The Rees Product of a Boolean Algebra and a Chain

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Abstract

For a simplicial complex \( \Sigma \) of dimension \( n-1 \), let \( \Gamma(\Sigma) \) be the partially ordered set of nonempty injective words \( \omega_1 \cdots \omega_r \) such that \( \{\omega_1, \ldots, \omega_r\} \) is a face of \( \Sigma \), the order being given by subword inclusion. We prove that the order complex of \( \Gamma(\Sigma) \) has the same Euler characteristic as the order complex of the Rees product of the face poset of \( \Sigma \) and an \( n \)-chain. In particular, we settle a conjecture due to Björner and Welker, stating that the two order complexes are homotopy equivalent. Shareshian and Wachs discovered another proof of the conjecture independently of our work.

1 Introduction

Let \( \Sigma \) be a simplicial complex. The purpose of this note is to prove that the order complexes of two different partially ordered sets, soon to be defined rigorously, have the same Euler characteristic, thereby settling a conjecture due to Björner and Welker [2]. The first poset is the set of injective words \( \omega_1 \cdots \omega_r \) such that \( \{\omega_1, \ldots, \omega_r\} \) is a face of \( \Sigma \). The second poset is the Rees product of the face poset of \( \Sigma \) with a chain of length \( \dim \Sigma + 1 \). By results due to on the one hand Farmer [3] and Jonsson and Welker [6] and on the other hand Björner and Welker [2], both posets are Cohen-Macaulay whenever \( \Sigma \) is Cohen-Macaulay. In particular, the two complexes are homotopy equivalent whenever \( \Sigma \) is Cohen-Macaulay.

First, we define the poset of injective words; we refer the reader to Farmer [3], Björner and Wachs [1], Hanlon and Hersh [5], Reiner and Webb [9], and Jonsson and Welker [6] for more information and further references. A word \( \omega \) over a finite alphabet \( S \) is called injective if no letter appears more than once, meaning that \( \omega = \omega_1 \cdots \omega_r \) for some \( \omega_1, \ldots, \omega_r \in S \) such that \( \omega_i \neq \omega_j \) for \( 1 \leq i < j \leq r \). For \( n = |S| \) we denote by \( \Gamma_n \) the set of all injective words on \( S \). A word \( \omega = \omega_1 \cdots \omega_r \) with \( r \) letters is said to be of length \( r \). A subword of a word \( \omega_1 \cdots \omega_r \) is a word \( \omega_{j_1} \cdots \omega_{j_s} \) such that \( 1 \leq j_1 < \cdots < j_s \leq r \). Clearly, a subword of an injective word is injective. We order \( \Gamma_n \) by saying that \( \rho_1 \cdots \rho_s \leq \omega_1 \cdots \omega_r \) if and only if \( \rho_1 \cdots \rho_s \) is a subword of \( \omega_1 \cdots \omega_r \). We write \( c(w) \) for the content \( \{\omega_1, \ldots, \omega_r\} \) of the word \( w = \omega_1 \cdots \omega_r \).
For a simplicial complex $\Sigma$ on the same vertex set as $\Gamma_n$, we define a subposet $\Gamma(\Sigma)$ of $\Gamma_n$ by restricting to injective words $w \in \Gamma_n$ such that the content $c(w)$ is a face of $\Sigma$. Note that $\Gamma(\Sigma)$ is an order ideal in $\Gamma_n$.

Next, we define the Rees product of two posets, introduced by Björner and Welker [2]. Let $P$ and $Q$ be posets of the same rank $r$. The Rees product of $P$ and $Q$ consists of all pairs $(p,q)$ such that the rank of $p$ is at least the rank of $q$, and we have a covering relation $(p,q) < (p',q')$ exactly when $p'$ covers $p$ in $P$ and either $q = q'$ or $q'$ covers $q$ in $Q$.

