On the Topology of Independence Complexes of Triangle-Free Graphs

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Abstract

The independence complex of a (finite) simple graph is the abstract simplicial complex consisting of all independent vertex sets in the graph. We show that the possible homotopy types of independence complexes of bipartite graphs are exactly those of suspensions of simplicial complexes. As a consequence, there exist bipartite graphs such that the integral homology of the associated independence complexes is not free. This answers a question by Engström. The smallest such bipartite graph found so far has 16 vertices and 30 edges.

Note. It has come to my attention that Uwe Nagel and Victor Reiner [8] published a proof of the main result of the present manuscript before I even started on the project. For this reason, I will not publish this manuscript.

1 Introduction

Throughout this note, we assume that all graphs and abstract simplicial complexes are finite. