HOM COMPLEXES OF SET SYSTEMS

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Abstract. A set system is a pair $S = (V(S), \Delta(S))$, where $\Delta(S)$ is a family of subsets of the set $V(S)$. We refer to the members of $\Delta(S)$ as the stable sets of $S$. A homomorphism between two set systems $S$ and $T$ is a map $f : V(S) \to V(T)$ such that the preimage under $f$ of every stable set of $T$ is a stable set of $S$. Inspired by a recent generalization due to Engström of Lovász’s Hom complex construction, we show how to associate a cell complex $\text{Hom}(S, T)$ to any two set systems $S$ and $T$. The main goal is to examine partitionability of set systems. Specifically, a partition of a set system $S$ is a partition of $V(S)$ into stable sets. A transversal of $S$ is a subset of $V(S)$ such that no two elements in the subset belong to a common stable set. We say that $S$ is partitionable if the size of a minimal partition coincides with the size of a maximal transversal. We show that the topology of the cell complex $\text{Hom}(S, T)$ is related to the partitionability of $T$. Loosely speaking, if $T$ is partitionable, then the homology of $\text{Hom}(S, T)$ must have certain properties. This yields an obstruction theory for partitionability.

1. Introduction

Recall that a graph homomorphism from a graph $G = (V, E)$ to a graph $G' = (V', E')$ is a map $f : V \to V'$ such that $f(x)$ and $f(y)$ are adjacent in $G'$ whenever $x$ and $y$ are adjacent in $G$. Lovász showed how to define a cell complex $\text{Hom}(G, G')$ with the property that the vertices are indexed by the graph homomorphisms from $G$ to $G'$; see Babson and Kozlov [1, 2].

A vertex set in a graph $G$ is independent if no two vertices in the set are joined by an edge. The independence complex $\text{Ind}(G)$ is the simplicial complex of independent sets in $G$. Note that a map $f : V \to V'$ is a graph homomorphism if and only if $f^{-1}(\sigma) \in \text{Ind}(G)$ for each $\sigma \in \text{Ind}(G')$. Inspired by this observation, Engström [7] generalized the construction to arbitrary pairs of simplicial complexes. Specifically, given any abstract simplicial complexes $\Delta$ and $\Delta'$ on the vertex sets $V$ and $V'$, respectively, say that a map $f : V \to V'$ is a homomorphism from $\Delta$ to $\Delta'$ if $f^{-1}(\sigma) \in \Delta$ for each $\sigma \in \Delta'$. This gives rise to a cell complex in which the vertices are indexed by the homomorphisms from $\Delta$ to $\Delta'$. Engström’s construction has interesting connections to Ramsey theory.

In this paper, we take Engström’s generalization one step further, dropping the requirement that $\Delta$ and $\Delta'$ are simplicial complexes. Instead, they can be arbitrary families of subsets of given ground sets $V$ and $V'$, respectively. We refer to the pairs $S = (V, \Delta)$ and $S' = (V', \Delta')$ as set systems. The members of $\Delta$ and $\Delta'$ are the stable sets of $S$ and $S'$, respectively. A straightforward adaptation of Lovász’ construction yields a cell complex $\text{Hom}(S, S')$ with one vertex for each
homomorphism from $S$ to $S'$. We discuss set systems in Section 2 and introduce the associated cell complexes in Section 5.

The main motivation is to introduce new tools for examining interval partitions of partially ordered sets (posets). For a given poset $P = (V, \leq)$, let $I(P)$ denote the family of closed intervals in $P$; see Section 2 for details. By convention, the empty set is an interval. We refer to the set system $I(P) = (V, I(P))$ as the interval system of $P$. For $n \geq 1$, let $A_n$ be the antichain on the set $\{1, \ldots, n\}$. There is a bijective correspondence between ordered partitions of $P$ into $n$ closed intervals and homomorphisms from the interval system of $P$ to the discrete system

\begin{equation}
D_n = (\{1, \ldots, n\}, I(A_n)) = (\{1, \ldots, n\}, \{\emptyset, \{1\}, \ldots, \{n\}\}).
\end{equation}

Specifically, a given partition $(\sigma_1, \ldots, \sigma_n)$ corresponds to the homomorphism $f$ defined by $f^{-1}(\{k\}) = \sigma_k$ for $k \in \{1, \ldots, n\}$. This is because the nonempty intervals in $A_n$ are the singleton sets $\{1\}, \ldots, \{n\}$.

We are particularly interested in finite simplicial complexes ordered by inclusion. For a given simplicial complex $\Lambda$, an important question is whether $\Lambda$ admits a partition into intervals such that each interval contains a maximal face. We say that $\Lambda$ is partitionable if it does admit such a partition. Stanley [19, Conjecture 2.7] conjectured that $\Lambda$ is partitionable whenever $\Lambda$ is a Cohen-Macaulay complex. We generalize the concept of partitionability to arbitrary set systems, saying that a set system $S$ is partitionable if there is an integer $n$ such that there are homomorphisms both from $D_n$ to $S$ and from $S$ back to $D_n$, where $D_n$ is the discrete system defined in (1). See Section 3 for details.

Let $\mathcal{X}$ and $\mathcal{S}$ be set systems, and assume that $m$ is maximal such that there is a homomorphism from $D_m$ to $\mathcal{S}$. Any homomorphism from $D_m$ to $\mathcal{S}$ gives rise to a cellular map from $\text{Hom}(\mathcal{X}, D_m)$ to $\text{Hom}(\mathcal{X}, \mathcal{S})$. In Section 5, we show that the induced map in reduced homology is a monomorphism whenever $\mathcal{S}$ is partitionable. This yields a necessary topological condition for a set system to be partitionable. In particular, we may use the result to show that a given set system is non-partitionable: if the homology map is not a monomorphism, then $\mathcal{S}$ is not partitionable. We say that $\mathcal{X}$ is a witness for the non-partitionability of $\mathcal{S}$.

The result has limitations. For the special case that $\mathcal{X}$ is monotone (the stable sets of $\mathcal{X}$ form a simplicial complex) and $\mathcal{S}$ is the interval system of a simplicial complex, it turns out that the above homology map is always a monomorphism. As a consequence, we cannot use monotone systems as witnesses in this case. See Section 6.2 for more information.

For this reason, much of our focus will be on the case that $\mathcal{X}$ is a non-monotone set system. A particularly nice example is the interval system $I(L_3)$ of the 3-chain $L_3$. This system benefits from having a quite simple structure, allowing for a detailed analysis of the associated Hom complexes. The following list summarizes some of the main properties of $I(L_3)$; see Sections 7, 8, and 9 for details.

1. The complex $\text{Hom}(I(L_3), D_n)$ is homotopy equivalent to a wedge of $n - 1$ spheres of dimension $n - 1$.
2. The system $I(L_3)$ is a witness for the non-partitionability of any simplicial complex containing a link equal to the disjoint union of two edges.
3. The system $I(L_3)$ is not a witness for the non-partitionability of a large class of simplicial complexes, including all Cohen-Macaulay complexes.
The above results do not really shed much light on the partitionability of set systems. The second result does imply that simplicial complexes with the given structure are not partitionable, but this is easy to prove directly. The third result may seem to provide some evidence for Stanley’s conjecture, but the result also holds for many complexes that are known to be non-partitionable.

What is indeed good news is rather the existence of explicit results for Hom complexes involving quite complicated set systems such as interval systems of simplicial complexes. An important goal for the future is to find alternatives to \( I(L_3) \) that provide better witnesses for non-partitionability, while still admitting explicit homology computations. In Section 10, we discuss two set systems and show that both systems detect the non-partitionability of the disjoint union of two edges. It seems worth exploring whether these systems can be used to obtain nontrivial results about non-partitionable set systems.

In Section 11, we round up the paper with a discussion on the categorical product and coproduct of set systems. The coproduct turns out to be similar to the join operation on simplicial complexes. The exact shape of the product depends on whether we consider the full category of all set systems or the subcategory of monotone systems.

2. SET SYSTEMS AND HOMOMORPHISMS

We define a set system to be a pair \( S = (V(S), \Delta(S)) \), where \( \Delta(S) \) is a family of subsets of the nonempty set \( V(S) \). We refer to \( V(S) \) as the ground set and to the members of \( \Delta(S) \) as the stable sets of \( S \). Following Engström’s approach [7], define a homomorphism between two set systems \( S \) and \( T \) to be a map \( f : V(S) \to V(T) \) such that \( f^{-1}(\tau) \in \Delta(S) \) for every \( \tau \in \Delta(T) \). In words, the preimage under \( f \) of every stable set of \( T \) is a stable set of \( S \). Let \( \text{Hom}_0(S,T) \) denote the set of homomorphisms from \( S \) to \( T \).

**Lemma 2.1.** We have that \( gf \in \text{Hom}_0(S,U) \) whenever \( f \in \text{Hom}_0(S,T) \) and \( g \in \text{Hom}_0(T,U) \).

**Proof.** Let \( \sigma \) be a stable set of \( U \). Then \( g^{-1}(\sigma) \) is stable in \( T \), because \( g \) is a homomorphism, and \( f^{-1}(g^{-1}(\sigma)) = (gf)^{-1}(\sigma) \) is stable in \( S \), because \( f \) is a homomorphism. \( \square \)

Let us list some basic set systems. Throughout the list, \( S \) denotes an arbitrary set system.

- To every nonempty set \( U \), we may associate the void system \((U, \emptyset)\). Every map \( f : V(S) \to U \) defines a homomorphism from \( S \) to \((U, \emptyset)\). There are no homomorphisms from \((U, \emptyset)\) to \( S \) unless \( S \) is also void.
- We also have the full system \((U, 2^U)\). Every map \( f : U \to V(S) \) defines a homomorphism from \((U, 2^U)\) to \( S \).
- Of particular importance is the discrete system \( D_n \) defined in (1). A map \( f : \{1, \ldots, n\} \to V(S) \) is a homomorphism from \( D_n \) to \( S \) if and only if every stable set of \( S \) contains at most one element from the image of \( f \). This is equivalent to saying that \( S \) admits an \( n \)-transversal; see Section 3. In the particular case \( S = D_n \), we have that \( f \) defines a homomorphism if and only if \( f \) is injective. A map \( f : V(S) \to \{1, \ldots, n\} \) is a homomorphism from \( S \) to \( D_n \) if and only if the preimage of every \( k \in \{1, \ldots, n\} \) is a stable
set of $S$. This is equivalent to saying that $S$ admits an $n$-partition; see Section 3.

We proceed with some interesting classes of set systems and homomorphisms.

- Let $G = (V, E)$ be a graph. The independence system of $G$ is the set system $(V, \text{Ind}(G))$, where $\text{Ind}(G)$ is the independence complex of $G$. As mentioned, a homomorphism between the independence systems of two graphs $G$ and $H$ is the same as a graph homomorphism between $G$ and $H$. Note that the discrete system $D_n$ is the independence system of the complete graph $K_n$.

- A set system $S = (V, \Delta)$ is monotone if $\Delta$ is a simplicial complex. This is equivalent to saying that $\rho \in \Delta$ whenever there exists some $\sigma \in \Delta$ such that $\rho \subset \sigma$. This is the class of set systems that Engström [7] examined.

- For the motivating example of this paper, let $P = (V(P), \leq_P)$ be a finite poset on the ground set $V(P)$. For any elements $a, b \in V(P)$ such that $a \leq_P b$, define

$$[a, b] = \{x \in V(P) : a \leq_P x \leq_P b\}.$$ 

This is the (closed) interval in $P$ between $a$ and $b$. Define the interval system of $P$ to be the set system $I(P) = (V(P), I(P))$, where

$$I(P) = \{\emptyset\} \cup \{[a, b] : a \leq_P b\}. \quad (2)$$

As already alluded to, $D_n$ is the interval system of the antichain $A_n$. Interval systems are typically not monotone. For example, if $P$ is the chain $x \leq y \leq z$, then $I(P)$ contains $\{x, y, z\}$ but not $\{x, z\}$.

- An important variant of the interval system of a poset $P$ is the reduced interval system of $P$, which we denote by $I^\text{red}(P)$. This is the set system on $V(P)$ consisting of the empty set and all intervals of $P$ that contain a coatom. By a coatom, we mean an element $x \in V(P)$ such that no element $y \neq x$ satisfies $x \leq_P y$.

3. PARTITIONS AND TRANSVERSALS

An interval partition of a poset $P$ is an ordered partition $(I_1, \ldots, I_n)$ of $V(P)$ such that each $I_j$ is a closed interval or empty. Equivalently, each $I_j$ is stable in the interval system $I(P)$. In particular, the existence of an interval partition of size $n$ is equivalent to the existence of a homomorphism $f \in \text{Hom}_0(I(P), D_n)$. Namely, such a homomorphism $f$ induces the interval partition $(f^{-1}(1), \ldots, f^{-1}(n))$.

More generally, we define an $n$-partition of a set system $S$ to be a homomorphism $f \in \text{Hom}_0(S, D_n)$. Define $\chi(S)$ to be the smallest $n \geq 1$ such that $S$ admits an $n$-partition. By convention, $\chi(S) = \infty$ if there is no $n$-partition for any $n$.

Dually, an $m$-transversal of $S$ is a homomorphism $f \in \text{Hom}_0(D_m, S)$. Equivalently, an $m$-transversal $f$ has the property that $f^{-1}(\sigma)$ consists of at most one element for each stable set $\sigma$. We identify an $m$-transversal $f$ with the sequence $(f(1), \ldots, f(m))$. Define $\omega(S)$ to be the greatest $m \geq 1$ such that $S$ admits an $m$-transversal. Note that $\omega(S) \geq 1$ for any $S$, because $D_1$ is a full system.

3.1. Basic properties of partitions and transversals. We state and prove some very basic facts about the numbers $\chi(S)$ and $\omega(S)$ for a given set system $S$.

**Proposition 3.1.** For any set system $S$, we have that $\omega(S) \leq \chi(S)$. 

Proof. Suppose that \( f \in \text{Hom}_0(D_m, S) \) and \( g \in \text{Hom}_0(S, D_n) \). Then we have that \( gf \in \text{Hom}_0(D_m, D_n) \) by Lemma 2.1. Since every homomorphism from \( D_m \) to \( D_n \) is injective, we deduce that \( m \leq n \). \( \square \)

**Proposition 3.2.** Let \( S \) and \( T \) be set systems. If \( \text{Hom}_0(S, T) \) is nonempty, then \( \omega(S) \leq \omega(T) \) and \( \chi(S) \leq \chi(T) \).

Proof. Let \( g \in \text{Hom}_0(S, T) \). If \( h \in \text{Hom}_0(T, D_n) \), then Lemma 2.1 implies that \( hg \in \text{Hom}_0(S, D_n) \); hence \( S \) admits an \( m \)-partition whenever \( T \) does. Analogously, if \( f \in \text{Hom}_0(D_m, S) \), then \( gf \in \text{Hom}_0(D_m, T) \); hence \( T \) admits an \( m \)-transversal whenever \( S \) does. \( \square \)

We say that \( S \) is partitionable if \( \omega(S) = \chi(S) \). For example, suppose that the stable sets of \( S \) are the independent sets in a given graph \( G \). Then \( \omega(S) \) is the size of a largest clique in \( G \), whereas \( \chi(S) \) is the chromatic number of \( G \). In particular, \( S \) is partitionable if and only if the clique number and the chromatic number of \( G \) coincide. An important problem is to classify partitionable set systems. In Section 3.2, we look more closely on the motivating case that the stable sets are intervals in a given poset.

We proceed with a partial characterization of set systems \( S \) such that \( \omega(S) \) is finite. Refer to a set system \( S \) as free if each \( v \in V(S) \) is contained in some stable set of \( S \). See Section 6.1 for some motivation for this terminology.

**Proposition 3.3.** Let \( S \) be a set system. If \( \omega(S) < \infty \), then \( S \) is free. The converse is true if \( |V(S)| < \infty \).

Proof. Suppose that \( S \) is not free. Let \( v \in V(S) \) be such that no stable set of \( S \) contains \( v \). Then the constant map \( k \mapsto v \) defines a homomorphism from \( D_m \) to \( S \) for any choice of \( m \). Conversely, suppose that \( S \) is free, and let \( f \in \text{Hom}_0(D_m, S) \). Then \( f \) is injective, because each \( v \in V(S) \) belongs to a stable set, whose preimage has size at most one. We conclude that \( m \) cannot exceed the cardinality of \( V(S) \). \( \square \)

Remark. One may generalize the theory by introducing infinite discrete systems, defining \( \omega(S) \) to be the smallest cardinal number \( c \) such that there is a homomorphism from the discrete system of cardinality \( c \) to \( S \), and defining \( \chi(S) \) analogously. We leave the details to the interested reader.

### 3.2. Interval maps between posets.

Let \( P = (V(P), \leq_P) \) be a finite poset, and consider the interval system \( \mathcal{I}(P) = (V(P), \mathcal{I}(P)) \) of a poset defined as in (2). Given two posets \( P \) and \( Q \), an interval map \( f \) from \( P \) to \( Q \) is a homomorphism from \( \mathcal{I}(P) \) to \( \mathcal{I}(Q) \). Hence, for every \( a' \leq_Q b' \) in \( Q \), we either have that \( f^{-1}([a', b']) \) is empty or that

\[
    f^{-1}([a', b']) = [a, b]
\]

for some \( a \leq_P b \) in \( P \).

It is very important to note that an interval map \( f \) does not need to be order-preserving or order-reversing. In fact, interval maps typically do not have either of these properties.

Let \( \text{Int}_0(P, Q) \) denote the set of interval maps from \( P \) to \( Q \); thus \( \text{Int}_0(P, Q) = \text{Hom}_0(\mathcal{I}(P), \mathcal{I}(Q)) \). By Lemma 2.1, we have that \( gf \in \text{Int}_0(P, R) \) whenever \( f \in \text{Int}_0(P, Q) \) and \( g \in \text{Int}_0(Q, R) \).

Let \( A_n \) denote the antichain on the set \( \{1, \ldots, n\} \). Recall that \( \mathcal{I}(A_n) \) coincides with the discrete system \( D_n \). Write \( \chi(P) = \chi(\mathcal{I}(P)) \); this is the minimum value
such that $\text{Int}_0(P, A_n)$ is nonempty. Moreover, write $\omega(P) = \omega(I(P))$: this is the maximum value $m$ such that $\text{Int}_0(A_m, P)$ is nonempty. By Proposition 3.1, we have that $\omega(P) \leq \chi(P)$. We say that $P$ is partitionable if $\omega(P) = \chi(P)$. This is equivalent to saying that the set system $I(P)$ is partitionable.

A poset $P$ is rooted if there exists a unique minimal element in $P$. Such an element $0$ has the property that $0 \leq P x$ for all $x$ in $P$.

**Proposition 3.4.** If $P$ is a rooted poset, then $\omega(P)$ equals the number of coatoms of $P$.

**Proof.** Let $c_1, \ldots, c_m$ be the coatoms of $P$. We obtain an interval map from $A_m$ to $P$ by defining $f(k) = c_k$ for each $k$. To see that this is indeed an interval map, note that any interval $\sigma$ in $P$ contains at most one coatom; hence $f^{-1}(\sigma)$ contains at most one element and is therefore stable in $A_m$. Conversely, suppose that we have an interval map from $A_n$ to $P$ for some $n$. Then $f^{-1}([0, c_i])$ contains at most one element for each $i$, where $0$ is the minimal element in $P$. Since $V(P)$ is the union of the intervals $[0, c_1], \ldots, [0, c_m]$, we deduce that $f^{-1}(V(P))$ contains at most $m$ elements; hence $n \leq m$. □

**Corollary 3.5.** Let $P$ be a rooted poset with minimal element $0$ and $m$ coatoms $c_1, \ldots, c_m$. Then $P$ is partitionable if and only if there exists an interval map $f$ from $P$ to $A_m$ and elements $b_1, \ldots, b_m$ in $P$ such that $b_k \leq_P c_k$ and $f^{-1}(k) = [b_k, c_k]$ for each $k$.

In Section 2, we introduced the reduced interval system $I_{\text{red}}(P)$ of a poset $P$. In Section 8, we will see that the reduced interval system is sometimes easier to handle than the full interval system.

**Corollary 3.6.** Let $P$ be a rooted poset. Then $I(P)$ is partitionable if and only if $I_{\text{red}}(P)$ is partitionable.

4. Some concepts from topological combinatorics

After having discussed basic properties of set systems and homomorphisms, we want to associate a cell complex to any given pair of set systems. Before doing that, we need to introduce some concepts from topological combinatorics. We also discuss some topological methods that will prove useful in later sections. We refer to Björner [3] for more information on the topic.

In this paper, all cell complexes are combinatorially defined. For example, a common situation is that each cell is indexed by a pair of sets $(X, Y)$, and the cells in the boundary of the cell are indexed by the pairs $(X_0, Y_0)$ satisfying $X_0 \subseteq X$ and $Y_0 \subseteq Y$. For convenience, we identify a cell complex with the family of
combinatorial objects indexing its cells. It will always be clear from context how the cells are glued together.

For the purposes of this paper, a cell complex always has a finite number of cells, and the boundary of each cell is a union of lower-dimensional cells. We refer to an $i$-dimensional cell as an $i$-cell. We will find it convenient to include a virtual $(-1)$-cell in each cell complex. This is mainly because we will consider augmented cellular chain complexes and reduced homology, which corresponds to adding such a cell. Most of our cell complexes are regular, meaning that all attaching maps are homeomorphisms. We will also encounter quotients of regular complexes by subcomplexes.

We will use various techniques to establish homotopy equivalences between different cell complexes. The following useful result is known as the Contractible Subcomplex Lemma; see Hatcher [10, Prop. 0.17].

**Lemma 4.1.** Let $\Gamma$ be a cell complex, and let $\Gamma_0$ be a contractible subcomplex. Then $\Gamma \simeq \Gamma / \Gamma_0$.

Recall that one may define the suspension $S\Gamma = S_{x,y}\Gamma$ of $\Gamma$ as the quotient of $\Gamma \times [-1,1]$ by collapsing $\Gamma \times \{-1\}$ to one point $x$ and $\Gamma \times \{1\}$ to another point $y$. Interpreting $[-1,1]$ as a cell complex with 0-cells at $-1$, 0, and 1 joined by the 1-cells $(-1,0)$ and $(0,1)$, we may view the suspension as a cell complex. Indeed, if $\Gamma$ is regular, then so is $S\Gamma$. We identify $\Gamma$ with the subcomplex $\Gamma \times \{0\}$ of $S\Gamma$.

The cone $C\Gamma = C_{x}\Gamma$ of $\Gamma$ is the quotient of $\Gamma \times [0,1]$ by collapsing $\Gamma \times \{1\}$ to a point $x$; this is a subcomplex of $S\Gamma$.

**Lemma 4.2.** Let $\Gamma$ be a contractible cell complex, and let $\Gamma_0$ be a subcomplex. Then $\Gamma / \Gamma_0 \simeq S\Gamma_0$.

**Proof.** By the Contractible Subcomplex Lemma 4.1, we have that

$$\frac{\Gamma}{\Gamma_0} \cong \frac{\Gamma \cup C_x\Gamma_0}{C_x\Gamma_0} \cong \Gamma \cup C_x\Gamma_0.$$

Further applications of the same lemma yield that

$$\Gamma \cup C_x\Gamma_0 \cong \frac{\Gamma \cup C_x\Gamma_0}{\Gamma} \cong \frac{C_x\Gamma_0}{\Gamma_0} \cong \frac{S_{x,y}\Gamma_0}{C_y\Gamma_0} \cong S_{x,y}\Gamma_0,$$

and we are done. \qed

Forman’s discrete Morse theory [9] is a useful technique for establishing homotopy equivalences between cell complexes. In this paper, we restrict our attention to the special case of collapses.

Let $F$ be a family of cells in a cell complex $\Gamma$, and assume that we may partition $F$ into pairs of cells, each of the form $(\sigma, \tau)$, where $\sigma$ is a face of $\tau$ of codimension one. Suppose that we may order the pairs as $(\sigma_1, \tau_1), \ldots, (\sigma_r, \tau_r)$ such that $\sigma_i$ is not contained in $\tau_j$ unless $i \leq j$. Then we refer to the partition as a perfect acyclic matching on $F$ and an acyclic matching on $\Gamma$.

**Lemma 4.3.** Let $\Gamma_0$ be a subcomplex of a cell complex $\Gamma$, and assume that there exists a perfect acyclic matching on $\Gamma \setminus \Gamma_0$. Then $\Gamma$ admits a collapse to $\Gamma_0$. In particular, $\Gamma \simeq \Gamma_0$.

**Proof.** Let $(\sigma_1, \tau_1), \ldots, (\sigma_r, \tau_r)$ be the pairs in the matching, ordered as described above. By assumption, $\sigma_r$ is a free face of $\tau_r$ in $\Gamma$, meaning that $\tau_r$ is the only cell
containing \( \sigma_r \). In particular, we may perform an elementary collapse, removing the two cells \( \sigma_r \) and \( \tau_r \). By an induction argument, the resulting cell complex admits a collapse to \( \Gamma_0 \). \( \square \)

We say that \( \Gamma \) is collapsible if \( \Gamma \) admits a collapse to a simplex (and hence to a single point). Collapsible complexes are contractible.

As already alluded to, a common situation in this paper is that the nonempty cells of the complex are indexed by pairs \((X, Y)\) of sets. Suppose that we are given a matching in which each pair is of the form \((X \setminus \{x\}, Y)\), \((X \cup \{x\}, Y)\), where \( x \) is a fixed element. Then the matching is acyclic; we leave the proof to the reader.

Given two posets \( P = (V(P), \leq_P) \) and \( Q = (V(Q), \leq_Q) \), a poset map \( f : P \to Q \) is a map \( f : V(P) \to V(Q) \) such that \( f(x) \leq_Q f(y) \) whenever \( x \leq_P y \). Using the following lemma, one may easily construct more complicated acyclic matchings; see Hersh [11] or Jonsson [12] for a proof.

**Lemma 4.4** (Cluster Lemma). Let \( F \) be a family of cells in a cell complex ordered by inclusion, and let \( f : F \to Q \) be a poset map, where \( Q \) is an arbitrary poset. For \( q \in Q \), let \( M_q \) be a perfect acyclic matching on \( f^{-1}(q) \). Let

\[
M = \bigcup_{q \in Q} M_q.
\]

Then \( M \) is a perfect acyclic matching on \( F \).

The face poset \( P(\Gamma) \) of a cell complex \( \Gamma \) is the poset consisting of all nonempty cells of \( \Gamma \) ordered by inclusion. The order complex \( \Delta(P) \) of a poset \( P \) is the simplicial complex consisting of all chains of \( P \). If \( \Gamma \) is a regular cell complex, then \( \Delta(P(\Gamma)) \) is the barycentric subdivision of \( \Gamma \), and the two complexes are homeomorphic.

Let \( P \) be a poset. Given \( x \in V(P) \), we let \( P^{\leq x} \) denote the set of all \( z \in V(P) \) such that \( z \leq x \). The following result is known as Quillen’s Fiber Lemma [17].

**Lemma 4.5.** Let \( f : P \to Q \) be a poset map such that the fiber \( \Delta(f^{-1}(Q^{\leq y})) \) is contractible for each \( y \in V(Q) \). Then \( \Delta(P) \) and \( \Delta(Q) \) are homotopy equivalent.

In this paper, the typical situation is that \( P = P(\Gamma) \) for some regular cell complex \( \Gamma \). In this case, \( f^{-1}(Q^{\leq y}) \) coincides with the face poset \( P(\Gamma_y) \) of some subcomplex \( \Gamma_y \) of \( \Gamma \), and \( \Delta(f^{-1}(Q^{\leq y})) = \Delta(P(\Gamma_y)) \cong \Gamma_y \).

**Corollary 4.6.** Let \( \Gamma \) be a regular cell complex, and let \( f : P(\Gamma) \to Q \) be a poset map such that \( P(\Gamma_y) \) is contractible for each \( y \in V(Q) \), where \( P(\Gamma_y) = f^{-1}(Q^{\leq y}) \). Then \( \Gamma \) and \( \Delta(Q) \) are homotopy equivalent. In particular, if \( Q \) is the face poset of the regular cell complex \( \Sigma \), then \( \Gamma \) and \( \Sigma \) are homotopy equivalent.

## 5. Hom complexes of general set systems

As already mentioned, Lovász introduced a cell complex \( \text{Hom}(G, H) \) with one vertex for each graph homomorphism from \( G \) to \( H \). In the language of the present paper, the vertices are indexed by homomorphisms between the independence systems of \( G \) and \( H \). Engström [7] generalized the construction to monotone systems, and the construction turns out to be straightforward to extend to any pair of set systems.

Let \( S \) and \( T \) be two set systems. We define \( \text{Hom}(S, T) \) to be a cell complex in which each nonempty cell is indexed by a set function \( F : V(S) \to 2^{V(T)} \setminus \{\emptyset\} \) with the following property.
We have that \( f \in \text{Hom}_0(S, T) \) for every map \( f : V(S) \to V(T) \) such that \( f(x) \in F(x) \) for each \( x \in V(S) \).

We write \( f \in F \) if \( f(x) \in F(x) \) for each \( x \in V(S) \). By convention, the empty \((-1)\)-dimensional cell \( \emptyset \) is always part of \( \text{Hom}(S, T) \).

Assuming \( V(S) = \{v_1, \ldots, v_m\} \), the closure of the cell indexed by \( F \) is a product \( \sigma_1 \times \cdots \times \sigma_n \), where \( \sigma_i \) is an \(|F(v_i)| - 1\)-simplex. The faces of codimension one of the cell indexed by \( F \) are those cells that are indexed by set functions obtained by removing one element from one of the sets \( F(v_i) \). Note that we do not obtain a cell when removing an element from a set with only one element. The dimension of a cell is given by \( \sum_{x \in V(S)} |F(x)| - |V(S)| \). See Babson and Kozlov [2] for more information about Hom complexes in the particular case of independence systems of graphs.

By Lemma 2.1 and the fact that composition of homomorphisms is associative, we may define a category \textit{Systems} in which objects are set systems and morphisms are set system homomorphisms. Let \textit{Top} be the category of topological spaces in which morphisms are continuous maps. By the following result, we have a covariant functor \( \text{Hom}(S, -) \) from \textit{Systems} to \textit{Top} for each set system \( S \).

\textbf{Proposition 5.1.} For any set systems \( S, T, U \) and for any homomorphism \( h \in \text{Hom}_0(T, U) \), we obtain a cellular map

\[ h_* : \text{Hom}(S, T) \to \text{Hom}(S, U) \]

by mapping the cell in \( \text{Hom}(S, T) \) indexed by \( F \) to the cell in \( \text{Hom}(S, U) \) indexed by \( hF \).

\textit{Proof.} Suppose that \( g \) is a map from \( S \) to \( U \) such that \( g \in hF \). This means that \( g(x) = h(\alpha_x) \) for some \( \alpha_x \in F(x) \) for each \( x \in V(S) \). By definition, we have that \( f \in F \), where \( f \) is the map defined by \( f(x) = \alpha_x \). As a consequence, \( f \in \text{Hom}_0(S, T) \), which yields that \( g = hf \in \text{Hom}_0(S, U) \). \( \square \)

We will be interested in the reduced cellular homology of \( \text{Hom}(S, T) \) with coefficients in \( \mathbb{Z} \). For this, we need to fix notation and agree on some conventions regarding the associated chain groups and boundary maps. For \( j \geq 0 \), we let \( C_j[S, T] \) denote the cellular chain group of degree \( j \) of \( \text{Hom}(S, T) \). This is a \( \mathbb{Z} \)-module with one generator, an \textit{oriented} \( j \)-cell, for each \( j \)-cell of \( \text{Hom}(S, T) \). A given cell \( F \) is a direct product of simplices, and the corresponding oriented cell is the tensor product of the associated oriented simplices. For example, if \( V(S) = \{1, 2, 3\} \) and \( (F(1), F(2), F(3)) = (\{a, b, c\}, \{d\}, \{e, f\}) \), then the corresponding oriented cell is \( a \wedge b \wedge c \otimes d \otimes e \wedge f \). We use the shorthand notation \( \sigma_1 \sigma_2 \cdots \sigma_k \) to denote the oriented cell \( \sigma_1 \sigma_2 \cdots \sigma_k \). The boundary map \( \partial \) is defined on a generator \( \sigma_1 \sigma_2 \cdots \sigma_k \) by the recursive rule

\[ \partial(\sigma_1 \sigma_2 \cdots \sigma_k) = \partial(\sigma_1) |\sigma_2| \cdots |\sigma_k| + (-1)^{j+1} \sigma_1 \partial(\sigma_2 \cdots |\sigma_k|) \]

where \( j \) is the degree of \( \sigma_1 \). The boundary of an oriented simplex \( a_0a_1a_2\cdots a_j = a_0 \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_j \) is given recursively by the usual rule \( \partial(a_0a_1a_2\cdots a_j) = a_1a_2\cdots a_j - a_0 \partial(a_1a_2\cdots a_j) \). By convention, this is zero if \( j = 0 \). For example,

\[ \partial(abc|de|f) = be|de|f - ac|de|f + ab|de|f - abc|e|f + abc|d|f. \]

We will find it convenient to work with the \textit{augmented} chain complex obtained by defining the chain group of degree \(-1\) to be an infinite cyclic group generated by an element \( e_0 \) corresponding to the empty cell. We redefine \( \partial \) on vertices, letting
a partition does not exist. Let \( \tilde{C}_j[S, T] \) denote the augmented chain group of degree \( j \); this group coincides with \( C_j[S, T] \) for \( j \geq 0 \). Let \( \tilde{H}_j[S, T] \) denote the associated homology.

We now state and prove one of the main results of the paper. The result relates the topology of certain \( \text{Hom} \) complexes to the partitionability of a set system. Let \( \pi_i[S, T] \) denote the \( i \)th homotopy group of \( \text{Hom}(S, T) \).

**Theorem 5.2.** Let \( S \) be a partitionable set system. For any homomorphism \( \varphi \in \text{Hom}_0(\omega(S), S) \) and any set system \( X \), we have that the following hold.

(a) \( \varphi \) induces a monomorphism from \( \pi_+[X, \omega(S)] \) to \( \pi_+[X, S] \). In particular, if \( \text{Hom}(X, S) \) is \( k \)-connected for a given \( k \), then so is \( \text{Hom}(X, \omega(S)) \).

(b) \( \varphi \) induces a monomorphism from \( \tilde{H}_*[X, \omega(S)] \) to \( \tilde{H}_*[X, S] \).

**Proof.** Write \( \omega = \omega(S) \). By definition of partitionability, there exists a homomorphism \( h \in \text{Hom}_0(S, \omega) \). We have that \( h\varphi \) is bijective, because \( h\varphi \) is a homomorphism from \( \omega \) to itself. By symmetry, we may assume that \( h\varphi \) is the identity.

By Proposition 5.1, \( h \) induces a cellular map \( h_* : \text{Hom}(X, S) \rightarrow \text{Hom}(X, \omega) \). Proposition 5.1 also yields that \( \varphi \) induces a cellular map \( \varphi_* : \text{Hom}(X, \omega) \rightarrow \text{Hom}(X, S) \). Since \( h\varphi \) is the identity map on \( \omega \), we obtain that \( h_*\varphi_* \) is the identity map on \( \text{Hom}(X, \omega) \). We conclude that \( \varphi_* \) induces monomorphisms as stated in the theorem. \( \square \)

For clarity, let us express part (b) of the theorem in the most practically useful form.

**Corollary 5.3.** Let \( S \) be a set system, and let \( \varphi \in \text{Hom}_0(\omega(S), S) \). Suppose that there exists a set system \( X \) such that the map from \( \tilde{H}_*(\text{Hom}(X, \omega(S))) \) to \( \tilde{H}_*(\text{Hom}(X, S)) \) induced by \( \varphi \) is not a monomorphism. Then \( S \) is not partitionable.

One benefit of considering reduced homology is that Corollary 5.3 then covers the case that \( \text{Hom}_0(X, \omega(S)) \) is empty and \( \text{Hom}_0(X, S) \) is nonempty. In this case, \( S \) is not partitionable by Lemma 2.1.

Let us say that \( X \) is a witness for a non-partitionable set system \( S \) if there is a homomorphism \( \varphi \in \text{Hom}_0(\omega(S), S) \) such that the induced homomorphism in homology given in Corollary 5.3 fails to be a monomorphism. Note that \( \omega(X) \leq \omega(S) \), because otherwise \( \text{Hom}_0(X, \omega(S)) \) and \( \text{Hom}_0(X, S) \) are both empty.

For the special case of rooted posets, the following result is immediate from the fact that the identity map on \( V(P) \) defines a homomorphism from the interval system \( I(P) \) to the reduced interval system \( I^{\text{red}}(P) \).

**Proposition 5.4.** Let \( X \) be a set system, and let \( P \) be a (finite) rooted poset. If \( X \) is a witness for \( I(P) \), then \( X \) is also a witness for \( I^{\text{red}}(P) \).

The converse is not always true. For example, assuming \( P \) is non-partitionable, we have that \( I^{\text{red}}(P) \) is a witness for itself; the identity is a homomorphism from \( I^{\text{red}}(P) \) to itself, whereas there is no homomorphism from \( I^{\text{red}}(P) \) to \( \omega(P) \). Yet, \( I^{\text{red}}(P) \) is not a witness for \( I(P) \), because there is no homomorphism from \( I^{\text{red}}(P) \) to \( I(P) \). Namely, given such a homomorphism \( \varphi \), the nonempty members of the family \( \{\varphi^{-1}(x) : x \in V(P)\} \) would constitute a partition of \( I^{\text{red}}(P) \) into \( \omega(P) \) stable sets; there are as many nonempty members as there are coatoms in \( P \). Such a partition does not exist.
While $\text{Hom}(\mathcal{S}, -)$ defines a covariant functor for each $\mathcal{S}$, the dual construction does not yield a contravariant functor in general. For example, let

$$
\mathcal{S} = (\{a, b\}, \emptyset, \{a, b\}), \\
\mathcal{T} = (\{c\}, \emptyset, \{c\}), \\
\mathcal{U} = (\{d, e\}, \emptyset, \{d\}, \{e\}).
$$

Define $f : \{a, b\} \to \{c\}$ and $H : \{c\} \to 2^{\{d, e\}} \setminus \emptyset$ by $f(a) = f(b) = c$ and $H(c) = \{d, e\}$. Then $H$ indexes a cell in $\text{Hom}(\mathcal{T}, \mathcal{U})$, and $f \in \text{Hom}_0(\mathcal{S}, \mathcal{T})$. However, $Hf(a) = Hf(b) = \{d, e\}$, which does not index a cell in $\text{Hom}(\mathcal{S}, \mathcal{U})$. For example, consider the map $g : \{a, b\} \to \{d, e\}$ given by $g(a) = d$ and $g(b) = e$. This map belongs to $Hf$, but $g^{-1}(\{d\}) = \{a\}$, which is not stable in $\mathcal{S}$, whereas $\{d\}$ is stable in $\mathcal{U}$.

As we will see in Section 6.2, we do get a contravariant functor $\text{Hom}(-, \mathcal{U})$ if we restrict to the subcategory $\text{MonSystems}$ of monotone systems.

6. Hom complexes on monotone systems

We consider Hom complexes $\text{Hom}(\mathcal{S}, \mathcal{T})$ in the case that $\mathcal{S}$ is a monotone system, starting with the case $\mathcal{S} = \mathcal{D}_m$ in Section 6.1 and proceeding with general monotone systems in Section 6.2.

6.1. Hom complexes on discrete systems. We give an overview of some basic properties of Hom complexes involving discrete systems $\mathcal{D}_m$. Before proceeding, let us recall that $\mathcal{D}_m$ is the independence system of the complete graph $K_m$. By the work of Lovász [14], $K_2$ is a useful “test graph” for establishing lower bounds on the chromatic number of a graph. Babson and Kozlov [2] have obtained analogous results for general complete graphs.

