatorname{im} d^m + \operatorname{im} d^{m-1}u + \cdots + \operatorname{im} du^{m-1} + \operatorname{im} u^m\right)$$

This yields the filtration (3). To see that $H^{(k,n-1)} = 0$, suppose that $x \in H^{(k-2r+n-1)}$, and consider $d^{n-1-r}u^r(x)$; this is an element in $H^{(k)}$. For the element $d^{n-1-r}(x)$ to be nonzero, it is necessary that $k > r$, because $d^{n-1-r}(x)$ lies in $H^{(k-r)}$. Yet, $u^r(x)$ lies in $H^{k-r+n-1}$; hence $u^r(x) = 0$ if $k > r$.

Let $i \in G$ and $1 \le k \le n-1$. For nonnegative integers q and r, consider the map

$$d^q u^r : H_{i+q}^{(k+q-r)} \to H_i^{(k)}.$$

Note that this map is zero if $r \ge k$ or $q \ge n-k$. Namely, $u^r(H^{(k+q-r)}) \subseteq H^{(k+q)}$, which is zero if $q \ge n-k$, and $d^q(H^{(k+q-r)}) \subseteq H^{(k-r)}$, which is zero if $r \ge k$. Moreover, $d^q u^r$ induces a map from $T_{i+q}^{(k+q-r)}$ to $H_i^{(k,q+r)}/H_i^{(k,q+r+1)}$. Namely,

$$d^q u^r(d(a) + u(b)) = d^{q+1}u^r(a) + d^q u^{r+1}(b) \in H^{(k,q+r+1)}$$

for $a \in H^{(k+q-r+1)}$ and $b \in H^{(k+q-r-1)}$, and

$$d^q u^r(H^{(k+q-r)}) = H^{(k)} \cap \operatorname{im} d^q u^r \subseteq H^{(k,q+r)}.$$

Let $I_{k,m}$ be the set of elements $0 \le q \le n-k-1$ such that $0 \le m-q \le k-1$. Let

$$\gamma_i^{(k,m)} : \bigoplus_{q \in I_{k,m}} H_{i+q}^{(k+q-(m-q))} \to H_i^{(k,m)}$$

be the map whose restriction to $H_{i+q}^{(k+q-(m-q))}$ is $d^q u^{m-q}$.

Theorem 2.4 *For $i \in G$, $1 \le k \le n-1$, and $0 \le m \le n-2$, we have that $\gamma_i^{(k,m)}$ induces an isomorphism*

$$\hat{\gamma}_i^{(k,m)} : \bigoplus_{q \in I_{k,m}} T_{i+q}^{(k+q-(m-q))} \to \frac{H_i^{(k,m)}}{H_i^{(k,m+1)}}. \tag{4}$$

Remark. Summing over m and i, note that the total number of occurrences of $T^{(p)}$ in (4) is

$$\begin{cases} p & \text{if } 1 \le p \le \min\{k, n-k\}, \\ \min\{k, n-k\} & \text{if } \min\{k, n-k\} \le p \le \max\{k, n-k\}, \\ n-p & \text{if } \max\{k, n-k\} \le p \le n-1. \end{cases}$$

	1	2	3	4	5	6
i	$(0,2)$	$(0,1)$	$(0,0)$			
$i+1$		$(1,2)$	$(1,1)$	$(1,0)$		
$i+2$			$(2,2)$	$(2,1)$	$(2,0)$	
$i+3$				$(3,2)$	$(3,1)$	$(3,0)$

Table 1: For $n = 7$ and $k = 3$, there is an entry (q, r) in row $i + q$ and column $k + q - r$ whenever $d^q u^r$ induces a monomorphism from $T^{(k+q-r)}_{i+q}$ to $H^{(k,q+r)}_i / H^{(k,q+r+1)}_i$.

In particular, the total number of occurrences of $T^{(p)}$ in $H^{(k)}$ is $\min\{p, n - p, k, n - k\}$, the quantity in (1). See Table 2.4 for an illustration.

Proof. By the discussion in this section, $\hat{\gamma}^{(k,m)}$ is a well-defined epimorphism.

It remains to show that $\hat{\gamma}^{(k,m)}$ is injective. Consider an element

$$a = \sum_{q \in I_{k,m}} x_{k+2q-m} \in \bigoplus_{q \in I_{k,m}} H^{(k+2q-m)}$$

such that

$$b = \gamma^{(k,m)}(a) = \sum_{q \in I_{k,m}} d^q u^{m-q}(x_{k+2q-m}) \in H^{(k,m+1)}.$$

We need to prove that $x_{k+2q-m} \in \operatorname{im} d + \operatorname{im} u$ for each $q \in I_{k,m}$.

For $q' \in I_{k,m}$, observe that

$$d^{k-(m-q)-1} d^{q'}(x_{k+2q'-m}) \in H^{(q'-q+1)}$$

and

$$u^{n-k-q-1} u^{m-q'}(x_{k+2q'-m}) \in H^{(n+(q'-q)-1)}.$$

The first group is zero if $q' - q < 0$, whereas the second group is zero if $q' - q > 0$. In particular,

$$d^{k-(m-q)-1} u^{n-k-q-1} d^{q'} u^{m-q'}(x_{k+2q'-m}) = 0$$

whenever $q \neq q'$. Writing $\ell = k + 2q - m$, we conclude that

$$\begin{aligned} c &= d^{k-(m-q)-1} u^{n-k-q-1}(b) \\ &= d^{k-(m-q)-1} u^{n-k-q-1} d^q u^{m-q}(x_\ell) \\ &= d^{\ell-1} u^{n-\ell-1}(x_\ell) \\ &= \psi^{(\ell)}(x_\ell), \end{aligned}$$

where $\psi^{(\ell)}$ is the map in Theorem 2.2. Yet, by assumption on b, we have that

$$c \in H^{(n-\ell, m+1+k-(m-q)-1+n-k-q-1)} = H^{(n-\ell,n-1)} = 0.$$

We may hence apply Theorem 2.2 to deduce that $x_\ell \in \operatorname{im} d + \operatorname{im} u$. $\square$

Corollary 2.5 *For $i \in G$, $0 \le k \le n-1$, and $0 \le m \le n-2$, we have a short exact sequence*

$$0 \longrightarrow H_i^{(k,m+1)} \longrightarrow H_i^{(k,m)} \longrightarrow \bigoplus_{q \in I_{m,k}} T_{i+q}^{(k+q-(m-q))} \longrightarrow 0.$$

Corollary 2.6 *The following are equivalent for an amplitude system $\mathcal{H}$.*

(i) $H^{(k)} = 0$ *for some k such that $1 \le k \le n-1$.*

(ii) $H^{(k)} = 0$ *for all k such that $1 \le k \le n-1$.*

(iii) $T^{(p)}(\mathcal{H}) = 0$ *for all p such that $1 \le p \le n-1$.*

Proof. For each k and p, Theorem 2.4 implies that $T^{(p)}(\mathcal{H})$ appears as a summand of $H^{(k,m)}/H^{(k,m+1)}$ for some m. This yields the corollary. □

See Kapranov [10, Prop. 1.5] and Dubois-Violette [3, Lemma 3] for alternative proofs of the equivalence (i) $\Longleftrightarrow$ (ii). Aligning with Kapranov [10], we say that an n-complex $\mathcal{C}$ is *n-exact* if the equivalent conditions (i)-(iii) in Corollary 2.6 are satisfied for the amplitude groups and train groups of $\mathcal{C}$.

2.5 Maps of amplitude systems

Let $\mathcal{H} = (H, d_H, u_H)$ and $\mathcal{L} = (L, d_L, u_L)$ be two G-graded amplitude systems. A *map of amplitude systems*

$$\varphi : \mathcal{H} \to \mathcal{L}$$

is a sequence $(\varphi^{(0)}, \dots, \varphi^{(n)})$ of degree-preserving maps

$$\varphi^{(k)} : H^{(k)} \to L^{(k)}$$

such that $u_L \varphi^{(k)} = \varphi^{(k+1)} u_H$ for $0 \le k \le n-1$ and $d_L \varphi^{(k)} = \varphi^{(k-1)} d_H$ for $1 \le k \le n$. Let $\varphi_i^{(k)}$ denote the restriction of $\varphi^{(k)}$ to $H_i^{(k)}$.

Let $\mathcal{C} = (C, d)$ and $\tilde{\mathcal{C}} = (\tilde{C}, \tilde{d})$ be two n-complexes. An *n-complex map* $\mu : \mathcal{C} \to \tilde{\mathcal{C}}$ is a homomorphism $\mu : C \to \tilde{C}$ such that $\tilde{\delta}\mu = \mu\delta$.

Proposition 2.7 *Let $\mu : \mathcal{C} \to \tilde{\mathcal{C}}$ be an n-complex map. Then μ induces a map φ of the corresponding amplitude systems, where*

$$\varphi^{(k)}(x + \operatorname{im} \delta^{n-k}) = \mu(x) + \operatorname{im} \tilde{\delta}^{n-k}.$$

Proof. If $x = \delta^{n-k}(y)$, then $\mu(x) = \tilde{\delta}^{n-k}\mu(y)$. Moreover, if $\delta^k(x) = 0$, then $\tilde{\delta}^k \mu(x) = \mu\delta^k(x) = 0$. As a consequence, $\varphi^{(k)}$ is well-defined. It is obvious that $\tilde{u}\varphi^{(k)} = \varphi^{(k+1)}u$ and $\tilde{d}\varphi^{(k)} = \varphi^{(k-1)}d$. □

For simplicity, we use the same symbol d to denote the map d_H in different amplitude systems, and similarly for u.

Proposition 2.8 *Let $\varphi : \mathcal{H} \to \mathcal{L}$ be a map of amplitude systems. For $1 \le k \le n-1$ and $i \in G$, we have that φ induces a map*

$$\hat{\varphi}_i^{(k)} : T_i^{(k)}(\mathcal{H}) \to T_i^{(k)}(\mathcal{L});$$

$\hat{\varphi}_i^{(k)}$ maps the class of an element $x \in H_i^{(k)}$ to the class of $\varphi^{(k)}(x) \in L_i^{(k)}$.

Proof. Given $a \in H^{(k+1)}$ and $b \in H^{(k-1)}$, we have that

$$\varphi^{(k)}(d(a) + u(b)) = d\varphi^{(k-1)}(a) + u\varphi^{(k+1)}(b) \in L^{(k)} \cap (\operatorname{im} d + \operatorname{im} u)$$

for $1 \le k \le n-1$. $\square$

Combining Propositions 2.7 and 2.8, we obtain the following fact.

Corollary 2.9 *Let $\mu : \mathcal{C} \to \mathcal{E}$ be an n-complex map. For $1 \le k \le n-1$ and $i \in G$, we have that μ induces a map*

$$\hat{\mu}_i^{(k)} : T_i^{(k)}(\mathcal{C}) \to T_i^{(k)}(\mathcal{E});$$

$\hat{\mu}_i^{(k)}$ maps the class of an element $x \in C_i$ to the class of the element $\mu(x) \in E_i$.

2.6 Amplitude systems and n-complexes

An interesting question regarding amplitude systems is whether every such system arises from an n-complex. Using Theorem 2.4, one can show that this is true whenever the short exact sequences in Corollary 2.5 all split, but the situation is more complicated if they do not split.

For example, let S be a ring, and let $R = S[x]/(x^2)$, where $S[x]$ denotes the polynomial ring over S in one variable. Let $H_0^{(1)} = H_2^{(2)} = R/(x)$ and $H_1^{(1)} = H_1^{(2)} = R$, and define $d_2(r) = u_1(r) = xr$ and $d_1(r) = r$; see the below diagram.

$$\begin{array}{ccccc}
0 \longrightarrow & H_2^{(2)} = R/(x) & & & \\
& \Big\downarrow{\scriptstyle \cdot x} & & & \\
& H_1^{(1)} = R & \xrightarrow{\cdot x} & H_1^{(2)} = R & \\
& & & \Big\downarrow{\scriptstyle \cdot 1} & \\
& & & H_0^{(1)} = R/(x) & \longrightarrow 0.
\end{array}$$

It is clear that this defines an amplitude system $\mathcal{H}$ with $n = 3$. Assume that (C, δ) is a 3-complex of R-modules such that the associated amplitude system coincides with $\mathcal{H}$. Let $a, b \in C_1$ be elements such that a belongs to the class 1 in $H_1^{(1)}$ and b belongs to the class 1 in $H_1^{(2)}$. Since $u_1(1) = x$, we have that $a - xb \in \operatorname{im} \delta_2$, which implies that $xa \in x \cdot \operatorname{im} \delta_2$. In $H_1^{(1)}$, we thus have that $x \in x \cdot \operatorname{im} d_2 = x^2 R = 0$, which is a contradiction.

It would be interesting to know whether there exist similar examples in the case that the underlying ring R is an integral domain. We have yet to find such a example. In the given example, the situation would be different if we viewed $H^{(1)}$ and $H^{(2)}$ as $S[x]$-modules. Specifically, let $C_3 = C_0 = S[x]$ and $C_2 = C_1 = S[x] \oplus S[x]$, and define $\delta_3(a) = (xa, 0)$, $\delta_2(a,b) = (xa + b, -xb)$, $\delta_1(a,b) = b$, and $\delta_0 = 0$. It is a straightforward exercise to check that the resulting amplitude system coincides with $\mathcal{H}$ (in the sense that the two systems are isomorphic as defined in Section 2.5).

3 Further properties of amplitude systems and train groups

We examine further properties of amplitude systems and train groups. The results of this section are not used in the remainder of the paper.

3.1 Free and projective groups

Say that an amplitude system is free (projective) if all amplitude groups are free (projective) as R-modules. We prove a result about the freeness and projectivity of amplitude systems and train groups. In the proof, we will use the fact that any terminating exact sequence of projective R-modules splits:

$$\begin{tikzcd}
\cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] \arrow[d, "\cong"'] & P_1 \arrow[r, "d_1"] \arrow[d, "\cong"'] & P_0 \arrow[r] \arrow[d, "\cong"'] & 0 \\
\cdots \arrow[r, "d_3"] & \operatorname{im} d_3 \oplus Q_2 \arrow[r, "d_2"] & \operatorname{im} d_2 \oplus Q_1 \arrow[r, "d_1"] & \operatorname{im} d_1 \arrow[r] & 0
\end{tikzcd}$$

In particular, $\operatorname{im} d_i$ is projective for each i. We refer to Eisenbud [5, §A3.2] for more information about projective modules.

Theorem 3.1 *The following hold for the train groups of an amplitude system $\mathcal{H}$.*

(i) *If $T^{(p)}(\mathcal{H})$ is free for $1 \leq p \leq n-1$, then $\mathcal{H}$ is free. Conversely, if R is a principal ideal domain and $H^{(n-k)}$ is free, then $T^{(k)}(\mathcal{H})$ is free.*

(ii) *If $T^{(p)}(\mathcal{H})$ is projective for $1 \leq p \leq n-1$, then $\mathcal{H}$ is projective. The converse is true if $\mathcal{H}$ is $\mathbb{Z}$-graded and there is a $b \in \mathbb{Z}$ such that $H_i^{(k)} = 0$ whenever $i < b$.*

Proof. (i) By Corollary 2.5 and downward induction on m, each group $H^{(k,m)}$ is free if the groups $T^{(p)}(\mathcal{H})$ are free for $1 \leq p \leq n-1$. Conversely, if R is a principal ideal domain and $H^{(n-k)}$ is free, then $T^{(k)}(\mathcal{H})$ is also free, being isomorphic to a submodule of the free R-module $H^{(n-k)}$.

(ii) The same argument as in (i) yields that each $H^{(k,m)}$ is projective if all $T^{(p)}(\mathcal{H})$ are projective. Conversely, consider the long exact sequence

$$\begin{array}{l}\cdots \xrightarrow{d^{n-k}} H_i^{(r)} \xrightarrow{u^{k-r}} H_i^{(k)} \xrightarrow{d^r} H_{i-r}^{(k-r)} \xrightarrow{u^{n-k}} \\ \xrightarrow{u^{n-k}} H_{i-r}^{(n-r)} \xrightarrow{d^{k-r}} H_{i-k}^{(n-k)} \xrightarrow{u^r} H_{i-k}^{(n-k+r)} \xrightarrow{d^{n-k}} \cdots\end{array}$$

By assumption, this sequence terminates. Since all modules in the sequence are projective, the sequence splits, and the image of each map is projective. In particular, $H_i^{(k)} \cap \operatorname{im} d^{n-k-\ell} = H_i^{(k)} \cap \ker u^\ell$ is projective for $1 \le \ell \le n-k-1$. Now, consider the exact sequence

$$H_i^{(1)} \cap \ker u^k \xrightarrow{u^{k-1}} H_i^{(k)} \cap \ker u \xrightarrow{d} H_{i-1}^{(k-1)} \cap \ker u \longrightarrow 0.$$

The sequence is indeed exact, because

$$\begin{aligned} H_i^{(k)} \cap \ker u \cap \ker d &= H_i^{(k)} \cap \ker u \cap \operatorname{im} u^{k-1} = H_i^{(k)} \cap u^{k-1}(\ker u^k), \\ H_{i-1}^{(k-1)} \cap \ker u &= H_{i-1}^{(k-1)} \cap \operatorname{im} d^{n-k} = H_{i-1}^{(k-1)} \cap d(\operatorname{im} d^{n-k-1}) = \\ &= H_{i-1}^{(k-1)} \cap d(\ker u). \end{aligned}$$

Moreover, all modules in the sequence are projective, which implies that $H_i^{(k)} \cap u^{k-1}(\ker u^k) = H_i^{(k)} \cap \ker u \cap \ker d$ is projective. By Corollary 2.3, this module is isomorphic to $T_{i+n-k-1}^{(n-k)}$. □

The second statement in Theorem 3.1 (i) was given in terms of a principal ideal domain R. The statement is no longer true for arbitrary domains. To give an example, recall that an R-module P is *stably free* if $F \oplus P$ is free for some free R-module F. For some rings R, there are stably free R-modules that are not free. For example, this is the case for the domain $R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$ [5, Example 19.17]. Pick such a ring R, and let F and P be such that F and $F \oplus P$ are free, whereas P is not free. Let $C_2 = F$, $C_1 = F \oplus F \oplus P$, and $C_0 = F$, and define $\delta_2(x) = (x,0,0)$, $\delta_1(x,y,z) = y$, and $\delta_0 = 0$. This yields a 3-complex, and we obtain that

$$\begin{array}{lll} H_0^{(1)} \cong F, & H_1^{(1)} \cong F \oplus P, & H_2^{(1)} \cong 0, \\ H_0^{(2)} \cong 0, & H_1^{(2)} \cong F \oplus P, & H_2^{(2)} \cong F, \end{array}$$

which are all free. Yet, $T_1^{(1)} \cong P$, which is not free.

3.2 Monomorphisms and epimorphisms

One may view the following result as a generalization of a result due to Dubois-Violette [3, §2, Prop. 1].

Theorem 3.2 *Let $\varphi : \mathcal{H} \to \mathcal{L}$ be a map of amplitude systems. Suppose that there is a $k \in \{1,\dots,n-1\}$ such that $\varphi^{(k)}$ is a monomorphism and $\varphi^{(n-k)}$ is an epimorphism. Then $\varphi^{(j)}$ is an isomorphism for each j.*

Proof. By an induction argument, it is sufficient to show that $\varphi^{(k-1)}$ and $\varphi^{(k+1)}$ are monomorphisms and that $\varphi^{(n-k+1)}$ and $\varphi^{(n-k-1)}$ are epimorphisms. By symmetry of d and u, it suffices to prove that $\varphi^{(k-1)}$ is a monomorphism and $\varphi^{(n-k+1)}$ is an epimorphism.

To see that $\varphi^{(k-1)}$ is a monomorphism, consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
H^{(n-k)} \arrow[r, "u^{k-1}"] \arrow[d, "\varphi^{(n-k)}"'] & H^{(n-1)} \arrow[r, "d^{n-k}"] \arrow[d, "\varphi^{(n-1)}"'] & H^{(k-1)} \arrow[r, "u"] \arrow[d, "\varphi^{(k-1)}"'] & H^{(k)} \arrow[d, "\varphi^{(k)}"'] \\
L^{(n-k)} \arrow[r, "u^{k-1}"] & L^{(n-1)} \arrow[r, "d^{n-k}"] & L^{(k-1)} \arrow[r, "u"] & L^{(k)}
\end{tikzcd}$$

Let $x \in H^{(k-1)}$ be such that $x' = \varphi^{(k-1)}(x) = 0$. Since $\varphi^{(k)}$ is a monomorphism, we have that $u(x) = 0$, which yields an element $y \in H^{(n-1)}$ such that $d^{n-k}(y) = x$. Since $y' = \varphi^{(n-1)}(y)$ has the property that $d^{n-k}(y') = \varphi^{(k-1)}d^{n-k}(y) = 0$, we obtain an element $z' \in L^{(n-k)}$ such that $u^{k-1}(z') = y'$. Now, $\varphi^{(n-k)}$ is an epimorphism, which yields an element $z \in H^{(n-k)}$ such that $\varphi^{(n-k)}(z) = z'$. Finally, note that

$$\varphi^{(k)}(d^{n-k-1}(y)) = d^{n-k-1}(y') = d^{n-k-1}u^{k-1}\varphi^{(n-k)}(z) = \varphi^{(k)}d^{n-k-1}u^{k-1}(z).$$

Since $\varphi^{(k)}$ is a monomorphism, we get that $d^{n-k-1}(y) = d^{n-k-1}u^{k-1}(z)$ and hence that

$$x = d^{n-k}(y) = d^{n-k}u^{k-1}(z) = u^{k-1}d^{n-k}(z) = 0.$$

To see that $\varphi^{(n-k+1)}$ is an epimorphism, we use a dual argument. Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
H^{(n-k)} \arrow[r, "u"] \arrow[d, "\varphi^{(n-k)}"'] & H^{(n-k+1)} \arrow[r, "d^{n-k}"] \arrow[d, "\varphi^{(n-1)}"'] & H^{(1)} \arrow[r, "u^{k-1}"] \arrow[d, "\varphi^{(k-1)}"'] & H^{(k)} \arrow[d, "\varphi^{(k)}"'] \\
L^{(n-k)} \arrow[r, "u"] & L^{(n-k+1)} \arrow[r, "d^{n-k}"] & L^{(1)} \arrow[r, "u^{k-1}"] & L^{(k)}
\end{tikzcd}$$

Let $x' \in L^{(n-k+1)}$, and define $w' = d(x') \in L^{(n-k)}$ and $y' = d^{n-k}(x') \in L^{(1)}$. Since $\varphi^{(n-k)}$ is an epimorphism, there is a $w \in H^{(n-k)}$ such that $\varphi^{(n-k)}(w) = w'$. Define $y = d^{n-k-1}(w) \in H^{(1)}$; note that $\varphi^{(1)}(y) = y'$. By the diagram, $u^{k-1}(y')$ is zero. Since $\varphi^{(k)}$ is a monomorphism, we obtain that $u^{k-1}(y)$ is also zero. As a consequence, there is an element $x_0 \in H^{(n-k+1)}$ such that $d^{n-k}(x_0) = y$. Write $x'_0 = \varphi^{(n-k+1)}(x_0)$. Since $d^{n-k}(x' - x'_0) = 0$, there is an element $z' \in L^{(n-k)}$ such that $u(z') = x' - x'_0$. Now, $\varphi^{(n-k)}$ is an epimorphism, which implies that $z' = \varphi^{(n-k)}(z)$ for some $z \in H^{(n-k)}$. To conclude, we note that $\varphi^{(n-k+1)}u(z) = x' - x'_0$, which yields that $x' = \varphi^{(n-k+1)}(u(z) + x_0)$. $\square$

For the remainder of the section, we compare a given map of amplitude systems to its induced maps of train groups.

Theorem 3.3 *Let $\varphi : \mathcal{H} \to \mathcal{L}$ be a map of amplitude systems. Let $1 \leq k \leq n-1$. Then the following hold.*

(i) *If* $\varphi_i^{(k)} : H_i^{(k)} \to L_i^{(k)}$ *is an epimorphism, then so is* $\hat{\varphi}_i^{(k)} : T_i^{(k)}(\mathcal{H}) \to T_i^{(k)}(\mathcal{L})$.

(ii) *If* $\varphi_{i-k+1}^{(n-k)} : H_{i-k+1}^{(n-k)} \to L_{i-k+1}^{(n-k)}$ *is a monomorphism, then so is* $\hat{\varphi}_i^{(k)} : T_i^{(k)}(\mathcal{H}) \to T_i^{(k)}(\mathcal{L})$.

As a consequence, if $\varphi_i^{(k)}$ *is an epimorphism and* $\varphi_{i-k+1}^{(n-k)}$ *is a monomorphism, then* φ *induces an isomorphism from* $T_i^{(k)}(\mathcal{H})$ *to* $T_i^{(k)}(\mathcal{L})$.

Proof. By Theorem 2.2, we may identify $T_i^{(k)}(\mathcal{H})$ with $\psi^{(k)}(H_i^{(k)}) \subseteq H_{i-k+1}^{(n-k)}$, where $\psi^{(k)} = d^{k-1}u^{n-k-1}$. We obtain the following commutative diagram, where on both rows $\psi^{(k)}$ is an epimorphism and $\iota^{(k)}$ is a monomorphism.

$$\begin{tikzcd}
H_i^{(k)} \arrow[r, "\psi^{(k)}"] \arrow[d, "\varphi^{(k)}"] & T_i^{(k)}(\mathcal{H}) \arrow[r, "\iota^{(k)}"] \arrow[d, "\hat{\varphi}^{(k)}"] & H_{i-k+1}^{(n-k)} \arrow[d, "\varphi^{(n-k)}"] \\
L_i^{(k)} \arrow[r, "\psi^{(k)}"] & T_i^{(k)}(\mathcal{L}) \arrow[r, "\iota^{(k)}"] & L_{i-k+1}^{(n-k)}
\end{tikzcd}$$

As a consequence, the theorem follows. □

One may view the next result as a partial converse of Theorem 3.3. For convenience, we state the result in the ungraded case, leaving the graded case to the interested reader.

Theorem 3.4 *Let* $\varphi : \mathcal{H} \to \mathcal{L}$ *be a map of amplitude systems. Then the following hold.*

(i) *If* $\hat{\varphi}^{(k)}$ *is an epimorphism for* $1 \le k \le n-1$*, then* $\varphi^{(k)}$ *is an epimorphism for* $1 \le k \le n-1$.

(ii) *If* $\hat{\varphi}^{(k)}$ *is a monomorphism for* $1 \le k \le n-1$*, then* $\varphi^{(k)}$ *is a monomorphism for* $1 \le k \le n-1$.

Proof. We apply Theorem 2.4. Let $\varphi^{(k,m)}$ be the restriction of $\varphi^{(k)}$ to $H^{(k,m)}$. For each m, we have that $\varphi^{(k,m)}(H^{(k,m)}) \subseteq L^{(k,m)}$, because $\varphi^{(k)} d^q u^{m-q} = d^q u^{m-q} \varphi^{(k+q-(m-q))}$ for $0 \le q \le m$. As a consequence, we have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & H^{(k,m+1)} \arrow[r] \arrow[d, "\varphi^{(k,m+1)}"] & H^{(k,m)} \arrow[r] \arrow[d, "\varphi^{(k,m)}"] & \bigoplus_{q \in I_{k,m}} T^{(k+q-(m-q))}(\mathcal{H}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L^{(k,m+1)} \arrow[r] & L^{(k,m)} \arrow[r] & \bigoplus_{q \in I_{k,m}} T^{(k+q-(m-q))}(\mathcal{L}) \arrow[r] & 0
\end{tikzcd}$$

To obtain (i), we use downward induction on m, starting with $m = n-2$, to prove that $\varphi^{(k,m)}$ is an epimorphism for $0 \le m \le n-2$; remember that $\varphi^{(k,0)} =$

$\varphi^{(k)}$. Now, the rightmost vertical arrow in the diagram is an epimorphism by assumption, and $\varphi^{(k,m+1)}$ is an epimorphism by induction. As a consequence, $\varphi^{(k,m)}$ is an epimorphism, and we are done. A similar argument yields (ii). $\square$

4 Pseudo-homotopic maps

Let $\mathcal{C} = (C, \delta)$ and $\mathcal{E} = (E, \epsilon)$ be n-complexes, and let φ be an n-complex map. By Corollary 2.9, φ induces maps $\hat{\varphi}^{(k)} : T^{(k)}(\mathcal{C}) \to T^{(k)}(\mathcal{E})$ for $1 \leq k \leq n-1$. Following Kassel and Wambst [11] and Dubois-Violette [3], we say that two n-complex maps $\mu, \lambda : \mathcal{C} \to \mathcal{E}$ are *homotopic* if there are maps $h_j : \operatorname{im} \delta^{j-1} \to E$ (typically not n-complex maps) such that

$$\mu - \lambda = \sum_{j=1}^{n} \epsilon^{n-j} h_j \delta^{j-1}.$$

Proposition 4.1 (Dubois-Violette [3]) *If μ and λ are homotopic n-complex maps from $\mathcal{C}$ to $\mathcal{E}$, then μ and λ induce the same map from $H^{(k)}(\mathcal{C})$ to $H^{(k)}(\mathcal{E})$ for $1 \leq k \leq n-1$.*

Proof. Suppose that $\mu - \lambda = \sum_{j=1}^{n} \epsilon^{n-j} h_j \delta^{j-1}$. Let $x \in \ker \delta^k$. We have that $\epsilon^{n-j} h_j \delta^{j-1}(x) = 0$ for $1 \leq j \leq k-1$ and $\epsilon^{n-j} h_j \delta^{j-1}(x) \in \operatorname{im} \epsilon^{n-k}$ for $k \leq j \leq n$. As a consequence, $\mu(x) - \lambda(x) \in \epsilon^{n-k}$, which yields the statement. $\square$

Corollary 4.2 *If μ and λ are homotopic n-complex maps from $\mathcal{C}$ to $\mathcal{E}$, then μ and λ induce the same map from $T^{(k)}(\mathcal{C})$ to $T^{(k)}(\mathcal{E})$ for $1 \leq k \leq n-1$.*

An important special case is that the identity map on C is homotopic to the zero map. By Corollary 4.2, we then have that $T^{(k)}(\mathcal{C}) = 0$ for $1 \leq k \leq n-1$.

We say that two n-complex maps λ, μ from $\mathcal{C}$ to $\mathcal{E}$ are *pseudo-homotopic* if

$$\mu - \lambda = \sum_{k=1}^{n} \sum_{j=1}^{k} \epsilon^{k-j} h_{jk} \delta^{j-1},$$

where the maps $h_{jk} : \operatorname{im} \delta^{j-1} \to E$ have the properties that $\operatorname{im} h_{jk} \subseteq \ker \epsilon^k$ and $\operatorname{im} \delta^k \subseteq \ker h_{jk}$. If $h_{jk} = 0$ whenever $k < \gamma$, then we say that μ and λ are pseudo-homotopic of degree γ. In particular, μ and λ are homotopic if they are pseudo-homotopic of degree n.

Theorem 4.3 *If μ and λ are pseudo-homotopic via the identity*

$$\mu - \lambda = g = \sum_{k=1}^{n} \sum_{j=1}^{k} \epsilon^{k-j} h_{jk} \delta^{j-1},$$

then the induced maps $\hat{\mu}^{(r)}, \hat{\lambda}^{(r)} : T^{(r)}(\mathcal{C}) \to T^{(r)}(\mathcal{E})$ satisfy the relation

$$\hat{\mu}^{(r)} - \hat{\lambda}^{(r)} = \hat{g}^{(r)}$$

for $1 \leq r \leq n-1$, where $\hat{g}^{(r)} : T^{(r)}(\mathcal{C}) \to T^{(r)}(\mathcal{E})$ is induced by $h_{rr}\delta^{r-1}$.

Proof. Write $g_{jk} = \epsilon^{k-j} h_{jk} \delta^{j-1}$. Note that $\operatorname{im} g_{rr} \subseteq \ker \epsilon^r$ and $\ker \delta^{r-1} + \operatorname{im} \delta \subseteq \ker g_{rr}$; hence g_{rr} indeed induces a map from $T^{(r)}(\mathcal{C})$ to $T^{(r)}(\mathcal{E})$. It follows that $f = g - g_{rr}$ also induces such a map. It remains to show that f induces the zero map, meaning that $f(x) \in \operatorname{im} \epsilon + \ker \epsilon^{r-1}$ whenever $x \in \ker \delta^r$. Now, if $j < k$, then $g_{jk} \in \operatorname{im} \epsilon$. Moreover, if $k > r$, then $\ker \delta^r \subseteq \ker \delta^{k-1} \subseteq \ker g_{jk}$; hence $g_{kk}(x) = 0$. Finally, if $k < r$, then $g_{kk}(x) \in \ker \epsilon^k \subseteq \ker \epsilon^{r-1}$. This yields the desired statement. □

We now consider n-complex endomorphisms that are pseudo-homotopic to the identity map. Under certain favorable conditions, we may use such endomorphisms to deduce information about the train groups. We will apply this fact in Section 5.

An n-complex $\mathcal{S} = (S, \delta)$ is an *n-subcomplex* of the n-complex $\mathcal{C} = (C, \delta)$ if $S \subseteq C$ and $\delta(S) \subseteq S$. Here, we use the same symbol δ to denote both δ itself and its restriction $\delta|_S$ to S. We write $\mathcal{S} \subseteq \mathcal{C}$.

If $\lambda : \mathcal{C} \to \mathcal{E}$ is an n-complex map inducing an isomorphism between the amplitude systems of $\mathcal{C}$ and $\mathcal{E}$, then we say that $\mathcal{C} \sim \mathcal{E}$ via λ.

Proposition 4.4 *Suppose that the n-complex map $\lambda : \mathcal{C} \to \mathcal{C}$ satisfies $\lambda^2 = \lambda$. Write $g = id - \lambda$. Then $(\operatorname{im} g, \delta)$ is an n-subcomplex of $\mathcal{C}$, and $\mathcal{C}$ splits as the direct sum of $(\operatorname{im} \lambda, \delta)$ and $(\operatorname{im} g, \delta)$. In particular, if $(\operatorname{im} g, \delta)$ is n-exact, then $\mathcal{C} \sim (\operatorname{im} \lambda, \delta)$ via λ and also via the inclusion map from $(\operatorname{im} \lambda, \delta)$ to $\mathcal{C}$.*

Proof. We have that $\delta g = \delta - \delta\lambda = \delta - \lambda\delta = g\delta$; hence $(\operatorname{im} g, \delta)$ is indeed an n-complex. Since $g^2 = g$ and $\mathrm{id} = \lambda + g$, we have that $C = \operatorname{im} \lambda \oplus \operatorname{im} g$. Since the restriction of δ to each of $\operatorname{im} \lambda$ and $\operatorname{im} g$ defines an n-complex, we obtain the desired result. □

Theorem 4.5 *Suppose that the n-complex map $\lambda : \mathcal{C} \to \mathcal{C}$ is pseudo-homotopic to the identity map via the identity*

$$\mathrm{id} - \lambda = g = \sum_{k=1}^{n} \sum_{j=1}^{k} \epsilon^{k-j} h_{jk} \delta^{j-1}.$$

Suppose that $\lambda^2 = \lambda$. Then

$$T^{(r)}(\mathcal{C}) \cong T^{(r)}(\operatorname{im} \lambda, \delta) \oplus \frac{\operatorname{im} g_{rr}}{\operatorname{im} g_{rr} \cap (\ker \delta^{r-1} + \operatorname{im} \delta)},$$

where $g_{rr} = h_{rr} \delta^{r-1}$. If $g_{rr}^2 = g_{rr}$, then this simplifies to

$$T^{(r)}(\mathcal{C}) \cong T^{(r)}(\operatorname{im} \lambda, \delta) \oplus \operatorname{im} g_{rr}$$

Proof. Proposition 4.4 yields that $\mathcal{C}$ splits as the direct sum of $(\operatorname{im} \lambda, \delta)$ and $(\operatorname{im} g, \delta)$. Since $g^2 = g$, the inclusion map from $\operatorname{im} g$ to C coincides with the restriction of g to $\operatorname{im} g$. By Theorem 4.3, the associated induced map $\hat{g}^{(r)} : T^{(r)}(\operatorname{im} g, \delta) \to T^{(r)}(C, \delta)$ is also induced by the map g_{rr}. We deduce that

$$T^{(r)}(\operatorname{im} g, \delta) \cong \frac{\operatorname{im} g_{rr}}{\operatorname{im} g_{rr} \cap (\ker \delta^{r-1} + \operatorname{im} \delta)}.$$

For the final statement, note that

$$\operatorname{im} g_{rr} \cap (\ker \delta^{r-1} + \operatorname{im} \delta) \subseteq \operatorname{im} g_{rr} \cap \ker g_{rr},$$

which is zero if $g_{rr}^2 = g_{rr}$. □

5 Discrete Morse theory on nilpotent operators

Forman's discrete Morse theory [6] is a method for reducing a chain complex to a smaller complex with the same homology. We want to generalize the method to n-complexes and show how one may use the method to simplify the computation of train groups.

First, we review the classical case, where we have a boundary operator ∂ satisfying $\partial^2 = 0$. We refer to the literature [6, 8, 9, 12, 16] for more details. The main idea of discrete Morse theory is to find a decomposition of C as a direct sum

$$C = A \oplus B \oplus U$$

such that $\pi_A \partial$ defines an isomorphism f from B to A, where $\pi_A(a + b + u) = a$ for $a \in A$, $b \in B$, and $u \in U$. Since f is an isomorphism, we may replace A with $\partial(B)$ in the decomposition, thereby obtaining the decomposition

$$C = \partial(B) \oplus B \oplus U.$$

Defining

$$\hat{U} = \{u - f^{-1}\pi_A \partial(u) : u \in U\},$$

we may replace U with $\hat{U}$ and obtain the decomposition

$$C = \partial(B) \oplus B \oplus \hat{U}.$$

By a straightforward computation, we obtain that $\partial(\hat{U}) \subseteq \hat{U}$. In particular,

$$\ker \partial = \partial(B) \oplus (\hat{U} \cap \ker \partial)$$

and

$$\ker \partial \cap \operatorname{im} \partial = \partial^2(B) \oplus \partial(B) \oplus \partial(\hat{U}) = \partial(B) \oplus (\hat{U} \cap \ker \partial \cap \operatorname{im} \partial).$$

This implies that

$$H(C, \partial) = T^{(1)}(C, \partial) \cong \frac{\hat{U} \cap \ker \partial}{\hat{U} \cap \ker \partial \cap \operatorname{im} \partial} = T^{(1)}(\hat{U}, \partial) = H(\hat{U}, \partial).$$

One may view the groups A and B as being paired; due to the isomorphism f, they cancel out nicely. Our goal is to generalize the construction to n-complexes, showing how to cancel out sequences of subgroups, while keeping control of the train groups. Similarly to the way Jöllenbeck and Welker [8] and Sköldberg [16] described discrete Morse theory in terms of homotopic maps, we describe the generalized version in terms of pseudo-homotopic maps.

For clarity, we divide the discussion into two parts. In Section 5.1, we restrict our attention to a generalization of the very special case that A and $\partial(B)$ coincide. In Section 5.2, we consider the full generalization.

5.1 Starting with a nice decomposition

Let $(\tilde{C}, \tilde{\delta})$ be an n-complex. Suppose that we are given a submodule

$$A = \bigoplus_{k=1}^{n} \bigoplus_{j=1}^{k} A^{jk} \tag{5}$$

of $\tilde{C}$ such that we may write

$$\tilde{C} = A \oplus U \tag{6}$$

for some submodule U. For every $j \leq k$, assume that the restriction of $\tilde{\delta}^{k-j}$ to A^{kk} defines an isomorphism

$$f_{jk} : A^{kk} \to A^{jk}.$$

Moreover, $\tilde{\delta}(A^{1k}) = 0$ for $1 \leq k \leq n$. The following diagram illustrates how $\tilde{\delta}$ maps the various submodules.

$$\begin{array}{ccccccccccc}
0 & \longleftarrow & A^{11} & & & & & & & & \\
0 & \longleftarrow & A^{12} & \xleftarrow{\cong} & A^{22} & & & & & & \\
0 & \longleftarrow & A^{13} & \xleftarrow{\cong} & A^{23} & \xleftarrow{\cong} & A^{33} & & & & \\
 & \cdots & & \cdots & & \cdots & & \cdots & & & \\
0 & \longleftarrow & A^{1n} & \xleftarrow{\cong} & A^{2n} & \xleftarrow{\cong} & A^{3n} & \xleftarrow{\cong} & \cdots & \xleftarrow{\cong} & A^{nn}
\end{array}$$

Remark. In the case of the usual boundary operator ∂, discussed at the beginning of Section 5, we have that $A^{22} = B$, $A^{12} = \partial(B)$, and $A^{11} = 0$.

Remark. In Section 5.2, we will leave more room for flexibility. Specifically, while we will still require that $\tilde{\delta}^{k-j}(A^{kk})$ is isomorphic to A^{jk}, we will no longer require that the two groups are identical.

We also need the following assumption:

(I) We have that $\pi_{A^{1k}}\tilde{\delta}^k = 0$ for $1 \leq k \leq n$, where $\pi_{A^{jk}}$ is the projection on A^{jk} with respect to decompositions (5) and (6).

The restriction of $\pi_{A^{1k}}\tilde{\delta}^k$ to A is always zero, which implies that (I) is equivalent to saying that the restriction of $\pi_{A^{1k}}\tilde{\delta}^k$ to U is zero for $1 \leq k \leq n$.

The *standard case* is the situation that $A^{jk} = 0$ whenever $k < n$. We are in the standard case if and only if the induced n-complex on A is n-exact. Note that (I) is trivially true in the standard case.

For $1 \leq k \leq n$ and $1 \leq j \leq k$, we define $h_{jk} = f_{1k}^{-1}\pi_{A^{1k}}$. Note that $\operatorname{im} h_{jk} \subseteq \ker \delta^k$ and $\operatorname{im} \delta^k \subseteq \ker h_{jk}$. Moreover, (I) is equivalent to saying that $h_{kk}\tilde{\delta}^k = 0$ for $1 \leq k \leq n$.

Write $g_{jk} = \tilde{\delta}^{k-j} h_{jk} \tilde{\delta}^{j-1}$. We have that $\operatorname{im} g_{jk} = A^{jk}$ and

$$g_{jk} g_{\ell m} = \begin{cases} g_{jk} & \text{if } (j,k) = (\ell, m), \\ 0 & \text{if } (j,k) \neq (\ell, m). \end{cases}$$

For example,

$$g_{jk}^2 = \tilde{\delta}^{k-j} f_{1k}^{-1} \pi_{A^{1k}} \left(\tilde{\delta}^{k-1} f_{1k}^{-1} \right) \pi_{A^{1k}} \tilde{\delta}^{j-1} = \tilde{\delta}^{k-j} f_{1k}^{-1} \pi_{A^{1k}} \tilde{\delta}^{j-1} = g_{jk}.$$

In particular, $g^2 = g$, where

$$g = \sum_{k=1}^{n} \sum_{j=1}^{k} g_{jk}.$$

Define

$$\tilde{\lambda} = \mathrm{id} - g. \tag{7}$$

Theorem 5.1 *We have that $\tilde{\lambda}$ defines an n-complex map from $\tilde{\mathcal{C}}$ to itself, and $\tilde{\lambda}|_U : U \to \operatorname{im} \tilde{\lambda}$ is an isomorphism of R-modules. Moreover,*

$$T^{(r)}(\tilde{C}, \tilde{\delta}) \cong T^{(r)}(\operatorname{im} \tilde{\lambda}, \tilde{\delta}) \oplus A^{rr}$$

for $1 \leq r \leq n-1$. In the standard case, $\tilde{\mathcal{C}} \sim (\operatorname{im} \tilde{\lambda}, \tilde{\delta})$ via $\tilde{\lambda}$ and also via the inclusion map from $\operatorname{im} \tilde{\lambda}$ to $\tilde{\mathcal{C}}$.

Proof. For $x \in A^{jk}$, note that

$$\tilde{\lambda}(x) = x - g_{jk}(x) = x - x = 0.$$

In particular, $\operatorname{im} \tilde{\lambda} = \tilde{\lambda}(U)$. Moreover, $\tilde{\lambda}|_U : U \to \operatorname{im} \tilde{\lambda}$ is indeed an isomorphism, because an inverse is given by taking the projection on U with respect to the decomposition (6).

Furthermore, $g^2 = g$ and $g_{rr}^2 = g_{rr}$ by the discussion preceding the theorem. In particular, assuming $\tilde{\lambda}$ is an n-complex map, we obtain that the conditions of Theorem 4.5 are satisfied, which implies the desired result; apply Proposition 4.4 for the very last statement of the theorem.

It remains to show that $\tilde{\lambda}$ is indeed an n-complex map. Equivalently, g is an n-complex map. Now,

$$\tilde{\delta} g_{jk} = \tilde{\delta}^{k-j+1} f^{-1} \pi_{A^{1k}} \tilde{\delta}^{j-1} = \begin{cases} g_{j-1,k} \tilde{\delta} & \text{if } 1 < j \leq k, \\ 0 & \text{if } j = 1. \end{cases}$$

As a consequence,

$$\tilde{\delta} g = \sum_{k=1}^{n} \sum_{j=1}^{k} \tilde{\delta} g_{jk} = \sum_{k=2}^{n} \sum_{j=2}^{k} g_{j-1,k} \tilde{\delta} = \sum_{k=2}^{n} \sum_{j=1}^{k-1} g_{jk} \tilde{\delta} = \sum_{k=1}^{n} \sum_{j=1}^{k} g_{jk} \tilde{\delta} = g \tilde{\delta},$$

where the next to last equality is a consequence of the assumption (I) that $g_{kk} \tilde{\delta} = 0$. This completes the proof. □

Write $A^{\{1\}} = \bigoplus_{k=1}^{n} A^{1k}$.

Theorem 5.2 *For each $u \in U$, the element $\tilde{\lambda}(u)$ is the unique element $t \in \tilde{C}$ with the properties that $\pi_U(t) = u$ and*

$$\pi_{A^{\{1\}}} \tilde{\delta}^m(t) = 0 \text{ for } m \geq 0. \tag{8}$$

Here, π_U and $\pi_{A^{\{1\}}}$ are projections with respect to the decompositions (5) *and* (6)*. In particular,* $\operatorname{im} \tilde{\lambda}$ *is the module consisting of all elements t satisfying* (8).

Proof. Since $g_{1k}(U) = 0$ for every k, we have that $\pi_{A^{\{1\}}} \tilde{\lambda} = 0$. By Theorem 5.1, $\operatorname{im} \tilde{\delta}^m \tilde{\lambda} \subseteq \operatorname{im} \tilde{\lambda}$, which yields that $\pi_{A^{\{1\}}} \tilde{\delta}^m \tilde{\lambda} = 0$.

Conversely, suppose that $\pi_U(t) = u$ and that $\pi_{A^{\{1\}}} \tilde{\delta}^m(t) = 0$ for all $m \geq 0$. Write

$$t = \tilde{\lambda}(u) + \sum_{k=1}^{n} \sum_{j=1}^{k} a_{jk},$$

where $a_{jk} \in A^{jk}$. We obtain that

$$0 = \pi_{A^{1k}} \tilde{\delta}^m(t) = \pi_{A^{1k}} \tilde{\delta}^m \tilde{\lambda}(u) + \tilde{\delta}^m(a_{m+1,k}) = \tilde{\delta}^m(a_{m+1,k}).$$

Since the restriction of $\tilde{\delta}^m$ to $A^{m+1,k}$ defines an isomorphism to A^{1k}, we obtain that $a_{m+1,k} = 0$ for all m and k. As a consequence, $t = \tilde{\lambda}(u)$. □

5.2 Starting with a more general decomposition

In applications of discrete Morse theory to ordinary complexes, the pair (A, B) typically does not have the property that $\partial(B) = A$. More precisely, the normal situation is that we have a standard basis for C and that A and B are generated by subsets of this basis [8, 9, 15, 16]. For example, C might be the chain complex associated to a simplicial complex, and A and B might be generated by subsets of oriented simplices.

Similarly for n-complexes, it seems reasonable to assume that it would not normally be straightforward to find groups A^{ik} such that δ^{k-i} defines an isomorphism from A^{kk} to A^{ik} for $1 \leq i \leq k$. In this section, we address this issue by considering a more general situation. See Section 7.1 for a simple illustrating example.

For an n-complex (C, δ), assume that we may write

$$C = B \oplus U,$$

where $B = B^{\{<\}} \oplus B^{\{=\}}$. Here,

$$B^{\{=\}} = \bigoplus_{k=1}^{n} B^k$$

for some R-modules $B^1, \ldots, B^n$, and

$$B^{\{<\}} \cong \bigoplus_{k=1}^{n} \bigoplus_{j=1}^{k-1} B^k.$$

We refer to the situation that $B^k = 0$ for $k < n$ as the *standard case.*

Remark. In the ideal case discussed in Section 5.1, one should think of B^k as A^{kk} and $B^{\{<\}}$ as $\bigoplus_{k=1}^{n} \bigoplus_{j=1}^{k-1} A^{jk}$.