For a simplicial complex $\Sigma$, let $P(\Sigma)$ denote the face poset of $\Sigma$; the poset $P(\Sigma)$ consists of all nonempty faces of $\Sigma$ ordered by inclusion. Let $R(\Sigma)$ be the Rees product of $P(\Sigma)$ and the chain $0 < 1 < \cdots < \dim \Sigma$. Note that elements in $R(\Sigma)$ are pairs $(\sigma,k)$ such that $\sigma \in \Sigma$ and $0 \leq k \leq |\sigma| - 1 = \dim \sigma$. For all $\sigma \in \Sigma \setminus \{\emptyset\}$, $k < |\sigma|$, and $a \notin \sigma$ such that $\sigma \cup \{a\} \in \Sigma$, we have the two covering relations $(\sigma,k) < (\sigma \cup \{a\},k)$ and $(\sigma,k) < (\sigma \cup \{a\},k+1)$.

Throughout this note, homology computations are carried out with coefficients in a principal ideal domain $R$. For a poset $P$, let $\Delta(P)$ be the order complex of $P$. For simplicity, we write $\tilde{\chi}(P) = \tilde{\chi}(\Delta(P))$ and $\tilde{H}_*(P;R) = \tilde{H}_*(\Delta(P);R)$.

We refer to the properties of being homotopically Cohen-Macaulay and Cohen-Macaulay over a given principal ideal domain as Cohen-Macaulay properties.

**Proposition 1.1 (Björner and Welker [2])** Let $\Sigma$ be a simplicial complex. If $\Sigma$ has a given Cohen-Macaulay property, then so has $R(\Sigma)$.

**Proposition 1.2 (Jonsson and Welker [6])** Let $\Sigma$ be a simplicial complex. If $\Sigma$ has a given Cohen-Macaulay property, then so has $\Gamma(\Sigma)$.

The following theorem is the main result of this note.

**Theorem 1.3** For any simplicial complex $\Sigma$, we have that

$$\tilde{\chi}(\Gamma(\Sigma)) = \tilde{\chi}(R(\Sigma)).$$

In particular, if $\Sigma$ is Cohen-Macaulay over $R$, then the groups $\tilde{H}_{\dim \Sigma}(\Gamma(\Sigma);R)$ and $\tilde{H}_{\dim \Sigma}(R(\Sigma);R)$ are isomorphic.

We prove Theorem 1.3 in Section 2. For an alternative proof discovered independently, see Shareshian and Wachs [7].

Let $\Sigma_n$ be the full simplex on $n$ vertices. Recall that $\Gamma_n = \Gamma(\Sigma_n)$, and define $R_n = R(\Sigma_n)$.

**Proposition 1.4 (Farmer [3])** $\Delta(\Gamma_n)$ is homotopy equivalent to a wedge of $D_n$ spheres of dimension $n - 1$, where $D_n$ is the derangement number; $D_0 = 1$ and $D_n = nD_{n-1} + (-1)^n$.

As an immediate consequence of Theorem 1.3 and Proposition 1.4, we obtain the following result, which was conjectured by Björner and Welker [2].
Corollary 1.5 We have that \( \tilde{\chi}(\Gamma_n) = \tilde{\chi}(R_n) \). In particular, \( \Delta(R_n) \) is homotopy equivalent to a wedge of \( D_n \) spheres of dimension \( n - 1 \), where \( D_n \) is defined as in Proposition 1.4.

One may examine the Rees product of other interesting posets with the chain. For example, Muldoon and Readdy [8] considered the cubical lattice, whereas Shareshian and Wachs [7] considered the face poset of the boundary of the crosspolytope, which is the dual of the cubical lattice.

2 Proof of Theorem 1.3

For a word \( x = x_1 \cdots x_s \), write \( c(x) = \{x_1, \ldots, x_s\} \) and \( d(x) = |\{i : x_i > x_{i+1}, 1 \leq i \leq s - 1\}| \). Define \( f : \Gamma(\Sigma) \to R(\Sigma) \) by mapping a word \( x = x_1 \cdots x_s \) to the pair \((c(x), d(x))\). To prove that the Euler characteristics coincide, it suffices to show that the order complex of the lower fiber

\[ \Gamma \leq f^{-1}(r) = f^{-1}(\{r' : r' \leq r\}) \]

has a vanishing reduced Euler characteristic for each \( r = (\sigma, k) \in R(\Sigma) \). This approach was suggested by Björner and Welker [2] for the special case that \( \Sigma \) is the full simplex.