For any set system $\mathcal{S}$, we may equip $\text{Hom}(\mathcal{D}_2, \mathcal{S})$ with the $\mathbb{Z}_2$-action given by $(X, Y) \mapsto (Y, X)$. This action is free if and only if each $v \in V(\mathcal{S})$ is contained in some stable set of $\mathcal{S}$. Recall from Section 3.1 that we refer to $\mathcal{S}$ itself as free if this is the case.

**Proposition 6.1.** For $m \leq n$, we have that $\text{Hom}(\mathcal{D}_m, \mathcal{D}_n)$ is homotopy equivalent to a wedge of $(n - m)$-dimensional spheres. In fact, $\text{Hom}(\mathcal{D}_2, \mathcal{D}_n)$ is $\mathbb{Z}_2$-homeomorphic to the $(n - 2)$-sphere equipped with the antipodal action.


For a graph $G$, let us identify the independence system of $G$ with the independence complex $\text{Ind}(G)$. It is well-known [5, 13] that $\text{Hom}(\text{Ind}(K_2), \text{Ind}(G))$ is homotopy equivalent to the neighborhood complex of $G$ for any graph $G$. The latter complex was instrumental in Lovász’s proof of Kneser’s conjecture [14]. Let us extend this result to general set systems.

For a set system $\mathcal{S}$, say that two elements $x, y \in V(\mathcal{S})$ are separated if no stable set of $\mathcal{S}$ contains both $x$ and $y$. Let $N(\mathcal{S})$ be the simplicial complex of all sets $\sigma \subseteq V(\mathcal{S})$ such that there is an element $x \in V(\mathcal{S})$ that is separated from all elements in $\sigma$. If $\mathcal{S}$ is the independence system of a graph $G$, then two elements are separated if and only if they are joined by an edge, and $N(\mathcal{S})$ coincides with the neighborhood complex of $G$. 

...
For a given set system $S$, it is convenient to identify a cell $F \in \text{Hom}(D_m, S)$ with the corresponding $m$-tuple $(F(1), \ldots, F(m))$.

**Proposition 6.2.** For any set system $S$, we have that $\text{Hom}(D_2, S)$ is homotopy equivalent to $N(S)$.

**Proof.** Write $\Sigma = \text{Hom}(D_2, S)$ and $Q = P(N(S))$. A surjective poset map $\varphi : P(\Sigma) \to Q$ is given by mapping $(X,Y)$ to $X$. Namely, the map is well-defined, because any element in $X$ must be separated from any element in $Y$, which is nonempty. Moreover, the map is surjective, because if $X$ is a nonempty face of $N(S)$, then there is an element $y \in V(S)$ separated from $X$, which means that $(X, \{y\}) \in \Sigma$.

By Corollary 4.6, it suffices to prove the following.

- For each $X \in N(S) \setminus \{\emptyset\}$, we have that $\Sigma_X$ is contractible, where $\Sigma_X$ is the subcomplex of $\Sigma$ with face poset $\varphi^{-1}(Q^\subseteq X) = \varphi^{-1}(2^X \setminus \{\emptyset\})$.

Let $y \in V(S)$ be any element such that $(X,\{y\}) \in \Sigma_X$. Then $(X_0, Y \cup \{y\}) \in \Sigma_X$ whenever $(X_0, Y) \in \Sigma_X$. Using Lemma 4.3, we may collapse $\Sigma_X$ to the subcomplex $\Sigma_X'$ consisting of all cells of the form $(X_0,\{y\})$ such that $X_0 \subseteq X$. Specifically, a perfect acyclic matching on $\Sigma_X \setminus \Sigma_X'$ is given by pairing $(X_0, Y \setminus \{y\})$ with $(X_0, Y \cup \{y\})$ for each $(X_0, Y) \in \Sigma_X$ such that $X_0 \subseteq X$ and $Y \setminus \{y\} \neq \emptyset$. Since $\Sigma_X'$ is an $|(X|-1)$-simplex, $\Sigma_X$ is collapsible.

**Corollary 6.3.** If $S$ is partitionable and $\chi(S) = n$, then $\tilde{H}_{n-2}(N(S); \mathbb{Z})$ is infinite.

**Proof.** This follows from Theorem 5.2 and Propositions 6.1 and 6.2.

The next result extends a famous theorem of Lovász [14] to general set systems.

**Proposition 6.4.** Let $S$ be a set system. If $N(S)$ is $(n-2)$-connected, then $\chi(S) > n$.

**Proof.** If $S$ is not free, then $\chi(S) = \infty$. Assume that $S$ is free; each $v \in V(S)$ is contained in some stable set of $S$. By Proposition 6.2, $N(S)$ is homotopy equivalent to $\text{Hom}(D_2, S)$. Suppose that there exists a map $f \in \text{Hom}(S, D_n)$. Then there is a map $f_* : \text{Hom}(D_2, S) \to \text{Hom}(D_2, D_n) \cong \mathbb{Z}_2 S^{n-2}$, and this map commutes with the $\mathbb{Z}_2$-action. Yet, since $\text{Hom}(D_2, S)$ is a free $(n-2)$-connected $\mathbb{Z}_2$-complex, there is a $\mathbb{Z}_2$-map from $S^{n-1}$ to $\text{Hom}(D_2, S)$; see Matoušek [15, Prop. 5.3.2 (iv)]. As a consequence, we have a $\mathbb{Z}_2$-map from $S^{n-1}$ to $S^{n-2}$, which contradicts the Borsuk-Ulam Theorem.

### 6.2. Hom complexes on general monotone systems

Given the discussion in Section 6.1, it seems reasonable to ask whether the cell complex $\text{Hom}(D_2, S) \cong N(S)$ tells us something useful about the partitionability of the set system $S$. Disappointingly, this does not seem to be the case for many interesting set systems $S$, including interval systems of rooted posets. Indeed, we will see that any monotone system fails to be a witness for interval systems. On the positive side, we show that the category $\text{MonSystems}$ of monotone systems has nice functorial properties.

Let $S$ be a set system, and write $\omega = \omega(S)$. We say that $S$ admits an optimal cover if there are $\omega$ stable sets $\sigma_1, \ldots, \sigma_\omega$ such that

$$V(S) = \bigcup_{j=1}^{\omega} \sigma_j.$$
Given a transversal \((c_1, \ldots, c_m)\), we have that each \(\sigma_j\) contains at most one \(c_i\). In particular, if \(m = \omega\), then each \(\sigma_j\) contains exactly one \(c_i\), and each \(c_i\) belongs to exactly one \(\sigma_j\).

Note that any partitionable set system admits an optimal cover, but the converse is not true in general. Let us consider two important special cases.

- Let \(S\) be a monotone system. Then \(S\) admits an optimal cover if and only if \(S\) is partitionable. Namely, given an optimal cover \((\sigma_1, \ldots, \sigma_{\omega(S)})\), we obtain a partition by removing elements from the sets \(\sigma_j\) until each element in the ground set appears in exactly one set \(\sigma_j\).
- Suppose that \(P\) is any rooted poset with minimal element 0. Then \(\mathcal{I}(P)\) admits an optimal cover. Namely, \(V(P)\) is the union over all coatoms \(c\) of the intervals \([0, c]\). By Proposition 3.4, the number of coatoms is \(\omega(P)\).

To conclude, the situation for monotone systems is completely different from that for set systems of rooted posets.

By the following result, monotone systems are not witnesses for systems admitting an optimal cover.

**Proposition 6.5.** Let \(\mathcal{X}\) be a monotone system, and let \(S\) be a set system admitting an optimal cover. Then any map \(\varphi \in \text{Hom}_0(\mathcal{D}_{\omega(S)}, S)\) induces monomorphisms from \(\pi_*[\mathcal{X}, \mathcal{D}_{\omega(S)}]\) to \(\pi_*[\mathcal{X}, S]\) and from \(\tilde{H}_*[\mathcal{X}, \mathcal{D}_{\omega(S)}]\) to \(\tilde{H}_*[\mathcal{X}, S]\).

**Proof.** If \(\text{Hom}_0(\mathcal{X}, S)\) is empty, then so is \(\text{Hom}_0(\mathcal{X}, \mathcal{D}_{\omega(S)})\); hence we may assume that \(\text{Hom}_0(\mathcal{X}, S)\) is nonempty.

Write \(\omega = \omega(S)\). Let \((\sigma_1, \ldots, \sigma_\omega)\) be an optimal cover of \(S\). Define a map \(h : V(S) \to \{1, \ldots, \omega\}\) in the following manner. For each \(v \in V(S)\), pick any element \(k \in \{1, \ldots, \omega\}\) such that \(v \in \sigma_k\), and define \(h(v) = k\). While \(h\) is typically not a homomorphism from \(S\) to \(\mathcal{D}_\omega\), we claim that \(h\) still induces a cellular map from \(\text{Hom}(\mathcal{X}, S)\) to \(\text{Hom}(\mathcal{X}, \mathcal{D}_\omega)\).

To prove the claim, consider a cell in \(\text{Hom}(\mathcal{X}, S)\) indexed by the set function \(F\). We need to show that every \(g \in hF\) is a homomorphism from \(\mathcal{X}\) to \(\mathcal{D}_\omega\), which is equivalent to saying that the preimage under \(g\) of every element in \(\{1, \ldots, \omega\}\) is stable in \(\mathcal{X}\). Now, write \(A_k = g^{-1}({\{k}\}}\). For each \(a \in A_k\), we have that \(k = h(b_a)\) for some \(b_a \in F(a)\). By construction, all elements in the set \(B_k = \{b_a : a \in A_k\}\) belong to \(\sigma_k\). Let \(f \in F\) be such that \(f(a) = b_a\) for \(a \in A_k\). Since \(\sigma_k\) is stable in \(S\) and \(f \in \text{Hom}_0(\mathcal{X}, S)\), we have that \(f^{-1}(\sigma_k)\) is stable in \(\mathcal{X}\). Yet,

\[ f^{-1}(\sigma_k) \supseteq f^{-1}(B_k) \supseteq A_k. \]

Since \(\mathcal{X}\) is monotone, we deduce that \(A_k\) is stable.

To conclude, \(h\) induces a cellular map \(h_* : \text{Hom}(\mathcal{X}, S) \to \text{Hom}(\mathcal{X}, \mathcal{D}_\omega)\). We also have that \(\varphi\) induces a cellular map \(\varphi_* : \text{Hom}(\mathcal{X}, \mathcal{D}_\omega) \to \text{Hom}(\mathcal{X}, S)\). Now, \(h\varphi\) is a bijection, because the restriction of \(h\) to the set of elements of any \(\omega\)-transversal is necessarily a bijection. As a consequence, the composition \(h_*\varphi_*\) must be an isomorphism, which yields the proposition.

The following result generalizes the fact that a monotone system \(S\) admits an optimal cover if and only if \(S\) is partitionable.

**Proposition 6.6.** Let \(\mathcal{X}\) be a monotone system, and let \(S\) be a set system admitting an optimal cover. If \(\text{Hom}_0(\mathcal{X}, S)\) is nonempty, then \(\chi(\mathcal{X}) \leq \omega(S)\).
Proof. Let $f \in \text{Hom}_0(X, S)$. Given an optimal cover $(\sigma_1, \ldots, \sigma_{\omega(S)})$ of $S$, we have that $V(X)$ is the union of the stable sets $\rho_i = f^{-1}(\sigma_i)$. Removing elements from the sets $\rho_i$ until each element appears in exactly one set, we obtain a partition of $X$ into $\omega(S)$ stable sets. \hfill \Box

In Section 5, we observed that $\text{Hom}(-, \mathcal{U})$ does not define a contravariant functor from Systems to Top. By the following result, the situation becomes much nicer if we restrict to the subcategory MonSystems of monotone systems.

**Proposition 6.7.** Let $S$ be any monotone system, let $\mathcal{T}$ and $\mathcal{U}$ be any set systems, and let $f \in \text{Hom}_0(S, \mathcal{T})$. Then we obtain a cellular map $f^* : \text{Hom}(\mathcal{T}, \mathcal{U}) \to \text{Hom}(S, \mathcal{U})$ by mapping the cell in $\text{Hom}(\mathcal{T}, \mathcal{U})$ indexed by $H$ to the cell in $\text{Hom}(S, \mathcal{U})$ indexed by $Hf$.

**Proof.** Suppose that $g$ is a map from $S$ to $\mathcal{U}$ such that $g \in Hf$. To prove the proposition, it suffices to show that $\delta = g^{-1}(\tau)$ is a stable set in $S$ for each stable set $\tau$ in $\mathcal{U}$.

Let $\{y_1, \ldots, y_k\}$ be the image of $f$, and define $\sigma_j = f^{-1}(y_j)$ for $1 \leq j \leq k$. Note that the ground set $X$ of $S$ is the disjoint union of the sets $\sigma_1, \ldots, \sigma_k$. Let $Y$ and $Z$ be the ground sets of $\mathcal{T}$ and $\mathcal{U}$, respectively, and define a map $h : Y \to Z$ via the following procedure. For each $j$ such that $\sigma_j \cap \delta \neq \emptyset$, let $x_j \in \sigma_j \cap \delta$. If $\sigma_j \cap \delta = \emptyset$, pick an arbitrary $x_j \in \sigma_j$. For $1 \leq j \leq k$, define

$$h(y_j) = g(x_j) \in Hf(x_j) = H(y_j).$$

For $y \in Y \setminus \{y_1, \ldots, y_k\}$, define $h(y)$ to be an arbitrary element in $H(y)$. It is clear that $h \in H$ and hence that $h \in \text{Hom}_0(\mathcal{T}, \mathcal{U})$.