For $1 \le j \le k \le n$, introduce a formal variable e_{jk}, and define $A^{jk} = B^k e_{jk}$. Let

$$\tilde{C} = \bigoplus_{k=1}^{n} \bigoplus_{j=1}^{k} A^{jk} \oplus U.$$

Define $\varphi : \tilde{C} \to C$ by

$$\varphi(be_{jk}) = \delta^{k-j}(b) \in \delta^{k-j}(B^k)$$

and $\varphi(u) = u$ for $u \in U$. Assume that $\varphi : \tilde{C} \to C$ defines an isomorphism. Then we may define an operator $\tilde{\delta} : \tilde{C} \to \tilde{C}$ by

$$\tilde{\delta} = \varphi^{-1}\delta\varphi.$$

Since φ is an n-complex map and an isomorphism, we have that

$$T^{(r)}(\tilde{C}, \tilde{\delta}) \cong T^{(r)}(C, \delta).$$

Now,

$$\begin{aligned} \tilde{\delta}(be_{jk}) &= \varphi^{-1}\delta\varphi(be_{jk}) = \varphi^{-1}\delta\delta^{k-j}(b) \\ &= \varphi^{-1}\delta^{k-j+1}(b) = \begin{cases} be_{j-1,k} & \text{if } 1 < j \le n, \\ 0 & \text{if } j = 1. \end{cases} \end{aligned}$$

In particular, we have the situation in Section 5.1, given that we again make the following assumption, which is trivially true in the standard case.

(I) We have that $\pi_{A^{1k}}\tilde{\delta}^k = 0$ for $1 \le k \le n$.

Note that $\tilde{\delta}^k = \varphi^{-1}\delta^k\varphi$.

Define $\tilde{\lambda}$ as in Section 5.1, and write $\lambda = \varphi\tilde{\lambda}\varphi^{-1}$. The map λ is an n-complex map, being the composition of n-complex maps. Also, note that $\lambda^2 = \varphi\tilde{\lambda}^2\varphi^{-1} = \varphi\tilde{\lambda}\varphi^{-1} = \lambda$. Theorem 5.1 yields the following result.

Theorem 5.3 *Let notation be as above. Suppose that φ is an n-complex isomorphism from $(\tilde{C}, \tilde{\delta})$ to (C, δ), and also suppose that $\pi_{A^{1k}}\tilde{\delta}^k = 0$ for $1 \le k \le n$. For $1 \le r \le n-1$, we have that*

$$\begin{aligned} T^{(r)}(C, \delta) &\cong T^{(r)}(\tilde{C}, \tilde{\delta}) \cong T^{(r)}(\operatorname{im}\tilde{\lambda}, \tilde{\delta}) \oplus A^{rr} \\ &\cong T^{(r)}(\operatorname{im}\lambda, \delta) \oplus B^r. \end{aligned}$$

In the standard case, $(C, \delta) \sim (\operatorname{im}\lambda, \delta)$ via λ and via the inclusion map from $\operatorname{im}\lambda$ to C.

Remark. Theorem 5.3 relies on φ being an isomorphism. Note that the restriction of φ to $\bigoplus_k A^{kk} \oplus U$ defines an isomorphism to $B^{\{=\}} \oplus U$. In particular, φ is an isomorphism if and only if $\pi_{B^{\{<\}}}\varphi$ defines an isomorphism from $A^{\{<\}} = \bigoplus_{j<k} A^{jk}$ to $B^{\{<\}}$, where the projection is with respect to the decomposition $C = B^{\{<\}} \oplus B^{\{=\}} \oplus U$.

The following special case is particularly nice. Recall that $A^{\{1\}} = \bigoplus_{k=1}^{n} A^{1k}$.

Theorem 5.4 *With notation and assumptions as above, suppose that we may write*

$$B = B^{\{1\}} \oplus B^{\{>\}}$$

such that $\pi_{A^{\{1\}}}(x) = 0$ *if and only if* $\pi_{B^{\{1\}}}\varphi(x) = 0$ *for every* $x \in \tilde{C}$*. Then* $\operatorname{im}\lambda$ *is the group consisting of all elements* $t \in C$ *such that* $\pi_{B^{\{1\}}}\delta^m(t) = 0$ *for all* $m \geq 0$*.*

Proof. By Theorem 5.2, an element $\tilde{t}$ belongs to $\operatorname{im}\tilde{\lambda}$ if and only if

$$\pi_{A^{\{1\}}}\tilde{\delta}^m(\tilde{t}) = 0 \text{ for } m \geq 0.$$

As a consequence, an element t belongs to $\operatorname{im}\lambda$ if and only if

$$\pi_{A^{\{1\}}}\tilde{\delta}^m\varphi^{-1}(t) = 0 \text{ for } m \geq 0.$$

By assumption, this is equivalent to saying that

$$\pi_{B^{\{1\}}}\varphi\tilde{\delta}^m\varphi^{-1}(t) = 0 \text{ for } m \geq 0.$$

Since $\pi_{B^{\{1\}}}\varphi\tilde{\delta}^m\varphi^{-1} = \pi_{B^{\{1\}}}\delta^m$, we are done. □

6 A cluster lemma

The main result of this section is Lemma 6.1, which is a generalization of the so-called Cluster Lemma of discrete Morse theory [7, 9]. The procedure of the lemma is to cut an n-complex into small pieces, apply discrete Morse theory to each piece, and finally glue all pieces together again. Lemma 6.2 comments on the final gluing process, which is by far the most complicated part.

Let (C, δ) be an n-complex, and let $P = (X, \leq_P)$ be a finite or countable partially ordered set. We assume that we may label the elements of X as $x_1, x_2, x_3, \ldots$ such that $i < j$ whenever $x_i < x_j$. This is true for all finite posets.

Suppose that we may write

$$C = \bigoplus_{x \in X} C^x \tag{9}$$

such that

$$\operatorname{im}\delta|_{C^x} \subseteq \bigoplus_{y \leq x} C^y. \tag{10}$$

for each $x \in X$. Define $\delta_x = \pi_x \delta|_{C^x}$, where π_x denotes the projection on C^x with respect to the decomposition (9). By (10), we have that (C^x, δ_x) defines an n-complex.

Lemma 6.1 *Suppose that each C^x admits a decomposition*

$$C^x = B^{\{<\},x} \oplus B^{\{=\},x} \oplus U^x \tag{11}$$

such that the conditions of Theorem 5.3 are satisfied for (C^x, δ_x), assuming the standard case (i.e., $B^r = 0$ for $r < n$ in the terminology of Section 5.2). Define

$$B^{\{<\}} = \bigoplus_{x \in X} B^{\{<\},x}, \quad B^{\{=\}} = \bigoplus_{x \in X} B^{\{=\},x}, \quad U = \bigoplus_{x \in X} U^x.$$

Then the conditions of Theorem 5.3 are satisfied for the decomposition

$$C = B^{\{<\}} \oplus B^{\{=\}} \oplus U. \tag{12}$$

Proof. For simplicity, we assume that $X = \mathbb{Z}_+ = \{1, 2, 3, \ldots\}$, where i is less than j as an integer whenever i is less than j in P. For each $k \in X$, let $(\tilde{C}^k, \tilde{\delta}_k)$ and φ_k be defined in terms of (C^k, δ_k) and the decomposition (11) as described in Section 5.2. In particular, φ_k is an n-complex isomorphism from $(\tilde{C}^k, \tilde{\delta}_k)$ to (C^k, δ_k). Similarly, let $(\tilde{C}, \tilde{\delta})$ and φ be defined in terms of (C, δ) and the decomposition (12). We may identify $\tilde{C}$ with $\bigoplus_{k \geq 1} \tilde{C}^k$. Since we are in the standard case, it suffices to prove that φ defines an isomorphism from $\tilde{C}$ to C.

For each $k \in \mathbb{Z}_+$, observe that φ_k coincides with the restriction of $\pi_k \varphi$ to $\tilde{C}^k$, where π_k is the projection onto C^k with respect to the decomposition (9). Using this fact and (10), we obtain that φ is a monomorphism. Namely, suppose that $\varphi(\tilde{c}) = 0$, where $\tilde{c} = \sum_{k \geq 1} \tilde{c}_k$ and $\tilde{c}_k \in \tilde{C}^k$. Assume that $\tilde{c} \neq 0$, and let k be maximal such that $\tilde{c}_k \neq 0$. By (10), $\pi_k \varphi(\tilde{c}_j) = 0$ for $j < k$, we obtain that

$$0 = \pi_k \varphi(\tilde{c}) = \varphi_k(\tilde{c}_k) + \sum_{j=1}^{k-1} \pi_k \varphi(\tilde{c}_j) = \varphi_k(\tilde{c}_k),$$

which yields that $\tilde{c}_k = 0$, because φ_k is a monomorphism. This is a contradiction; hence $\tilde{c} = 0$.

It remains to prove that φ is an epimorphism. Consider an element

$$c = \sum_{k \geq 1} c_k \in C,$$

where $c_k \in C^k$ for each k. We want to find an element $\tilde{c} \in \tilde{C}$ such that $\varphi(\tilde{c}) = c$.

If $c = 0$, then we may pick $\tilde{c} = 0$. Otherwise, let $i = i(c)$ be maximal such that $c_i \neq 0$. Since φ is an epimorphism, there is then a (unique) element $\tilde{c}_i \in \tilde{C}^i$ such that

$$\pi_i \varphi(\tilde{c}_i) = \varphi_i(\tilde{c}_i) = c_i.$$

Define

$$c' = c - \varphi(\tilde{c}_i).$$

Since $\varphi(\tilde{c}_i) \in \bigoplus_{j\leq i} C^j$ by (10), we have that $\pi_i(c') = 0$, and $\pi_j(c') = \pi_j(c) = 0$ whenever $j > i$. By induction on $i(c)$, there is an element $\tilde{c}'$ such that $\varphi(\tilde{c}') = c'$. As a consequence,

$$\varphi(\tilde{c}_i + \tilde{c}') = \varphi(\tilde{c}_i) + c' = c,$$

which implies that $c = c_i + c'$ has desired properties. □

For each $x \in X$, there is an n-complex map $\lambda_x = \varphi_x \tilde{\lambda}_x \varphi_x^{-1} : \mathcal{C}^x \to \mathcal{C}^x$ associated to the decomposition (11) as described in Section 5.2; see Theorem 5.3. Similarly, there is an n-complex map $\lambda = \varphi\tilde{\lambda}\varphi^{-1} : \mathcal{C} \to \mathcal{C}$ associated to the big decomposition (12). The theory in Section 5.2 gives a formula for computing the n-complex $(\operatorname{im}\lambda, \delta)$ from φ. One may ask whether we may obtain $(\operatorname{im}\lambda, \delta)$ by "gluing" the smaller n-complexes (λ_x, δ_x) in some cunning manner. While this seems hard in general, we do have one result in this direction. In fact, Lemma 6.2 below shows that it suffices to find an approximation U of $\operatorname{im}\lambda$, provided U aligns with each $\operatorname{im}\lambda_x$ in a certain sense.

Lemma 6.2 *Assume that $\mathcal{C} = (C, \delta)$ satisfies* (9) *and* (10)*. For each $x \in X$, let (U^x, δ_x) be an n-subcomplex of (C^x, δ_x) such that $(U^x, \delta_x) \sim (C^x, \delta_x)$ via the inclusion map. Writing $U = \bigoplus_{x\in X} U^x$, suppose that there is a map $\alpha : U \to C$ satisfying the following properties, where we define $\alpha_x = \alpha|_{U^x}$.*

(i) $\delta(\operatorname{im}\alpha) \subseteq \operatorname{im}\alpha$.

(ii) *For each $x \in X$, we have that* $\operatorname{im}\alpha_x \subseteq \bigoplus_{y\leq x} C^y$.

(iii) *For each $x \in X$, we have that $\pi_{C^x}\alpha_x$ is the inclusion map from U_x to C^x. Here, π_{C^x} is the projection with respect to the decomposition $C = \bigoplus_y C^y$.*

Then $(\operatorname{im}\alpha, \delta) \sim \mathcal{C}$ *via the inclusion map.*

Remark. To apply the lemma to the situation in Lemma 6.1, define $U^x = \operatorname{im}\lambda_x$.

Proof. As in the proof of Lemma 6.1, assume that $X = \{1, 2, 3, \ldots\}$, where $i < j$ whenever i is less than j in P. Write $C^{\leq i} = \bigoplus_{j=1}^{i} C^j$.

Observe that α is a monomorphism. Namely, let u be a nonzero element in U, and write $u = \sum_{i\geq 1} u_i$, where $u_i \in U^i$. Let j be maximal such that $u_j \neq 0$. By property (ii), we obtain that $\pi_{C^j}\alpha_i = 0$ for all i such that $i < j$. We deduce that

$$\pi_{C^j}\alpha(u) = \pi_{C^j}\alpha_j(u_j),$$

which equals $u_j \neq 0$ by property (iii). We conclude that $\alpha(u) \neq 0$.

As a consequence, we may write

$$\operatorname{im}\alpha = \bigoplus_{i\geq 1} \operatorname{im}\alpha_i, \tag{13}$$

and α_i defines an isomorphism from U^i to $\operatorname{im}\alpha_i$ for each i. Define $\beta_i : \operatorname{im}\alpha_i \to U^i$ as the inverse of α_i; we have that β_i coincides with the restriction of π_{C^i}

to $\operatorname{im}\alpha_i$. We obtain an isomorphism $\beta : \operatorname{im}\alpha \to U$ by defining $\beta|_{\operatorname{im}\alpha_i} = \beta_i$ and extending linearly. Note that the composition $\alpha\beta$ is the inclusion map from $\operatorname{im}\alpha$ to C.

Next, note that

$$\delta(\operatorname{im}\alpha_j) \subseteq \bigoplus_{i\le j} \operatorname{im}\alpha_i$$

for each j. Namely, let $a \in \operatorname{im}\alpha_j$, and write $\delta(a) = \sum_i \alpha_i(b_i)$, where $b_i \in U^i$. Assuming $\delta(a) \neq 0$, let k be maximal such that $b_k \neq 0$. Then

$$\pi_{C^k}\delta(a) = \pi_{C^k}\alpha_k(b_k) = b_k.$$

Since $a \in \operatorname{im}\alpha_j \subseteq C^{\le j}$, we have that $\delta(a) \in C^{\le j}$, which yields that $k \le j$.

Most importantly, writing $\operatorname{im}\alpha_{\le j} = \bigoplus_{i\le j} \operatorname{im}\alpha_i$, we have that $(\operatorname{im}\alpha_{\le j}, \delta)$ is an n-complex for each j. In fact, we have the following short exact sequence of n-complexes:

$$0 \longrightarrow (\operatorname{im}\alpha_{\le i-1}, \delta) \longrightarrow (\operatorname{im}\alpha_{\le i}, \delta) \xrightarrow{\pi_{C^i}} (U^i, \kappa_i) \longrightarrow 0.$$

This sequence is indeed exact, because π_{C^i} vanishes on $\operatorname{im}\alpha_{\le i-1}$, and the restriction of π_{C^i} to $\operatorname{im}\alpha_i$ coincides with the map β_i, which defines an isomorphism to U^i.

We use induction on i to show that $(\operatorname{im}\alpha_{\le i}, \delta) \sim \mathcal{C}^{\le i}$ via the inclusion map for each $i \ge 1$. Observing that $\{(\operatorname{im}\alpha_{\le i}, \delta) : i \ge 1\}$ and $\{\mathcal{C}^{\le i} : i \ge 1\}$ are direct systems with direct limits $(\operatorname{im}\alpha, \delta)$ and $\mathcal{C}$, respectively, we obtain the desired result; direct limits commute with the homology functor [18, Th. 4.1.7].

The statement is clear for $i = 1$, because $\alpha_{\le 1} = \alpha_1$ is the inclusion map from U^1 to C^1 by (iii). By assumption, this yields that $(\operatorname{im}\alpha_{\le 1}, \delta) \sim \mathcal{C}^{\le 1}$ via the inclusion map.

Assume that $i > 1$. By the above discussion, the following commutative diagram is well-defined and has exact rows, and all maps commute with the boundary maps.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}\alpha_{\le i-1} \arrow[r] \arrow[d, "\iota"'] & \operatorname{im}\alpha_{\le i} \arrow[r, "\pi_{C^i}"] \arrow[d, "\iota"'] & U^i \arrow[r] \arrow[d, "\iota"'] & 0 \\
0 \arrow[r] & C^{\le i-1} \arrow[r] & C^{\le i} \arrow[r, "\pi_{C^i}"] & C^i \arrow[r] & 0
\end{tikzcd}$$

Here, ι denotes inclusion maps. By the discussion in Section 1.2 and standard homology theory (see also Dubois-Violette [3] and Kassel and Wambst [11]), each row gives rise to a long exact sequence of amplitude homology groups, and the maps induced by the vertical maps commute with the maps induced by the horizontal maps. More precisely, we have the following commutative diagram for $1 \le k \le n-1$, where both columns are exact.

$$
\begin{array}{ccc}
H^{(n-k)}(U^i,\delta_i) & \xrightarrow{\cong} & H^{(n-k)}(\mathcal{C}^i) \\
\downarrow & & \downarrow \\
H^{(k)}(\operatorname{im}\alpha_{\leq i-1},\delta) & \xrightarrow{\cong} & H^{(k)}(\mathcal{C}^{\leq i-1}) \\
\downarrow & & \downarrow \\
H^{(k)}(\operatorname{im}\alpha_{\leq i},\delta) & \longrightarrow & H^{(k)}(\mathcal{C}^{\leq i}) \\
\downarrow & & \downarrow \\
H^{(k)}(U^i,\delta_i) & \xrightarrow{\cong} & H^{(k)}(\mathcal{C}^i) \\
\downarrow & & \downarrow \\
H^{(n-k)}(\operatorname{im}\alpha_{\leq i-1},\delta) & \xrightarrow{\cong} & H^{(n-k)}(\mathcal{C}^{\leq i-1})
\end{array}
$$

By assumption and induction, we have the indicated isomorphisms. The Five Lemma yields an isomorphism also in the middle, which concludes the proof. □

7 Illustrating examples

We provide examples to illustrate some methods of this paper. All applications of discrete Morse theory are in the standard case; the group A^{jk} is zero unless $k = n$. In particular, the map λ in (7) is always homotopic to the identity map, and assumption (I) in Section 5.1 is trivially true.

7.1 Baby application of discrete Morse theory

First, we give a simple example to illustrate the details of the generalized discrete Morse theory introduced in Section 5.

Let p be any prime, and view the polynomial ring $\mathbb{Z}_p[x, y]$ as a vector space over $\mathbb{Z}_p$. Define an operator $\hat{\delta}$ on $\mathbb{Z}_p[x, y]$ by $\hat{\delta} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. It is straightforward to prove that $\hat{\delta}^p = 0$.

Let $Q = \mathbb{Z}_p[x, y]/(\mathbb{Z}_p[x] + \mathbb{Z}_p[y])$, and let C be the subspace of Q generated by all monomials $x^a y^b$ such that $1 \leq a \leq p-1$ and $1 \leq b \leq p-1$. Using the substitution $x^a y^b \mapsto x^a/a! \cdot y^b/b!$, we obtain that the p-complex $(C, \hat{\delta})$ is isomorphic to the p-complex (C, δ), where

$$\delta(x^a y^b) = x^{a-1} y^b + x^a y^{b-1}$$

(modulo $\mathbb{Z}_p[x] + \mathbb{Z}_p[y]$). Note that C admits a grading $C = \bigoplus_{i=2}^{2p-2} C_i$, where C_i is the subspace of C generated by monomials $x^a y^b$ such that $a + b = i$.

In this section, we let $p = 3$. In this case, $C = \langle xy, x^2y, xy^2, x^2y^2\rangle$. See Section 7.2 for the general case. Let us use the same notation as in Section 5.2. Define

$$B^{\{=\}} = \langle x^2y^2\rangle, \quad B^{\{<\}} = \langle xy, xy^2\rangle, \quad U = \langle x^2y\rangle.$$

For $1 \le i \le 3$, write $A^i = A^{i3} = B^{\{=\}} \cdot e_i$, where $e_i = e_{i3}$ is a formal variable. Define $\tilde{C} = A^1 \oplus A^2 \oplus A^3 \oplus U$. We have a map $\varphi : \tilde{C} \to C$ given by

$$\begin{aligned} \varphi(x^2y^2 \cdot e_3) &= x^2y^2, \\ \varphi(x^2y^2 \cdot e_2) &= \delta(x^2y^2) = x^2y + xy^2, \\ \varphi(x^2y^2 \cdot e_1) &= \delta^2(x^2y^2) = 2xy = -xy, \\ \varphi(x^2y) &= x^2y. \end{aligned}$$

Clearly, φ is an isomorphism. Writing $f = f_{13} = \pi_{A^1}\tilde{\delta}^2$, note that

$$\begin{aligned} \tilde{\lambda}(x^2y) &= x^2y - g_{13}(x^2y) - g_{23}(x^2y) - g_{33}(x^2y) \\ &= x^2y - \tilde{\delta}^2 f^{-1}\pi_{A^1}(x^2y) - \tilde{\delta} f^{-1}\pi_{A^1}\tilde{\delta}(x^2y) - f^{-1}\pi_{A^1}\tilde{\delta}^2(x^2y) \\ &= x^2y - 0 - \tilde{\delta} f^{-1}\pi_{A^1}(\varphi^{-1}(xy)) - 0 \\ &= x^2y - \tilde{\delta} f^{-1}\pi_{A^1}(-x^2y^2 \cdot e_1) \\ &= x^2y - \tilde{\delta} f^{-1}(-x^2y^2 \cdot e_1) \\ &= x^2y + x^2y^2 \cdot e_2. \end{aligned}$$

As a consequence,

$$\varphi\tilde{\lambda}(x^2y) = \varphi(x^2y + x^2y^2 \cdot e_2) = xy^2 - x^2y.$$

By Theorem 5.3, the above computations imply that the original 3-complex (C, δ) is homotopic to the 3-complex

$$0 \longrightarrow \langle xy^2 - x^2y \rangle \xrightarrow{\delta} 0.$$

In particular, $T_3^{(1)} \cong \mathbb{Z}_3$, and $T_i^{(k)} = 0$ if $(i, k) \neq (3, 1)$.

7.2 Baby application for general p

The computations in Section 7.1 were quite explicit. In this section, we show that it is sometimes possible to determine $\lambda(U)$ without carrying out every detail in the computations. We will make use of the following simple result.

Lemma 7.1 *Let $n \ge 2$, and let $\mathcal{C} = (C, \delta)$ be a $\mathbb{Z}$-graded n-complex. For $i \in \mathbb{Z}$, let $S_i \subseteq C_i$ be such that the restriction of δ^{n-1} to S_i is a monomorphism. Define*

$$A_i = S_i + \delta(S_{i+1}) + \cdots + \delta^{n-1}(S_{i+n-1}).$$

Then

$$A_i \cong S_i \oplus \delta(S_{i+1}) \oplus \cdots \oplus \delta^{n-1}(S_{i+n-1}), \tag{14}$$

and

$$\mathcal{A} : \cdots \xrightarrow{\delta} A_{i+1} \xrightarrow{\delta} A_i \xrightarrow{\delta} A_{i-1} \xrightarrow{\delta} A_{i-2} \xrightarrow{\delta} \cdots$$

forms an n-exact n-subcomplex of $\mathcal{C}$.

Proof. Consider a linear combination

$$a = \sum_{k=0}^{n-1} \delta^k(s_{i+k}) = 0,$$

where $s_{i+k} \in S_{i+k}$. Suppose the combination is nontrivial, and let r be minimal such that $\delta^r(s_{i+r})$ is nonzero. We get that

$$0 = \delta^{n-1-r}(a) = \sum_{k=r}^{n-1} \delta^{n-1-r+k}(s_{i+k}) = \delta^{n-1}(s_{i+k}).$$

Since the restriction of δ^{n-1} to S_{i+k} is a monomorphism, we conclude that $s_{i+k} = 0$, which is a contradiction. Hence the combination is trivial, which implies (14).

To prove n-exactness of $\mathcal{A}$, simply observe that the n-complex splits into n-exact subcomplexes

$$S_i \xrightarrow{\delta} \delta(S_i) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \delta^{n-2}(S_i) \xrightarrow{\delta} \delta^{n-1}(S_i).$$

□

Consider any prime p, and let C be defined as in Section 7.1. Define

$$\begin{aligned} B^{\{=\}} &= \langle x^{p-1}y^j : 2 \le j \le p-1 \rangle, \\ B^{\{<\}} &= \langle x^i y^j : 1 \le i \le p-2, 1 \le j \le p-1 \rangle, \\ U &= \langle x^{p-1}y \rangle. \end{aligned}$$

For $1 \le i \le p$, write $A^i = A^{ip} = B^{\{=\}} \cdot e_i$ where $e_i = e_{ip}$ is a formal variable. Define $\tilde{C} = \bigoplus_{i=1}^p A^i \oplus U$. We have a map $\varphi : \tilde{C} \to C$ given by

$$\begin{aligned} \varphi(b \cdot e_i) &= \delta^{p-i}(b) \quad \text{for } b \cdot e_i \in A^i, \\ \varphi(x^{p-1}y) &= x^{p-1}y. \end{aligned}$$

For the time being, let us assume without proof that φ is an isomorphism. To compute $\tilde{\lambda}(x^{p-1}y)$, note that $\tilde{\delta}\tilde{\lambda}(x^{p-1}y) = 0$, because $\tilde{\delta}\tilde{\lambda}(x^{p-1}y) \in \tilde{\lambda}(U) \cap \tilde{C}_{p-1}$, which is zero. As a consequence, $\delta\varphi\tilde{\lambda}(x^{p-1}y) = 0$. Writing $\varphi\tilde{\lambda}(x^{p-1}y) = \sum_{i=1}^{p-1} \mu_i x^{p-i} y^i$, we obtain that

$$\begin{aligned} \delta\varphi\tilde{\lambda}(x^{p-1}y) &= \mu_1 x^{p-2}y + \sum_{i=2}^{p-2} \mu_i (x^{p-i-1}y^i + x^{p-i}y^{i-1}) + \mu_{p-1} x y^{p-2} \\ &= \sum_{i=1}^{p-2} (\mu_i + \mu_{i+1}) x^{p-i-1} y^i, \end{aligned}$$

which yields that $\mu_i = (-1)^{i-1}\mu_1$. Since $\tilde{\lambda}(x^{p-1}y)$ is nonzero, we deduce that $\varphi\tilde{\lambda}(U)$ is generated by

$$\gamma = \sum_{i=1}^{p-1} (-1)^i x^{p-i} y^i. \tag{15}$$

As a consequence, the original p-complex (C, δ) is homotopic to the p-complex

$$0 \longrightarrow \langle \gamma \rangle \xrightarrow{\delta} 0.$$

In particular, $T_p^{(1)} \cong \mathbb{Z}_p$, and $T_i^{(k)} = 0$ if $(i,k) \neq (p,1)$.

It remains to prove that φ is an isomorphism. This amounts to proving that we may write

$$C = \bigoplus_{j=0}^{p-1} \delta^j (B^{\{=\}}) \oplus \langle x^{p-1} y \rangle.$$

For $p+1 \leq i \leq 2p-2$, define $S_i = \langle x^{p-1} y^{i-p+1} \rangle$. Note that

$$B^{\{=\}} = \bigoplus_{i=p+1}^{2p-2} S_i.$$

In particular, it suffices to prove that

$$C = \bigoplus_{i=p+1}^{2p-2} \bigoplus_{j=0}^{p-1} \delta^j (S_i) \oplus \langle x^{p-1} y \rangle.$$

We have that the restriction of δ^{p-1} to S_i is a monomorphism. Namely, the coefficient of xy^{i-p} in $\delta^{p-1}(x^{p-1}y^{i-p+1})$ is $p-1 \neq 0$. By Lemma 7.1, we may hence view

$$C_i' = \bigoplus_{j=\max\{1,i-p\}}^{\min\{p-2,i-1\}} \delta^{j+p-i}(S_{j+p})$$

as a subspace of C_i. For $i \neq p$, we note that $\dim C_i' = \dim C_i$; thus the two spaces coincide. For $i = p$, we have that $\dim C_i = \dim C_i' + 1 = p-1$. For $0 \leq j \leq p-2$, note that

$$\delta^j (x^{p-1} y^{j+1}) = x^{p-j-1} y^{j+1} + m_j,$$

where m_j is a linear combination of elements $x^{p-r}y^r$ such that $r \leq j$. In particular,

$$C_p = C_p' \oplus \langle x^{p-1} y \rangle,$$

which concludes the proof that φ is an isomorphism.

7.3 A tensor product

The purpose of this section is twofold. We provide an application of Lemma 6.2, and we prove a result that will be of use in Section 7.4.

Given two graded chain complexes $\mathcal{C} = (C, \partial_C)$ and $\mathcal{E} = (E, \partial_E)$, one may define a chain complex structure on the tensor product $C \otimes E$ by

$$\partial(c \otimes e) = \partial_C(c) \otimes e + (-1)^{|c|} c \otimes \partial_E(e),$$

where $|c|$ denotes the degree of c. Similar constructions exist for n-complexes, and a common approach is to replace the factor $(-1)^{|c|}$ with $q^{|c|}$, where q is a primitive nth root of unity; see Kapranov [10] and Dubois-Violette [3]. For simplicity, we consider a more basic construction due to Mayer [13, 14].

Let n be a prime p, and let $R = \mathbb{Z}_p$. Given two p-complexes $\mathcal{C} = (C, \delta)$ and $\mathcal{E} = (E, \epsilon)$ of vector spaces over $\mathbb{Z}_p$, we define a tensor product

$$\mathcal{C} \otimes \mathcal{E} = (C \otimes E, \delta \otimes \epsilon),$$

where

$$(\delta \otimes \epsilon)(c \otimes e) = \delta(c) \otimes e + c \otimes \epsilon(e).$$

Since

$$(\delta \otimes \epsilon)^k (c \otimes e) = \sum_{i=0}^{k} \binom{k}{i} \delta^i(c) \otimes \epsilon^{k-i}(e),$$

we note that the tensor product of two p-complexes is again a p-complex.

From now on, we assume that $\mathcal{C}$ and $\mathcal{E}$ are $\mathbb{Z}$-graded and that $C_i = E_i = 0$ whenever $i < 0$.

Theorem 7.2 *Suppose that $\mathcal{C}' = (C', \delta)$ and $\mathcal{E}' = (E, \epsilon)$ are p-subcomplexes of $\mathcal{C}$ and $\mathcal{E}$, respectively, such that $\mathcal{C}' \sim \mathcal{C}$ and $\mathcal{E}' \sim \mathcal{E}$ via the inclusion map. Then $\mathcal{C}' \otimes \mathcal{E}' \sim \mathcal{C} \otimes \mathcal{E}$ via the inclusion map.*

Proof. Defining $X = \{0, 1, 2, \ldots\}$, and equipping X with the natural order, we have that the decomposition $C \otimes E = \bigoplus_{i \geq 0} C_i \otimes E$ is of the form (9) and satisfies (10).

For $i \geq 0$, define $(\delta \otimes \epsilon)_i : C_i \otimes E \to C_i \otimes E$ by

$$(\delta \otimes \epsilon)_i (c \otimes e) = c \otimes \epsilon(e).$$

It is straightforward to check that $C_i \otimes \mathcal{E}' \sim C_i \otimes \mathcal{E}$ via the inclusion map. Indeed, since we are working over a field, we have that

$$H^{(k)}(C_i \otimes \mathcal{E}') \cong C_i \otimes H^{(k)}(\mathcal{E}') \cong C_i \otimes H^{(k)}(\mathcal{E}) \cong H^{(k)}(C_i \otimes \mathcal{E}). \tag{16}$$

Let α denote the inclusion map from $\mathcal{C} \otimes \mathcal{E}'$ to $\mathcal{C} \otimes \mathcal{E}$. For each i, we note that the restriction of α to $C_i \otimes \mathcal{E}'$ defines an inclusion map to $C_i \otimes \mathcal{E}$. In particular, (ii) and (iii) in Lemma 6.2 are satisfied. Moreover, if $c \in C_i$ and $e \in E'$, then

$$(\delta \otimes \epsilon)(c \otimes e) = \delta(c) \otimes \epsilon + c \otimes \epsilon(e) \subseteq C_{i-1} \otimes E' + C_i \otimes E' \subseteq C \otimes E' = \operatorname{im} \alpha.$$

As a consequence, $(\delta \otimes \epsilon)(\operatorname{im} \alpha) \subseteq \operatorname{im} \alpha$, which means that (i) in Lemma 6.2 is satisfied. By Lemma 6.2, we get that $\mathcal{C} \otimes \mathcal{E}' \sim \mathcal{C} \otimes \mathcal{E}$ via the inclusion map.

By a similar argument, we obtain that $\mathcal{C}' \otimes \mathcal{E}' \sim \mathcal{C} \otimes \mathcal{E}'$ via the inclusion map. □

Remark. When adapting Theorem 7.2 to other tensor products, one should keep in mind that the above proof relies on (16) being true. This is not the case in general if the underlying ring R is not a field. For example, it is not true that $H_*(\mathbb{Z}_2 \otimes \mathcal{E}) \cong \mathbb{Z}_2 \otimes H_*(\mathcal{E})$ for every chain complex $\mathcal{E}$ of $\mathbb{Z}$-modules.

7.4 Mayer's variant of simplicial homology

Mayer [13, 14] examined a p-complex over $\mathbb{Z}_p$ defined in terms of a simplicial complex Σ. Spanier [17] showed that the amplitude homology of the p-complex coincides (up to a degree shift) with the simplicial homology of Σ over $\mathbb{Z}_p$. We use the theory of the present paper to reestablish Spanier's result. Dubois-Violette [3] obtained a similar result for a slightly different construction.

For an integer m, consider the polynomial ring $P_m = \mathbb{Z}_p[x_1, \ldots, x_m]$, viewed as a vector space over $\mathbb{Z}_p$. Define an operator $\hat{\delta} : P_m \to P_m$ by

$$\hat{\delta} = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_m}.$$

We have that $\hat{\delta}^p = 0$. Namely, let $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_m^{a_m}$ be a monomial. For any integers $b_1, \ldots, b_m$ summing to t such that $0 \le b_j \le a_j$, we have that the coefficient of $\mathbf{x}^{\mathbf{a}-\mathbf{b}} = x_1^{a_1-b_1} \cdots x_m^{a_m-b_m}$ in the expansion of $\hat{\delta}^t(\mathbf{x}^{\mathbf{a}})$ is $t! \cdot \prod_{j=1}^m \binom{a_j}{b_j}$.

Now, let Σ be an abstract simplicial complex on the set $\{1, \ldots, m\}$. Equivalently, Σ is a family of subsets of $\{1, \ldots, m\}$ such that $\sigma \in \Sigma$ whenever $\tau \in \Sigma$ for some $\tau \supset \sigma$. Define C^Σ to be the subspace of P_m generated by all monomials $\mathbf{x}^{\mathbf{a}}$ such that the following hold.

- $0 \le a_i \le p - 1$ for $1 \le i \le m$.
- The set $\mathrm{Supp}(\mathbf{x}^{\mathbf{a}}) = \{i : a_i > 0\}$ belongs to Σ.

As in Section 7.1, we may replace the operator $\hat{\delta}$ with the operator $\delta = \sum_{i=1}^m \delta^{(i)}$ on C^Σ defined by

$$\delta^{(i)}(\mathbf{x}^{\mathbf{a}}) = \begin{cases} 1/x_i \cdot \mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_i^{a_i - 1} x_{i+1}^{a_{i+1}} \cdots x_m^{a_m} & \text{if } a_i > 0, \\ 0 & \text{otherwise,} \end{cases}$$

for each monomial $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_m^{a_m}$ in C^Σ. This yields a p-complex $\mathcal{C}^\Sigma = (C^\Sigma, \delta)$ isomorphic to $(C^\Sigma, \hat{\delta})$.

Define $[\emptyset] = 1$, $[k] = x_k^{p-1}$, and

$$\begin{aligned} [k_1, \ldots, k_{2r-1}, k_{2r}] &= \sum_{i=1}^{p-1} (-1)^i \cdot \delta^{i-1}([k_1, \ldots, k_{2r-1}]) \cdot x_{k_{2r}}^i, \\ [k_1, \ldots, k_{2r}, k_{2r+1}] &= [k_1, \ldots, k_{2r}] \cdot x_{k_{2r+1}}^{p-1} \end{aligned} \tag{17}$$

for $r \ge 1$, where $k_i \in \{1, \ldots, m\}$. Note that

$$[k, \ell] = \sum_{i=1}^{p-1} (-1)^i x_k^{p-i} x_\ell^i$$

and

$$\delta([k, \ell]) = x_\ell^{p-1} - x_k^{p-1}. \tag{18}$$

Later on, we will show that (C^{Σ}, δ) has the same amplitude homology as the p-complex described in the following proposition. Let $\tilde{H}_i(\Sigma; \mathbb{Z}_p)$ denote the ith reduced simplicial homology group of Σ.

Proposition 7.3 *for $r \geq 1$. Let V^{Σ} be the subspace of C^{Σ} generated by the following elements:*

- *For each even face $\{k_1, \ldots, k_{2r}\} \in \Sigma$ such that $k_1 < \cdots < k_{2r}$, the element $[k_1, \ldots, k_{2r}]$.*
- *For each odd face $\{k_1, \ldots, k_{2r+1}\} \in \Sigma$ such that $k_1 < \cdots < k_{2r+1}$, the elements $\delta^i([k_1, \ldots, k_{2r+1}])$ where $0 \leq i \leq p-2$.*

Then $\mathcal{V}^{\Sigma} = (V^{\Sigma}, \delta)$ is an p-subcomplex of $\mathcal{C}^{\Sigma}$, and

$$\begin{array}{rcll} T^{(1)}_{pj}(\mathcal{V}^{\Sigma}) & \cong & \tilde{H}_{2j-1}(\Sigma; \mathbb{Z}_p) & \textit{for } j \geq 0, \\ T^{(p-1)}_{pj+p-1}(\mathcal{V}^{\Sigma}) & \cong & \tilde{H}_{2j}(\Sigma; \mathbb{Z}_p) & \textit{for } j \geq 0, \\ T^{(k)}_{i}(\mathcal{V}^{\Sigma}) & = & 0 & \textit{for other values of } (k, i). \end{array}$$

Proof. To show that $\mathcal{V}^{\Sigma}$ is an p-complex, we compute the boundary of each generator. Let $\sigma = \{k_1, \ldots, k_s\}$ be a face of Σ. We use induction on $|\sigma|$ to prove the following statements.

- For even s, we have that $\delta([\sigma]) = \displaystyle\sum_{j=1}^{s} (-1)^{j-1}[\sigma \setminus \{k_j\}]$.
- For odd s, we have that $\delta^{p-1}([\sigma]) = \displaystyle\sum_{j=1}^{s} (-1)^{j-1}[\sigma \setminus \{k_j\}]$.

The statements are clearly true for $s \leq 1$; thus assume that $s \geq 2$. Write $\sigma' = \sigma \setminus \{k_s\}$.

First, assume that $s = 2r$. We obtain that

$$\begin{aligned} \delta([\sigma]) &= \sum_{i=1}^{p-1} (-1)^i \cdot \delta(\delta^{i-1}([\sigma']) \cdot x^i_{k_{2r}}) \\ &= -[\sigma'] + (-1)^{p-1}\delta^{p-1}([\sigma']) \cdot x^{p-1}_{k_{2r}} \\ &= -[\sigma'] + \sum_{j=1}^{2r-1} (-1)^{j-1}[\sigma' \setminus \{k_j\}] \cdot x^{p-1}_{k_{2r}} \\ &= -[\sigma'] + \sum_{j=1}^{2r-1} (-1)^{j-1}[\sigma \setminus \{k_j\}] \\ &= \sum_{j=1}^{2r} (-1)^{j-1}[\sigma \setminus \{k_j\}]. \end{aligned}$$

Here, we use induction on the third line.

Next, assume that $s = 2r-1$. Remember that $\binom{p-1}{i} = (-1)^i$ in $\mathbb{Z}_p$ for $1 \le i \le p-1$. We obtain that

$$\begin{aligned}
\delta^{p-1}([\sigma]) &= \delta^{p-1}\left([\sigma'] \cdot x_{k_{2r-1}}^{p-1}\right) \\
&= \sum_{i=0}^{p-1} \binom{p-1}{i} \delta^i([\sigma']) \cdot x_{k_{2r-1}}^i \\
&= [\sigma'] + \sum_{i=1}^{p-1} (-1)^i \delta^i \left(\sum_{j=1}^{2r-2} (-1)^{j-1} [\sigma' \setminus \{k_j\}] \right) \cdot x_{k_{2r-1}}^i \\
&= [\sigma'] + \sum_{j=1}^{2r-2} (-1)^{j-1} \sum_{i=1}^{p-1} (-1)^i \delta^{i-1}([\sigma' \setminus \{k_j\}]) \cdot x_{k_{2r-1}}^i \\
&= [\sigma'] + \sum_{j=1}^{2r-2} (-1)^{j-1} [\sigma \setminus \{k_j\}] \\
&= \sum_{j=1}^{2r-1} (-1)^{j-1} [\sigma \setminus \{k_j\}].
\end{aligned}$$

Again, we use induction on the third line.

We deduce that $\mathcal{V}^\Sigma$ is a p-complex. It remains to compute the train groups of $\mathcal{V}^\Sigma$.

First, we make a few observations.

(a) We have that δ defines an isomorphism from V_r^Σ to V_{r-1}^Σ whenever $jp - p + 2 \le r \le jp - 1$ for some j.

(b) We may identify V_{jp}^Σ with $\tilde{C}_{2j-1}(\Sigma; \mathbb{Z}_p)$ and V_{jp-i}^Σ with $\tilde{C}_{2j-2}(\Sigma; \mathbb{Z}_p)$ for $1 \le i \le p-1$, identifying $[k_1, \ldots, k_s]$ and $\delta^{i-1}([k_1, \ldots, k_s])$ with the oriented simplex corresponding to the face $k_1 \cdots k_s$. Under this identification, we may identify $\delta^i : V_{jp}^\Sigma \to V_{jp-i}^\Sigma$ with the simplicial boundary map from $\tilde{C}_{j-1}(\Sigma; \mathbb{Z}_p)$ to $\tilde{C}_{j-2}(\Sigma; \mathbb{Z}_p)$, and we may identify $\delta^i : V_{jp+i}^\Sigma \to V_{jp}^\Sigma$ with the simplicial boundary map from $\tilde{C}_j(\Sigma; \mathbb{Z}_p)$ to $\tilde{C}_{j-1}(\Sigma; \mathbb{Z}_p)$,

If $i \bmod p \notin \{0, p-1\}$, then V_i^Σ is generated by boundaries. In particular, $T_i^{(k)} = T_i^{(k)}(\mathcal{V}^\Sigma) = 0$ for all k.

Suppose $i = jp$ for some j. By property (a), we have that $T_{jp}^{(k)} = 0$ for $k > 1$, because $\delta(c) = 0$ if and only if $\delta^i(c) = 0$, where $1 \le i \le p-1$ and $c \in V_{jp}^\Sigma$. By properties (a) and (b), we have that

$$T_{jp}^{(1)} = \frac{V_{jp} \cap \ker \delta}{V_{jp} \cap \delta(\ker \delta^2)} = \frac{V_{jp} \cap \ker \delta}{V_{jp} \cap \delta^{p-1}(\ker \delta^p)} = \frac{V_{jp} \cap \ker \delta}{V_{jp} \cap \operatorname{im} \delta^{p-1}} \cong \tilde{H}_{2j-1}(\Sigma; \mathbb{Z}_p).$$

Finally, let $i = jp-1$ for some j. By property (a), we have that $T^{(k)}_{jp-1} = 0$ for $k < p-1$. By properties (a) and (b), we have that

$$T^{(p-1)}_{jp-1} = \frac{V_{jp-1} \cap \ker \delta^{p-1}}{V_{jp-1} \cap \delta(\ker \delta^{p})} = \frac{V_{jp-1} \cap \ker \delta^{p-1}}{V_{jp-1} \cap \operatorname{im} \delta} \cong \tilde{H}_{2j-2}(\Sigma; \mathbb{Z}_p).$$

This concludes the proof. $\square$

Theorem 7.4 *We have that $\mathcal{V}^{\Sigma} \sim \mathcal{C}^{\Sigma}$ via the inclusion map from $\mathcal{V}^{\Sigma}$ to $\mathcal{C}^{\Sigma}$.*

Proof. For any given face σ of Σ, let C^{σ} be the subspace of C^{Σ} generated by all $\mathbf{x}^{\mathbf{a}}$ such that $\mathrm{Supp}(\mathbf{x}^{\mathbf{a}}) = \sigma$. It is clear that

$$C^{\Sigma} = \bigoplus_{\sigma \in \Sigma} C^{\sigma}. \tag{19}$$

We obtain a p-complex structure on C^{σ} by defining $\delta_{\sigma} : C^{\sigma} \to C^{\sigma}$ as the projection of $\delta|_{C^{\sigma}}$ on C^{σ} with respect to the decomposition (19). Viewing Σ as a partially ordered set under inclusion, we note that the condition (10) in Section 6 is satisfied.

Now, consider a face $\sigma \in \Sigma$.

- If $\sigma = \{k_1, \ldots, k_{2r}\}$, then let $\mathcal{U}^{\sigma} = (U^{\sigma}, \delta_{\sigma})$ be the p-subcomplex of $\mathcal{C}^{\sigma}$ generated by the single element

  $$\gamma_{\sigma} = [k_1, k_2] \cdot [k_3, k_4] \cdots [k_{2r-1}, k_{2r}].$$

  One easily checks that this indeed defines a p-complex; by (18), the product rule yields that $\delta(\gamma_{\sigma})$ lies in $\bigoplus_{\rho \subsetneq \sigma} C^{\rho}$.

- If $\sigma = \{k_1, \ldots, k_{2r+1}\} = \sigma' \cup \{k_{2r+1}\}$, then let $\mathcal{U}^{\sigma} = (U^{\sigma}, \delta_{\sigma})$ be the p-subcomplex generated by the $p-1$ elements

  $$\gamma_{\sigma,i} = \gamma_{\sigma'} \cdot x^{i}_{k_{2r+1}} = [k_1, k_2] \cdot [k_3, k_4] \cdots [k_{2r-1}, k_{2r}] \cdot x^{i}_{k_{2r+1}},$$

  where $1 \leq i \leq p-1$. Again, one easily checks that this indeed defines a p-complex; we have that $\delta_{\sigma}(\gamma_{\sigma,i}) = \gamma_{\sigma,i-1}$ for $2 \leq i \leq p-1$ and $\delta_{\sigma}(\gamma_{\sigma,1}) = 0$.

Lemma 7.5 *For each $\sigma \in \Delta$, we have that $\mathcal{U}^{\sigma} \sim \mathcal{C}^{\sigma}$ via the inclusion map.*

Proof. For $\sigma = \{k_1, \ldots, k_{2r}\}$, note that

$$\mathcal{C}^{\sigma} \cong \mathcal{C}^{\{k_1,k_2\}} \otimes \mathcal{C}^{\{k_3,k_4\}} \otimes \cdots \otimes \mathcal{C}^{\{k_{2r-1},k_{2r}\}}.$$

By the discussion in Section 7.2 and Theorem 5.3, $\mathcal{U}^{\{k_i,k_j\}}$ is homotopic to $\mathcal{C}^{\{k_i,k_j\}}$ via the inclusion map. Applying Theorem 7.2 repeatedly, we deduce that

$$\mathcal{C}^{\sigma} \sim \mathcal{U}^{\{k_1,k_2\}} \otimes \mathcal{U}^{\{k_3,k_4\}} \otimes \cdots \otimes \mathcal{U}^{\{k_{2r-1},k_{2r}\}} \cong \mathcal{U}^{\sigma}$$

via the inclusion map. For $\sigma = \{k_1, \ldots, k_{2r+1}\} = \sigma' \cup \{k_{2r+1}\}$, we have that

$$\mathcal{C}^{\sigma} \cong \mathcal{C}^{\sigma'} \otimes \mathcal{C}^{\{k_{2r+1}\}}.$$

Another application of Theorem 7.2 yields that

$$\mathcal{C}^{\sigma} \sim \mathcal{U}^{\sigma'} \otimes \mathcal{C}^{\{k_{2r+1}\}} \cong \mathcal{U}^{\sigma}$$

via the inclusion map. □

Proceeding with the proof of Theorem 7.4, we want to apply Lemma 6.2. For any σ with an even number of elements, define

$$\alpha_\sigma(\gamma_\sigma) = [\sigma].$$

For any σ with an odd number of elements and $1 \leq i \leq p-1$, define

$$\alpha_\sigma(\gamma_{\sigma,i}) = \delta^{p-1-i}([\sigma]).$$

Extend linearly to a map $\alpha : U^\Sigma \to C^\Sigma$, where $U^\Sigma = \bigoplus_{\sigma \in \Sigma} U^\sigma$. Note that $\operatorname{im}\alpha = V^\Sigma$. We want to prove that Lemma 6.2 applies.

(i) $\delta(\operatorname{im}\alpha) \subseteq \operatorname{im}\alpha$. This follows from Proposition 7.3 and the fact that $\operatorname{im}\alpha = V^\Sigma$.

(ii) $\operatorname{im}\alpha_\sigma \subseteq \bigoplus_{\tau \subseteq \sigma} C^\tau$. This is clearly true.

(iii) $\pi_{C^\sigma}\alpha_\sigma(u) = u$ for all $u \in U^\sigma$. This is straightforward to prove by induction using (17) and (18).

Hence Lemma 6.2 indeed applies, and we are done. □

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