Write \( \Gamma_{\sigma,k} = \Gamma \leq f^{-1}(\sigma,k) \) (\( \Sigma \)); this poset does not depend on \( \Sigma \). Note that \( \Gamma_{\sigma,k} \) is the family of all nonempty injective words \( x \) on the set \( \sigma \) such that

\[ |x| - |\sigma| + k \leq d(x) \leq k. \]

Moreover, \( \Gamma_{\sigma,k} \) is the face poset of a boolean complex \( B_{\sigma,k} \). Obviously, the Euler characteristic of \( B_{\sigma,k} \) coincides with that of \( \Delta(\Gamma_{\sigma,k}) \); the complexes are homeomorphic.

To summarize, it suffices to show that the reduced Euler characteristic of \( B_{\sigma,k} \) is zero. We prove this by defining a perfect matching on \( B_{\sigma,k} \) such that each matched pair is of the form \( \{xy, xay\} \) for some words \( x \) and \( y \) and some symbol \( a \); let us refer to such a matching as an element matching. In terms of discrete Morse theory [4], our matching is not acyclic in general. In fact, it cannot be, as the complex \( B_{\sigma,k} \) is not contractible in general; see Björner and Welker [2]. For example, \( B_{3,1} \) is not contractible, where \( B_{m,k} = B_{[m],k} \).

First, note that \( B_{1,0} = \{\phi, 1\} \), \( B_{2,0} = \{\phi, 1, 2, 12\} \), and \( B_{2,1} = \{\phi, 1, 2, 21\} \); these complexes clearly admit perfect element matchings.

Next, assume that \( |\sigma| \geq 3 \). For simplicity, assume that \( \sigma = [m] \). We divide \( B_{m,k} \) into subfamilies as follows:

- Let \( A_{1} \) be the family of words \( x \) in \( B_{m,k} \) not containing \( m \) such that \( d(x) = k \). Let \( A_{2} \) be the family of words \( xm \) in \( B_{m,k} \) such that \( d(x) = k \).

Note that \( d(xm) = d(x) \).
• Let $B_1$ be the family of words $y$ in $B_{m,k}$ not containing $m$ such that $d(y) < k$. Let $B_2$ be the family of words $my$ in $B_{m,k}$ such that $d(y) < k$. Note that $d(my) = d(y) + 1$.

• Let $C$ be the family of words in $B_{m,k}$ not contained in $A_1 \cup A_2 \cup B_1 \cup B_2$.

It is immediate that we have perfect element matchings on $A = A_1 \cup A_2$ (match $x$ and $xm$) and $B = B_1 \cup B_2$ (match $y$ and $my$).

What remains is the family $C$. This family consists of all faces $xmy \in B_{m,k}$ such that $x \neq \phi$ and $d(x) < k$. Namely, $d(x) = k$ would imply that $y = \phi$ and hence that $xm \in A_2$; $x = \phi$ would imply that $my \in B_2$, as $d(y) < d(my) \leq k$.

Now, for each $x$ such that $d(x) < k$ and $xm \in B_{m,k}$, define

$$C(x) = \{xmy : xmy \in C\}.$$ 

For each $xmy \in C(x)$, we have that $[m-1] \setminus c(x) \neq \emptyset$, because if $xm$ is a word of full length $m$ in $B_{m,k}$, then $d(x) = d(xm) = k$ and hence $xm \in A_2$. Also, note that $d(xmy) = d(x) + d(y) + 1$. We conclude that

$$C(x) = \{xmy : y \in B_{[m-1] \setminus c(x), k-d(x)-1}\}.$$ 

By induction on $m = |\sigma|$, $B_{[m-1] \setminus c(x), k-d(x)-1}$ admits a perfect element matching, which implies that the same is true for $C(x)$.

Taking the union of the matchings just defined, we obtain a perfect element matching on $C$. The union of this matching and the above matchings on $A$ and $B$ is a perfect element matching on $B_{m,k}$, which concludes the proof.

The last statement in the theorem is an immediate consequence of the first statement and Propositions 1.1 and 1.2.

References