Let $\tilde{g} = hf \in \text{Hom}_0(S, \mathcal{U})$. For each $x \in \sigma_j$, we have that $\tilde{g}(x) = hf(x) = Hf(x_j) = g(x_j)$. In particular, $\tilde{g}(x) \in \tau$ if and only if $\delta \cap \sigma_j \neq \emptyset$. We conclude that

$$\tilde{g}^{-1}(\tau) = \bigcup_{\delta \cap \sigma_j \neq \emptyset} \sigma_j \supseteq \delta.$$  

Now, $\tilde{g}$ is a homomorphism, which yields that $\tilde{g}^{-1}(\tau)$ is stable in $S$. Since $S$ is monotone, we deduce that the subset $\delta$ is also stable in $S$. \hfill \Box

### 7. Hom complexes on the 3-chain

By Proposition 6.5, only non-monotone set systems can be witnesses for set systems admitting an optimal cover. In this and the following two sections, we provide a detailed analysis of the interval system of the 3-chain $L_3$, which is the poset on the ground set $\{1, 2, 3\}$ with the total order $1 < 2 < 3$. The interval system $\mathcal{I}(L_3)$ of $L_3$ is non-monotone, the stable sets being all subsets of $\{1, 2, 3\}$ except $\{1, 3\}$.

In Section 8, we show that $\mathcal{I}(L_3)$ is a witness for infinitely many systems admitting an optimal cover. This makes $\mathcal{I}(L_3)$ fundamentally different from monotone systems. However, as we show in Section 9, $\mathcal{I}(L_3)$ fails to be a witness for a large class of set systems, including all interval systems of Cohen-Macaulay complexes. In particular, we cannot use $\mathcal{I}(L_3)$ to detect a counterexample to Stanley’s conjecture.

Note that a map $f : \{1, 2, 3\} \to V(S)$ defines a homomorphism from $\mathcal{I}(L_3)$ to the set system $S$ if and only if every stable set of $S$ containing $f(1)$ and $f(3)$ also
contains \( f(2) \). We may identify each nonempty cell of \( \text{Hom}(I(L_3), S) \) with a triple \((X_1, X_2, X_3)\), where \( X_1, X_2, X_3 \) are nonempty subsets of \( V(S) \) such that each choice of elements \( a_i \in X_i \) defines a homomorphism \( i \mapsto a_i \) from \( I(L_3) \) to \( S \). Equivalently, for every stable set \( \sigma \) of \( S \), if there are elements \( a_1 \in X_1 \) and \( a_3 \in X_3 \) such that \( a_1, a_3 \in \sigma \), then \( X_2 \not\subseteq \sigma \).

7.1. **Basic properties.** For a given set system \( S \), define \( \Gamma(S) \) to be the family of all pairs \((X_1, X_3)\) such that \((X_1, X_2, X_3) \in \text{Hom}(I(L_3), S) \) for some \( X_2 \). The members of \( \Gamma(S) \) index the nonempty cells of a cell complex. This cell complex is a subcomplex of the product of two \(|V(S)|-1\)-simplices; the cell indexed by a given pair \((X_1, X_3)\) is the product of an \(|X_1|\-1\)-simplex and an \(|X_3|\-1\)-simplex.

We identify \( \Gamma(S) \) with the corresponding cell complex.

A surjective cell complex map \( \varphi : \text{Hom}(I(L_3), S) \to \Gamma(S) \) is given by
\[
\varphi(X_1, X_2, X_3) = (X_1, X_3).
\]

**Proposition 7.1.** The map \( \varphi \) induces a homotopy equivalence between the complexes \( \text{Hom}(I(L_3), S) \) and \( \Gamma(S) \).

**Proof.** Write \( \Sigma = \text{Hom}(I(L_3), S) \) and \( \Gamma = \Gamma(S) \). We may view \( \varphi \) as a poset map from \( P(\Sigma) \) and \( P(\Gamma) \). Corollary 4.6 yields that it suffices to prove the following.

- For every \( \gamma \in \Gamma \setminus \{\emptyset\} \), we have that \( \Sigma_{\gamma} \) is contractible, where \( \Sigma_{\gamma} = \varphi^{-1}(\{\rho \in \Gamma : \rho \subseteq \gamma\}) \).

Consider \( \gamma = (X_1, X_3) \). Let \( \mu \in V \) be any element such that \((X_1, \{\mu\}, X_3) \in \Sigma_{\gamma} \). Then \((Y_1, Y_2 \cup \{\mu\}, Y_3) \in \Sigma_{\gamma} \) whenever \((Y_1, Y_2, Y_3) \in \Sigma_{\gamma} \). Using Lemma 4.3, we may collapse \( \Sigma_{\gamma} \) to the subcomplex \( \Sigma'_{\gamma} \) consisting of all cells of the form \((Y_1, \{\mu\}, Y_3) \) such that \( Y_1 \subseteq X_1 \) and \( Y_3 \subseteq X_3 \). Specifically, a perfect acyclic matching on \( \Sigma'_{\gamma} \) is given by pairing \((Y_1, Y_2 \setminus \{\mu\}, Y_3) \) with \((Y_1, Y_2 \cup \{\mu\}, Y_3) \) for each \((Y_1, Y_2, Y_3) \in \Sigma_{\gamma} \) such that \( Y_2 \setminus \{\mu\} \neq \emptyset \). Since \( \Sigma'_{\gamma} \) is the product of an \(|X_1|-1\)-simplex and an \(|X_3|-1\)-simplex, we deduce that \( \Sigma'_{\gamma} \) is collapsible. As a consequence, the same is true for \( \Sigma_{\gamma} \). \( \square \)

From now on, we consider \( \Gamma(S) \) instead of \( \text{Hom}(I(L_3), S) \). Let \( \Gamma_0(D_n) \) be the subcomplex of \( \Gamma(D_n) \) consisting of all pairs \((X, Y)\) such that \( X \cap Y = \emptyset \). By Proposition 6.1, we have that
\[
\Gamma_0(D_n) = \text{Hom}(D_2, D_n) \cong S^{n-2}.
\]
Define
\[
Q(D_n) = \Gamma(D_n)/\Gamma_0(D_n).
\]

**Proposition 7.2.** We have that
\[
\Gamma(D_n) \cong \bigvee_{n-1} S^n.
\]

Hence
\[
\tilde{H}_i(\Gamma(D_n); \mathbb{Z}) \cong \left\{ \begin{array}{ll}
\mathbb{Z}^{n-1} & \text{if } i = n - 1, \\
0 & \text{if } i \neq n - 1.
\end{array} \right.
\]

Moreover,
\[
Q(D_n) \cong \bigvee_{n} S^{n-1}.
\]
Proof. Note that \( \Gamma(\mathcal{D}_n) \) consists of all pairs \((X,Y)\) such that \(|X \cap Y| \leq 1\). For \(i \in \{1, \ldots, n\}\), let \(\Gamma_i\) be the subcomplex of \(\Gamma(\mathcal{D}_n)\) consisting of all pairs \((X,Y)\) such that \(X \cap Y \subseteq \{i\}\).

We have that \(\Gamma_i\) is collapsible. Namely, let \(\Sigma_i\) be the subcomplex consisting of all cells \((A,\{i\})\) such that \(A\) is a nonempty subset of \(\{1, \ldots, n\}\). Using Lemma 4.3, we obtain a collapse from \(\Gamma_i\) to \(\Sigma_i\) by pairing \((A,B \setminus \{i\})\) with \((A,B \cup \{i\})\) whenever \(B \setminus \{i\} \neq \emptyset\). Since \(\Sigma_i\) is an \((n-1)\)-simplex, we obtain the desired result.

Observing that \(\Gamma_i \cap \Gamma_j = \Gamma_0 = \Gamma_0(\mathcal{D}_n)\) whenever \(i \neq j\), we conclude that

\[
\Gamma \cong \Gamma_0 \cong \bigvee_{i=1}^{n} \Gamma_i / \Gamma_0 \cong \bigvee_{n-1}^{n} S\Gamma_0 \cong \bigvee_{n-1}^{n} S^{n-1}.
\]

the homotopy equivalences are consequences of Lemmas 4.1 and 4.2. Similarly,

\[
Q(\mathcal{D}_n) = \Gamma / \Gamma_0 \cong \bigvee_{i=1}^{n} \Gamma_i / \Gamma_0 \cong \bigvee_{n} S\Gamma_0 \cong \bigvee_{n} S^{n-1},
\]

which concludes the proof. \(\square\)

7.2. Strong covers. Given a set system \(S\), an optimal cover \((\sigma_1, \ldots, \sigma_m(S))\) is a strong cover of \(S\) if the members of the cover are the inclusion-maximal stable sets of \(S\). That is, each member of the cover is maximal, and there are no other maximal stable sets. Such a cover is clearly unique up to the order of the sets in the cover.

For example, let \(P\) be a rooted poset with coatoms \(c_1, \ldots, c_m\) and minimum element \(0\). Then the intervals \([0, c_1], \ldots, [0, c_m]\) form a strong cover of the interval system and also of the reduced interval system of \(P\).

For a set system \(S\), say that two sets \(A, B \subseteq V(S)\) are separated if \(a\) and \(b\) are separated for each \(a \in A\) and \(b \in B\); there is no stable set of \(S\) that intersects both \(A\) and \(B\). Let \(\Gamma_0(S)\) be the subcomplex of \(\Gamma(S)\) consisting of all faces \((A,B)\) such that \(A\) and \(B\) are separated.

Lemma 7.3. Let \(S\) be a set system with a strong cover \((\sigma_1, \ldots, \sigma_m)\). Then

\[
\Gamma_0(S) \cong \Gamma_0(\mathcal{D}_m) = \text{Hom}(\mathcal{D}_2, \mathcal{D}_m) \cong S^{m-2}.
\]

Moreover, any homomorphism \(\varphi \in \text{Hom}_0(\mathcal{D}_m, S)\) induces a homotopy equivalence between \(\Gamma_0(\mathcal{D}_m)\) and \(\Gamma_0(S)\).

Proof. We define a poset map \(f : P(\Gamma_0(S)) \to P(\Gamma_0(\mathcal{D}_m))\) by

\[
f(A, B) = ([i : A \cap \sigma_i \neq \emptyset], [j : B \cap \sigma_j \neq \emptyset]).
\]

This map is indeed a well-defined poset map, because \(A \cap \sigma_i\) and \(B \cap \sigma_i\) cannot both be empty if \(A\) and \(B\) are separated. Moreover, the map is surjective. Namely, let \((c_1, \ldots, c_m)\) be a transversal such that \(c_i \in \sigma_i\) for each \(i\). If \((X,Y)\) is a cell in \(\Gamma_0(D_m)\), then \(f\) maps \(([c_i : i \in X], [c_j : j \in Y])\) to \((X,Y)\).

We want to apply Corollary 4.6 to deduce that \(\Gamma_0(S)\) and \(\Gamma_0(D_m)\) are homotopy equivalent. Given a nonempty cell \((X,Y)\) in \(\Gamma_0(\mathcal{D}_m)\), we have that a cell \((A,B)\) in \(\Gamma_0(\mathcal{D}_m)\) is mapped to a face of \((X,Y)\) if and only if

\[
A \subseteq \alpha(X) = \{a : a \notin \sigma_i\text{ if }i \notin X\},
B \subseteq \alpha(Y) = \{b : b \notin \sigma_j\text{ if }j \notin Y\}.
\]

We cannot have that some \(\sigma_i\) intersects both \(\alpha(X)\) and \(\alpha(Y)\), because then \(i \in X \cap Y\). Since \((\sigma_1, \ldots, \sigma_m)\) is a strong cover, this implies that \(\alpha(X)\) and \(\alpha(Y)\) are
Let $\psi \in \text{Hom}_0(\Gamma_0(S),Z)$, where $m = \omega(S)$. We have that $\psi$ induces a monomorphism from $\tilde{H}_{m-1}(\Gamma(D_m);Z) \cong \mathbb{Z}^{m-1}$ to $\tilde{H}_{m-1}(\Gamma(S);Z)$ if and only if $\psi$ induces a monomorphism from $\tilde{H}_{m-1}(\Gamma(D_m);Z) \cong \mathbb{Z}^m$ to $\tilde{H}_{m-1}(\Gamma(S);Z)$.

**Proof.** We have that $\psi$ gives rise to a commutative diagram associating the long exact sequences for the pairs $(\Gamma(D_m),\Gamma_0(D_m))$ and $(\Gamma(S),\Gamma_0(S))$: see the diagram on the left in Figure 1. We want to show that $\varphi_\#$ is a monomorphism if and only if $\psi_\#$ is a monomorphism. By Proposition 7.2 and Lemma 7.3, the diagram simplifies to the diagram on the right in Figure 1. As a consequence, the proposition follows.

Note that $\tau$ is the zero map in the diagram on the right in Figure 1. In particular, we have the following result.
Figure 2. The cycle \(a|ac + ac|a - c|ac - ac|c\) is a boundary in the chain complex of \(\Gamma(\mathcal{I}_{\text{red}}(K^2_2))\).

**Corollary 7.5.** Let \(\mathcal{S}\) be a set system admitting a strong cover, and let \(m = \omega(\mathcal{S})\). Then

\[
\tilde{H}_i(Q(\mathcal{S}); \mathbb{Z}) \cong \begin{cases} 
\tilde{H}_i(\Gamma(\mathcal{S}); \mathbb{Z}) \oplus \mathbb{Z} & \text{if } i = m - 1, \\
\tilde{H}_i(\Gamma(\mathcal{S}); \mathbb{Z}) & \text{if } i \neq m - 1.
\end{cases}
\]

8. The 3-chain as a witness

We show that the 3-chain is a witness for an infinite family of set systems. Specifically, \(\mathcal{I}(L_3)\) is a witness for the reduced system \(\mathcal{I}_{\text{red}}(\Lambda)\) whenever \(\Lambda\) is a simplicial complex in which some link is the disjoint union of two edges. In Section 8.1, we show this for the particular case that \(\Lambda\) itself is such a disjoint union. In Section 8.2, we develop the necessary theory to deduce the result for general \(\Lambda\). We leave open whether \(\mathcal{I}(L_3)\) is also a witness for the full interval system \(\mathcal{I}(\Lambda)\).

We remark that each \(\Lambda\) as above is trivially non-partitionable. It remains an open problem whether the 3-chain is a witness for any set system not already known to be non-partitionable. In Section 9, we show that the 3-chain is not a witness for a large class of interval systems, including interval systems of Cohen-Macaulay complexes.

8.1. A small example. In this section, we show that the interval system of the 3-chain is a witness for the reduced interval system of the simplicial complex consisting of two disjoint edges.

**Proposition 8.1.** Let \(K^2_2\) be the simplicial complex with maximal faces \(\{a, b\}\) and \(\{c, d\}\). Then \(\mathcal{I}(L_3)\) is a witness for \(\mathcal{I}_{\text{red}}(K^2_2)\).

**Proof.** As described in Section 5, we let \(x_1 \cdots x_k|y_1 \cdots y_\ell\) denote the oriented cell corresponding to a given cell \((\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_\ell\})\) in \(\Gamma(\mathcal{S})\). We have that \(\Gamma(D_2)\) is the boundary of a square, and the fundamental cycle is \(z = 112 + 121 - 212 - 122\); see Section 5 for the exact definition of the boundary operator \(\partial\). Let \(f \in \text{Hom}_0(D_2, \mathcal{I}_{\text{red}}(K^2_2))\) be defined by \(f(1) = a\) and \(f(2) = c\). Note that \(f\) maps \(z\) to the cycle \(z' = a|ac + ac|a - c|ac - ac|c\) in the chain complex of \(\Gamma(\mathcal{I}_{\text{red}}(K^2_2))\).

Yet, \(z'\) is the boundary of the element

\[a|abc - ad|bc - acd|c + cd|bc - c|abc + cd|ab + acd|a - ad|ab.\]

See Figure 2 for a geometric illustration. As a consequence, \(f\) does not induce a monomorphism in homology; hence \(\mathcal{I}(L_3)\) is a witness for \(\mathcal{I}_{\text{red}}(K^2_2)\). \(\square\)
Lemma 8.2. Let \( \tilde{f} \) from \( \text{It} \) suffices to show the result for 

\[
\begin{align*}
\mathbb{Z}^m &\cong \hat{H}_i(\Gamma(D_m + D_1)) \xrightarrow{\varphi_x''} \hat{H}_i(\Gamma(T + D_1)) \\
&\cong \hat{H}_{i-1}(\Sigma(D_m)) \xrightarrow{\varphi_x} \hat{H}_{i-1}(\Sigma(T))
\end{align*}
\]

Figure 3. Commutative diagram in the proof of Lemma 8.2.

Using Pilarczyk’s computer program homchain [16], one can verify that

\[
\hat{H}_2(\Gamma(T^{\text{red}}(K_2^2)); \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}^2 & \text{if } i = 2, \\
0 & \text{otherwise.}
\end{cases}
\]

One may also obtain this result via a rather complicated procedure using discrete Morse theory. We omit the details, as we will not need the result.

8.2. Building further examples. Using the example in Section 8.1 as starting point, we construct further examples of non-partitionable reduced interval systems for which \( I(L_3) \) is a witness.

For two sets \( V \) and \( W \), let \( V \cup W \) denote the disjoint union of \( V \) and \( W \). For set systems \( S_1 \) and \( S_2 \), define

\[
S_1 + S_2 = (V(S_1) \cup V(S_2), \Delta(S_1) \cup \Delta(S_2)).
\]

Note that \( D_m + D_r \cong D_{m+r} \).

Given set systems \( T_1 \) and \( T_2 \) and maps \( f_i \in \text{Hom}_0(S_i, T_i) \), we obtain a map \( f = f_1 \circ f_2 \in \text{Hom}_0(S_1 + S_2, T_1 + T_2) \) by defining \( f(x) = f_1(x) \) if \( x \in V(S_1) \) and \( f(x) = f_2(x) \) if \( x \in V(S_2) \).

Lemma 8.2. Let \( T \) be a set system admitting a strong cover \( (\sigma_1, \ldots, \sigma_m) \). Then

\[
\hat{H}_i(\Gamma(T + D_1); \mathbb{Z}) \cong \begin{cases} 
\hat{H}_{i-r}(\Gamma(T); \mathbb{Z}) \oplus \mathbb{Z}^r & \text{if } i = m + r - 1, \\
\hat{H}_{i-r}(\Gamma(T); \mathbb{Z}) & \text{if } i \neq m + r - 1.
\end{cases}
\]

Moreover, for every \( \varphi \in \text{Hom}_0(D_m, T) \), we have that \( \varphi \) induces a monomorphism from \( \hat{H}_{m-1}(\Gamma(D_m); \mathbb{Z}) \cong \mathbb{Z}^{m-1} \) to \( \hat{H}_{m-1}(\Gamma(T); \mathbb{Z}) \) if and only if \( \varphi \circ \text{id}_{D_m} \) induces a monomorphism from \( \hat{H}_{m+r-1}(\Gamma(D_m + D_r); \mathbb{Z}) \cong \mathbb{Z}^{m+r-1} \) to \( \hat{H}_{m+r-1}(\Gamma(T + D_r); \mathbb{Z}) \).

Proof. It suffices to show the result for \( r = 1 \). Namely, for \( r > 1 \), we then obtain by induction that

\[
\begin{align*}
\hat{H}_{m+r-1}(\Gamma(T + D_r); \mathbb{Z}) &= \hat{H}_{m+r-1}(\Gamma((T + D_{r-1}) + D_1); \mathbb{Z}) \\
&\cong \hat{H}_{m+r-2}(\Gamma(T + D_{r-1}); \mathbb{Z}) \oplus \mathbb{Z} \\
&\cong \hat{H}_{m-1}(\Gamma(T); \mathbb{Z}) \oplus \mathbb{Z}^r \oplus \mathbb{Z}^r \oplus \mathbb{Z}.
\end{align*}
\]

Moreover, \( \varphi \circ \text{id}_{D_m} = (\varphi \circ \text{id}_{D_{m-1}}) \circ \text{id}_{D_1} \), which yields the second statement via a simple induction argument.

Let \( x \) denote the single element in \( D_1 \). We may write \( \Gamma = \Gamma(T + D_1) \) as a union

\[
\Gamma = \Gamma_L(T) \cup \Gamma_R(T).
\]

Here, \( \Gamma_L = \Gamma_L(T) \) consists of all cells \((A, B)\) in \( \Gamma \) such that \((A \cup \{x\}, B)\) is a cell in \( \Gamma' \). Analogously, \( \Gamma_R = \Gamma_R(T) \) consists of all cells in \( \Gamma \) such that \((A, B \cup \{x\})\) is a cell in \( \Gamma' \).
We have that $\Gamma_L$ is contractible to a point. Namely, define $\Gamma'_L$ to be the subcomplex of $\Gamma_L$ consisting of all cells of the form $(\{x\}, B)$. We have a perfect acyclic matching on $\Gamma_L \setminus \Gamma'_L$ given by pairing $(A \setminus \{x\}, B)$ with $(A \cup \{x\}, B)$ for each $(A, B)$ in $\Gamma_L$ such that $A \setminus \{x\} \neq \emptyset$. Since $\Gamma'_L$ is contractible, $\Gamma_L$ is also contractible. Analogously, $\Gamma_R$ is contractible. As a consequence, a Mayer-Vietoris argument yields that

$$\tilde{H}_i(\Gamma; \mathbb{Z}) \cong \tilde{H}_{i-1}(\Gamma_L \cap \Gamma_R)$$

for every $i$. Moreover, writing

$$\Sigma(T) = \Gamma_L(T) \cap \Gamma_R(T),$$

we have the commutative diagram in Figure 3, and $\varphi''_#$ is a monomorphism if and only if $\varphi'_#$ is a monomorphism.

Note that

$$\Sigma(T) = \Gamma(T) \cup \Sigma_0(T),$$

where

$$\Sigma_0(T) = \{(A, B) \in \Gamma(T + D_1) : A \setminus \{x\} \text{ and } B \setminus \{x\} \text{ are separated}\}.$$  

This means that $\Gamma(T) \setminus \Sigma_0(T)$ consists of all $(A, B) \in \Gamma(T)$ such that $A$ and $B$ are not separated. In particular,

$$(3) \quad \frac{\Sigma(T)}{\Sigma_0(T)} \cong \frac{\Gamma(T)}{\Gamma_0(T)} = Q(T).$$

By an argument very similar to the one above for $\Gamma_L(T)$ and $\Gamma_R(T)$, we get that $\Sigma_0(T)$ is contractible. Using the Contractible Subcomplex Lemma 4.1 and (3), we conclude that

$$\Sigma(T) \cong \frac{\Sigma(T)}{\Sigma_0(T)} \cong Q(T).$$

This yields the commutative diagram in Figure 4, and $\varphi'_#$ is a monomorphism if and only if $\varphi'_#$ is a monomorphism. By Proposition 7.4 and Corollary 7.5, we are done.\hfill \Box

For a simplicial complex $\Lambda$ and a face $\sigma$ of $\Lambda$ (including the possibility $\sigma = \emptyset$), we define $\text{link}_\Lambda(\sigma)$ to be the subcomplex consisting of all $\rho$ such that $\rho \cap \sigma = \emptyset$ and $\rho \cup \sigma \in \Lambda$.

**Theorem 8.3.** Let $\Lambda$ be a simplicial complex, and let $\sigma$ be a face of $\Lambda$. If $\mathcal{I}(L_{\Lambda})$ is a witness for the reduced interval system of $\text{link}_\Lambda(\sigma)$, then $\mathcal{I}(L_{\Lambda \setminus \sigma})$ is also a witness for the reduced interval system of $\Lambda$.\hfill \Box
Proof. Let \( \tau_1, \ldots, \tau_r \) be the maximal faces of \( \Lambda \) that do not contain \( \sigma \). Write \( S = I^{\text{red}}(\text{link}_A(\sigma)) + D_r \).

We want to define a map \( f \in \text{Hom}_0(S, I^{\text{red}}(\Lambda)) \). For this, let \( f(\rho) = \rho \cup \sigma \) if \( \rho \in \text{link}_A(\sigma) \) and \( f(k) = \tau_k \) if \( k \in \{1, \ldots, r\} \). Consider an interval \([\alpha, \tau]\) in \( \Lambda \), where \( \tau \) is maximal. If \( \sigma \subseteq \tau \), then

\[
  f^{-1}([\alpha, \tau]) = f^{-1}([\alpha \cup \sigma, \tau]) = [\alpha \setminus \sigma, \tau \setminus \sigma],
\]

which is stable in \( I^{\text{red}}(\text{link}_A(\sigma)) \). If \( \sigma \not\subseteq \tau \), then \( \tau = \tau_i \) for some \( i \), and

\[
  f^{-1}([\alpha, \tau]) = f^{-1}(\tau_i) = \{\{i\}\},
\]

which is stable in \( D_r \). We deduce that \( f \in \text{Hom}_0(S, I^{\text{red}}(\Lambda)) \).

Now, let \( m \) be the number of maximal faces of \( \text{link}_A(\sigma) \). Suppose that there is a map \( g \in \text{Hom}_0(D_m, I^{\text{red}}(\text{link}_A(\sigma))) \) such that the induced cellular map from \( \Gamma(D_m) \) to \( \Gamma(I^{\text{red}}(\text{link}_A(\sigma))) \) does not induce a monomorphism in homology. By Lemma 8.2, the same is true for the map \( g' = g \circ \text{id}_{D_r} \in \text{Hom}_0(D_{m+r}, S) \). We conclude that the cellular map from \( \Gamma(D_{m+r}) \) to \( \Gamma(I^{\text{red}}(\Lambda)) \) induced by \( f \circ g' \) does not induce a monomorphism in homology. \( \square \)

Corollary 8.4. Let \( \Lambda \) be a simplicial complex, and suppose that \( \sigma \) is a face of \( \Lambda \) such that \( \text{link}_A(\sigma) \) is isomorphic to the simplicial complex \( K^2 \) with maximal faces \( ab \) and \( cd \). Then \( I(L_3) \) is a witness for the reduced interval system of \( \text{link}_A(\sigma) \). As a consequence, we are done by Theorem 8.3. \( \square \)

9. The 3-chain as a non-witness

The main result of this section is that the 3-chain is not a witness for certain non-partitionable “benchmark” systems. This turns out to imply that the 3-chain is not a witness for unreduced or reduced interval systems of Cohen-Macaulay (CM) complexes. The reason is that systems of the latter kind admit homomorphisms to benchmark systems. Recall from Section 3.2 that it is not known whether there exist any non-partitionable CM complexes.

We refer to Björner [3] for information about CM complexes. The one property we need in the present paper is that a \( d \)-dimensional CM complex has the property that the link of each face of dimension at most \( d - 2 \) is connected.

We define the benchmark system \( B_m \) in the following manner. \( V(B_m) \) is the family of nonempty subsets of the set \( \{1, \ldots, m\} \). The stable sets of \( B_m \) are the singleton sets \( \{1\}, \ldots, \{m\} \) and the sets

\[
  \tau_i = \{I \subseteq \{1, \ldots, m\} : i \in I\}.
\]

Note that \( B_m \) is non-partitionable for \( m \geq 3 \).

Theorem 9.1. Let \( m \geq 2 \). For any homomorphism \( f : D_m \rightarrow B_m \), the induced map \( f : \Gamma(D_m) \rightarrow \Gamma(B_m) \) induces a monomorphism between the associated homology groups.

Proof. Up to permutations, the only \( m \)-transversal of \( B_m \) is \( \{\{1\}, \ldots, \{m\}\} \). By symmetry, we may assume that \( f(i) = \{i\} \) for each \( i \in \{1, \ldots, m\} \). Note that \( f_* \) defines an embedding of \( \Gamma(D_m) \) into \( \Gamma(B_m) \).

The goal of the proof is to show that \( \Gamma(D_m) \) admits a collapse to a subcomplex \( \Pi \) containing the image of \( \Gamma(D_m) \) under \( f_* \). If we can show that every maximal face of
this image is also a maximal face of II, then we are done. Namely, all nonvanishing homology of \( \Gamma(D_m) \) is in top dimension \( m - 1 \). The collapse is complicated and relies on the Cluster Lemma 4.4.

To define the collapse, we first need some notation. For any \((A, B) \in \Gamma(B_m)\), define

\[
\Delta_{A, B} = \{ \sigma \in \Delta(S) : \sigma \cap A \neq \emptyset, \sigma \cap B \neq \emptyset \},
\]

and let \( \bigcap \Delta_{A, B} \) denote the intersection of all families in \( \Delta_{A, B} \). By convention, \( \bigcap \emptyset = V(B_m) \). We have that \((A, B) \in \Gamma(B_m)\) if and only if \( \bigcap \Delta_{A, B} \neq \emptyset \).

For \( 1 \leq i \leq m \), let \( F_i \) be the family of cells \((A, B) \in B_m\) such that the following hold.

(a) The singleton set \( \{i\} \) belongs to both \( A \) and \( B \).

(b) The family \( A \) contains some set of size at least two.

For each \((A, B) \in F_i\) and each \( j \neq i \), we have that neither \( \{j\} \) nor \( \tau_j \) belongs to \( \Delta_{A, B} \), because \( \bigcap \Delta_{A, B} \neq \emptyset \). In particular, there is no \( j \neq i \) such that both \( A \) and \( B \) contain sets containing the element \( j \). Define \( F = \bigcup_{i=1}^m F_i \). No cell in \( \Gamma(B_m) \setminus F \) contains a cell in \( F \); hence \( \Gamma(B_m) \setminus F \) defines a cell complex.

For any \( A \subseteq V(B_m) \) and any \( i \in \{1, \ldots, m\} \), we define \( \alpha_i(A) \) to be the subset of \( \{1, \ldots, m\} \setminus \{i\} \) consisting of all \( j \neq i \) such that \( j \) belongs to a set of size at least two in \( A \). For example, if \( A = \{2, 4, 6\}, \{1, 3\}, \{3\}, \{5\} \), then \( \alpha_3(A) = \{1, 2, 4, 6\} \).

By property (b), \( \alpha_i(A) \neq \emptyset \) whenever \((A, B) \in F_i\).

For each \( i \) \( \in \{1, \ldots, m\} \) and each nonempty subset \( \alpha \) of \( \{1, \ldots, m\} \setminus \{i\} \), let \( F_{i, \alpha} \) be the family of all cells \((A, B) \in F_i\) such that \( \alpha_i(A) = \alpha \). We have that

\[
F_i = \bigcup_{\alpha} F_{i, \alpha},
\]

where the union is over all nonempty \( \alpha \subseteq \{1, \ldots, m\} \setminus \{i\} \).

We introduce a poset \( Q \) consisting of all pairs \((i, \alpha)\) such that \( i \in \{1, \ldots, m\} \) and \( \emptyset \neq \alpha \subseteq \{1, \ldots, m\} \setminus \{i\} \). We define \((i, \alpha) \leq (j, \beta)\) if and only if \( i = j \) and \( \alpha \subseteq \beta \).

Define a map \( \gamma : F \to Q \) by \( \gamma^{-1}(i, \alpha) = F_{i, \alpha} \). It is clear that \( \gamma \) is a poset map; \( \alpha_i(A) \subseteq \alpha_i(A') \) if \((A, B) \) is a face of \((A', B') \). By the Cluster Lemma 4.4, a perfect acyclic matching on each \( F_{i, \alpha} \) yields a perfect acyclic matching on \( F \).

Now, consider a given \( F_{i, \alpha} \), and pick \( j \in \alpha \). Then a perfect acyclic matching on \( F_{i, \alpha} \) is given by pairing \((A \setminus \{j\}, B)\) and \((A \cup \{j\}, B)\) for each \((A, B) \in F_{i, \alpha} \). Namely, \( \alpha_i(A) \) remains unmodified when adding or removing singleton sets to or from \( A \). Moreover, \( B \) cannot contain any set containing the element \( j \), because \( A \) contains such a set with at least two elements.

To conclude, we do have a perfect acyclic matching on \( F \). By Lemma 4.3, there is hence a collapse from \( \Gamma(B_m) \) to \( \Sigma = \Gamma(B_m) \setminus F \).

Using exactly the same argument, we may define a perfect acyclic matching on the subfamily \( G \) of \( \Sigma \) consisting of all cells \((A, B)\) satisfying the following conditions for some \( i \).

(a) The singleton set \( \{i\} \) belongs to both \( A \) and \( B \).

(b) \( B \) contains some set of size at least two.

Define \( \Pi = \Sigma \setminus G \). To finish the proof, it remains to check that the image under \( f_\ast \) of every maximal face of \( \Gamma(B_m) \) is maximal in \( \Pi \). Let \((A, B) \) be such an image;
thus
\[ A \cup B = \{\{j\} : j \in \{1, \ldots, m\}\}, \]
\[ A \cap B = \{\{i\}\} \]
for some \(i\). Suppose that \((A \cup \{I\}, B) \in \Pi\) for some \(I \notin A\). By properties of \(\Pi\), we cannot have that \(|I| \geq 2\); hence \(I\) is a singleton set \(\{j\}\). Yet, \(\{j\} \notin A\), we have that \(\{j\} \in B\), which means that \(\{i\}, \{j\} \in A \cap B\), which is impossible. Similarly, it is impossible that \((A, B \cup \{I\}) \in \Pi\) for some \(I \notin B\). Thus \((A, B)\) is maximal in \(\Pi\), which concludes the proof. 

Corollary 9.2. Let \(S\) be a set system with a strong cover \((\sigma_1, \ldots, \sigma_m)\), and let \(\mu_i = \sigma_i \setminus \bigcup_{j \neq i} \sigma_j\). For each \(i\), suppose that \(\mu_i\) is stable. Then there exists a homomorphism \(g\) from \(S\) to the benchmark system \(B_m\). In particular, for each \(f \in \text{Hom}_1(B_m, S)\), the induced map \(f_* : \Gamma(D_m) \to \Gamma(S)\) induces a homomorphism between the associated homology groups.

Proof. Define \(g : V(S) \to V(B_m)\) by
\[ g(x) = \{i : x \in \sigma_i\}. \]
We have that \(g\) defines a homomorphism from \(S\) to \(B_m\), because
\[ g^{-1}(\{i\}) = \mu_i, \]
\[ g^{-1}(\tau_i) = \sigma_i. \]
For the final statement of the corollary, Theorem 9.1 yields that the composition \(g_* f_*\) induces a monomorphism from the homology of \(\Gamma(D_m)\) to the homology of \(\Gamma(B_m)\). As a consequence, the same is true for \(f_*\). 

Corollary 9.3. Let \(P\) be a rooted poset with minimal element 0 and coatoms \(c_1, \ldots, c_m\). Let \(\mu_i\) be the set of elements \(x\) such that \(x \leq c_i\) and \(x \not\leq c_j\) for \(j \neq i\). Suppose \(\mu_i\) has a unique minimal element for each \(i\). Then there exists a homomorphism \(g\) from \(\mathcal{I}(P)\) to the benchmark system \(B_m\). In particular, for each \(f \in \text{Hom}_1(B_m, \mathcal{I}(P))\), the induced map \(f_* : \Gamma(D_m) \to \Gamma(\mathcal{I}(P))\) induces a monomorphism between the associated homology groups. This remains true if we replace \(\mathcal{I}(P)\) with the reduced interval system \(\mathcal{I}^\text{red}(P)\).

Proof. Let us show that the conditions of Corollary 9.2 are satisfied for \(\mathcal{I}(P)\) and \(\mathcal{I}^\text{red}(P)\). For both systems, a strong cover is given by the intervals \([0, c_1], \ldots, [0, c_m]\).

The set \(\mu_i\) having a unique minimal element \(q_i\) means that \(\mu_i = [q_i, c_i]\); hence \(\mu_i\) is stable. Applying Corollary 9.2, we are done. 

In a pure simplicial complex, all maximal faces have the same dimension. In the following corollary, we view the empty set as a \((-1)\)-dimensional face of the simplicial complex under consideration. By the discussion at the beginning of the section, every Cohen–Macaulay complex satisfies the assumptions of the corollary.

Corollary 9.4. Let \(\Lambda\) be a pure \(d\)-dimensional simplicial complex, viewed as a poset ordered by inclusion, and assume that \(m = \omega(\Lambda)\). Assume that \(\text{link}_\Lambda(q)\) is connected for every face \(q\) of \(\Lambda\) of dimension at most \(d - 2\). Then there exists a homomorphism \(g\) from \(\mathcal{I}(\Lambda)\) to the benchmark system \(B_m\). In particular, for each \(f \in \text{Hom}_1(B_m, \mathcal{I}(\Lambda))\), the induced map \(f_* : \Gamma(D_m) \to \Gamma(\mathcal{I}(\Lambda))\) induces a monomorphism between the associated homology groups. This remains true if we replace \(\mathcal{I}(\Lambda)\) with the reduced interval system \(\mathcal{I}^\text{red}(\Lambda)\).
Proof. Let \( c \) be a maximal face of \( \Lambda \), and let \( \mu \) be the family of faces of \( \Lambda \) that are contained in \( c \) but not in any other maximal face of \( \Lambda \). Let \( q \in \Lambda \) be maximal such that \( q \) is contained in every member of \( \mu \). We want to show that \( q \in \mu \); this will imply that the conditions of Corollary 9.3 are satisfied for \( \mathcal{I}(\Lambda) \) and \( \mathcal{I}^{\text{red}}(\Lambda) \).

Suppose \( q \notin \mu \). Let \( y \) be maximal in the interval \([q, c]\) such that \( y \notin \mu \). Then \( \text{link}_\Lambda(y) \) contains the full simplex on \( c \setminus y \) as a proper subcomplex. Yet, there are no edges in \( \text{link}_\Lambda(y) \) of the form \( \{a, b\} \) such that \( a \in c \setminus y \) and \( b \notin c \setminus y \). Namely, that would imply that \( y \cup \{a\} \subseteq y \cup \{a, b\} \in \Lambda \setminus \mu \) and hence that \( y \cup \{a\} \notin \mu \), contradicting the maximality of \( y \).

In particular, \( \text{link}_\Lambda(y) \) is disconnected. By assumption, this implies that \( y \) must be a face of \( c \) of codimension one and hence of the form \( c \setminus \{a\} \) for some \( a \in c \). We deduce that every member of \( \mu \) contains the element \( a \). Yet, by maximality of \( q \), we then have that \( a \in q \), which contradicts the assumption that \( q \subseteq y \). \( \square \)

10. Other potential witnesses

In this section, we provide a few examples of other set systems that might be potentially useful as witnesses. Indeed, it turns out that both systems are witnesses for the reduced interval system of the simplicial complex \( K_2^2 \) with maximal faces \( ab \) and \( cd \), and the proofs are even simpler than the proof of Lemma 8.1, which settled the analogous fact for the reduced interval system of the 3-chain. Getting beyond this result seems harder, and we have yet to discover counterparts to any of the other results in Sections 8 and 9.

Our first example is the interval system of the poset \( X \) consisting of the five elements \( x_1, x_2, y, z_1, z_2 \) satisfying \( x_i \leq y \) and \( y \leq z_i \) for all \( i \in \{1, 2\} \); the only incomparable pairs of elements are \( \{x_1, x_2\} \) and \( \{z_1, z_2\} \). One may check that we obtain an interval map \( f \) from \( \mathcal{I}(X) \) to \( \mathcal{I}^{\text{red}}(K_2^2) \) by defining

\[
f(x_1) = a, f(x_2) = c, f(y) = \emptyset, f(z_1) = b, f(z_2) = d.
\]

In particular, \( \text{Hom}(\mathcal{I}(X), \mathcal{I}^{\text{red}}(K_2^2)) \) is not the empty complex, which implies that there is no nonzero reduced homology in degree \(-1\). Yet, it is easy to see that \( X \) is not partitionable, which is equivalent to saying that \( \text{Hom}(\mathcal{I}(X), D_2) \) is the empty complex. We conclude that this complex has nonvanishing reduced homology in degree \(-1\). As a consequence, \( \mathcal{I}(X) \) is a witness for the reduced interval system of \( K_2^2 \).

Our second example is the set system \( \mathcal{F} \) on the ground set \( \{1, 2, 3\} \) with stable sets

\[
\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3\}.
\]

Equivalently, \( \{1, 3\} \) and \( \{2, 3\} \) are the non-stable sets. Given another set system \( \mathcal{T} \), we may identify a cell \( F \in \text{Hom}(\mathcal{F}, \mathcal{T}) \) (and the associated oriented cell) with the triple \( F(1)|F(2)|F(3) \). One may note that \( \text{Hom}(\mathcal{F}, D_2) \) is the disjoint union of two edges, one with endpoints \( 1|1\{1\} \) and \( 1|1\{2\} \) and one with endpoints \( 2|2\{1\} \) and \( 2|2\{2\} \). In particular, there is nonvanishing homology in degree \( 0 \), and the homology is generated by the homology class of the cycle \( z = 1|1\{1\} - 2|2\{2\} \). Let \( f \in \text{Hom}_0(D_2, \mathcal{I}^{\text{red}}(K_2^2)) \) be defined by \( f(1) = a \) and \( f(2) = c \). We have that \( f \) induces a map from \( \text{Hom}(\mathcal{F}, D_2) \) to \( \text{Hom}(\mathcal{F}, \mathcal{I}^{\text{red}}(K_2^2)) \), and the associated chain map sends \( z \) to the cycle \( z' = a|a - c|c \). Now, \( z' \) is the boundary of

\[
c|c|cd - c|c|d + c|0|d + + 0|d db - a|0|b + a|0|b - a|a|ab.
\]
In particular, the map in homology induced by $f$ is not a monomorphism, which means that $F$ is a witness for $\Lambda$.

11. Categorical properties of set systems

Recall that Systems is the category in which objects are set systems and morphisms are set system homomorphisms. In this section, we discuss properties of this category. To start with, we observe that any void system of the form $\{\{x\}, \emptyset\}$ is a terminal object. There is no initial object, but we could introduce one by adding the system $\{\emptyset, \{\emptyset\}\}$ and introducing an “empty” homomorphism from this system to any other system. For the purposes of the present section, we will not need any initial object.

For two families $\Delta_1$ and $\Delta_2$ of subsets of $V$ and $W$, respectively, define

$$\Delta_1 \star \Delta_2 = \{\delta_1 \cup \delta_2 : \delta_1 \in \Delta_1, \delta_2 \in \Delta_2\} \subseteq 2^{V \cup W}.$$ 

**Proposition 11.1.** The categorical coproduct of two set systems $S_1$ and $S_2$ is the set system

$$S_1 \star S_2 = (V(S_1) \sqcup V(S_2), \Delta(S_1) \star \Delta(S_2)).$$

**Proof.** Consider the inclusion map $i_i : V(S_i) \to V(S_1) \sqcup V(S_2)$ for $i \in \{1, 2\}$. The preimage under $i_i$ of a stable set $\sigma_1 \sqcup \sigma_2 \in \Delta(S_1) \star \Delta(S_2)$ is $\sigma_i$, which is stable in $S_i$; hence $i_i \in \text{Hom}_0(S_i, S_1 \star S_2)$ for $i \in \{1, 2\}$.

Next, let $\mathcal{T}$ be any set system. Given morphisms $f_i \in \text{Hom}_0(S_i, \mathcal{T})$, define $f : V(S_1) \sqcup V(S_2) \to V(\mathcal{T})$ by $f(v) = f_1(v)$ if $v \in V(S_1)$ and $f(v) = f_2(v)$ if $v \in V(S_2)$. This is the unique morphism in $\text{Hom}_0(S_1 \star S_2, \mathcal{T})$ satisfying $f_{i_1} = f_i$ for $i \in \{1, 2\}$. Namely, uniqueness is clear, and given any stable set $\sigma$ of $\mathcal{T}$, the preimage of $\sigma$ is $f^{-1}(\sigma) = f_1^{-1}(\sigma) \sqcup f_2^{-1}(\sigma)$, hence stable. □

Note that $S_1 \star S_2$ is not the same set system as the system $S_1 + S_2$ discussed in Section 8.2.

We have that a map $f : S \star \mathcal{T} \to \mathcal{U}$ is a homomorphism if and only if the restriction of $f$ to each of $S$ and $\mathcal{T}$ is a homomorphism. As a consequence, we have the following result.

**Proposition 11.2.** For any set systems $S$, $\mathcal{T}$, and $\mathcal{U}$, we have that the cell complexes $\text{Hom}(S \star \mathcal{T}, \mathcal{U})$ and $\text{Hom}(S, \mathcal{U}) \times \text{Hom}(\mathcal{T}, \mathcal{U})$ are isomorphic.

The shape of $\text{Hom}(S, \mathcal{T} \star \mathcal{U})$ depends on the involved set systems. The separability graph of a set system $S$ is the graph on the vertex set $V(S)$ in which two vertices $u$ and $v$ are joined by an edge if and only if $u$ and $v$ are separated.

**Proposition 11.3.** If the separability graph of $S$ is connected and $|V(S)| \geq 2$, then the cell complex $\text{Hom}(S, \mathcal{T} \star \mathcal{U})$ is isomorphic to the disjoint union of $\text{Hom}(S, \mathcal{T})$ and $\text{Hom}(S, \mathcal{U})$.

**Proof.** Let $F \in \text{Hom}(S, \mathcal{T} \star \mathcal{U})$. Suppose there is an $x$ such that $F(x) \cap V(\mathcal{T}) \neq \emptyset$, and let $y$ be a neighbor of $x$ in the separability graph. Then we must have that $F(y) \subseteq V(\mathcal{T})$. Since the separability graph is connected, we deduce that $F(z) \subseteq V(\mathcal{T})$ for all $z \in V(S)$. The discussion is analogous for the case that $F(x) \cap V(\mathcal{U}) \neq \emptyset$.

Conversely, suppose that $F \in \text{Hom}(S, \mathcal{T})$. Then we may view $F$ as a cell of $\text{Hom}(S, \mathcal{T} \star \mathcal{U})$, because given a stable set $\sigma_1 \sqcup \sigma_2$ of $\mathcal{T} \star \mathcal{U}$ and $f \in F$, we have
that $f^{-1}(\sigma_1 \uplus \sigma_2) = f^{-1}(\sigma_1)$, which is stable in $S$. Again, the discussion remains analogous for $U$ instead of $T$. \hfill \Box

**Proposition 11.5.** Let $S_1$ and $S_2$ be set systems such that the empty set is stable in both systems. Then

$$
\begin{align*}
\omega(S_1 \ast S_2) &= \max \{\omega(S_1), \omega(S_2)\}, \\
\chi(S_1 \ast S_2) &= \max \{\chi(S_1), \chi(S_2)\}.
\end{align*}
$$

**Proof.** By the existence of a homomorphism in $\text{Hom}(S_i, S_i \ast S_i)$, any homomorphism in $\text{Hom}_i(S_i \ast S_i, D_n)$ extends to a homomorphism in $\text{Hom}_i(S_i, D_n)$ for $i = 1, 2$. Thus $\chi(S_1 \ast S_2) = \max \{\chi(S_1), \chi(S_2)\}$. Conversely, any two homomorphisms in $\text{Hom}_0(S_1, D_n)$ and $\text{Hom}_0(S_2, D_n)$ extend to a homomorphism in $\text{Hom}_0(S_1 \ast S_2, D_n)$. Since the empty set belongs to $\Delta(S_i)$, we have that $\text{Hom}_0(S_i, D_n)$ is nonempty whenever $n \geq \chi(S_i)$. Thus $\chi(S_1 \ast S_2) \leq \max \{\chi(S_1), \chi(S_2)\}$.

By the existence of a homomorphism in $\text{Hom}_0(S_i, S_i \ast S_i)$, any homomorphism in $\text{Hom}_0(D_n, S_i)$ extends to a homomorphism in $\text{Hom}_0(D_n, S_i \ast S_i)$ for $i = 1, 2$. Thus $\omega(S_1 \ast S_2) = \max \{\omega(S_1), \omega(S_2)\}$. Conversely, first note that $S_1 \ast S_2$ is nonfree (see Section 3.1) whenever $S_1$ or $S_2$ is nonfree; hence $\omega(S_1 \ast S_2) = \max \{\omega(S_1), \omega(S_2)\} = \infty$ by Proposition 3.3. Suppose that $S_1$ and $S_2$ are both free. Let $f \in \text{Hom}_0(D_n, S_1 \ast S_2)$. Since both set systems are free, any two elements $y_1 \in V(S_1)$ and $y_2 \in V(S_2)$ appear in a common stable set in $\Delta_1 \ast \Delta_2$. In particular, the image of $f$ must be a subset of either $V(S_1)$ or $V(S_2)$, meaning that we may view $f$ as a member of $\text{Hom}_0(D_m, S_i)$ for either $i = 1$ or $i = 2$. Thus $\omega(S_1 \ast S_2) \leq \max \{\omega(S_1), \omega(S_2)\}$. \hfill \Box

**Proposition 11.5.** The categorical product of two set systems $T_1 = (W_1, \Sigma_1)$ and $T_2 = (W_2, \Sigma_2)$ is the set system

$$
T_1 \times T_2 = (W_1 \times W_2, \{\sigma_1 \times W_2 : \sigma_1 \in \Sigma_1\} \cup \{W_1 \times \sigma_2 : \sigma_2 \in \Sigma_2\}).
$$

**Proof.** Consider the projection map $\pi_i : W_1 \times W_2 \Rightarrow W_i$ for $i \in \{1, 2\}$. The preimage under $\pi_1$ of a stable set $\sigma_1 \in \Sigma_1$ is $\sigma_1 \times W_2$, which is stable in $T_1 \times T_2$. Similarly, the preimage under $\pi_2$ of every stable set is stable. In particular, $\pi_i \in \text{Hom}_0(T_1 \times T_2, T_i)$ for $i \in \{1, 2\}$.

Next, let $S$ be any set system. Given morphisms $f_i \in \text{Hom}_0(S, T_i)$, define $f : V(S) \Rightarrow W_1 \times W_2$ by $f(v) = (f_1(v), f_2(v))$. This is the unique morphism in $\text{Hom}_0(S, T_1 \times T_2)$ satisfying $f_i f = f_i$ for $i \in \{1, 2\}$. Namely, uniqueness is clear, and given any stable sets $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, the preimages of $\sigma_1 \times W_2$ and $W_1 \times \sigma_2$ are $f_1^{-1}(\sigma_1)$ and $f_2^{-1}(\sigma_2)$, respectively, hence stable. \hfill \Box

**Proposition 11.5.** Let $T_1$ and $T_2$ be set systems. Then

$$
\begin{align*}
\omega(T_1 \times T_2) &= \min \{\omega(T_1), \omega(T_2)\}, \\
\chi(T_1 \times T_2) &= \min \{\chi(T_1), \chi(T_2)\}.
\end{align*}
$$

**Proof.** By the existence of a homomorphism in $\text{Hom}_0(T_1 \times T_2, T_i)$, any homomorphism in $\text{Hom}_0(D_m, T_1 \times T_2)$ extends to a homomorphism in $\text{Hom}_0(D_m, T_i)$ for $i = 1, 2$. Thus $\omega(T_1 \times T_2) \leq \min \{\omega(T_1), \omega(T_2)\}$.

Conversely, any two homomorphisms in $\text{Hom}_0(D_m, T_1)$ and $\text{Hom}_0(D_m, T_2)$ extend to a homomorphism in $\text{Hom}_0(D_m, T_1 \times T_2)$. As a consequence, $\omega(T_1 \times T_2) \leq \min \{\omega(T_1), \omega(T_2)\}$. \hfill \Box
By the existence of a homomorphism in \( \text{Hom}_0(T_1 \times T_2, T_i) \), any homomorphism in \( \text{Hom}_0(T_i, D_n) \) extends to a homomorphism in \( \text{Hom}_0(T_i \times T_2, D_n) \) for \( i = 1, 2 \).

Thus \( \chi(T_1 \times T_2) \leq \min \{ \chi(T_1), \chi(T_2) \} \).

Conversely, let \( f \in \text{Hom}_0(T_1 \times T_2, D_n) \). Suppose that \( k, \ell \in \{1, \ldots, n\} \) are elements such that \( f^{-1}(k) = V(T_2) \times V(T_1) \) and \( f^{-1}(\ell) = V(T_2) \times V(T_1) \) for some nonempty \( \sigma_1 \subseteq V(T_1) \) and \( \sigma_2 \subseteq V(T_2) \). Then \( f^{-1}(k) \cap f^{-1}(\ell) = \sigma_1 \times \sigma_2 \neq \emptyset \), a contradiction. We conclude that the preimages \( f^{-1}(k) \) are either all of the form \( \sigma_1 \times V(T_2) \) or all of the form \( V(T_1) \times \sigma_2 \). In the former case, we obtain a homomorphism \( f_1 \in \text{Hom}_0(T_1, D_n) \) by defining \( f_1(x) = f(x, y) \), where we may pick \( y \) arbitrarily. The latter case is treated analogously. Thus \( \chi(T_1 \times T_2) \geq \min \{ \chi(T_1), \chi(T_2) \} \). \( \square \)

The categorical product is different if we restrict our attention to the subcategory \textbf{MonSystems} of monotone systems. More precisely, in this category, the product of two monotone systems \( T_1 = (W_1, \Sigma_1) \) and \( T_2 = (W_2, \Sigma_2) \) is the monotone system \( (W_1 \times W_2, \{ E \subseteq W_1 \times W_2 : \pi_1(E) \in \Sigma_1 \text{ or } \pi_2(E) \in \Sigma_2 \}) \).

Indeed, this is the monotone system on \( W_1 \times W_2 \) in which the simplicial complex of stable sets is minimal with the property that all stable sets of \( T_1 \times T_2 \) belong to the complex.

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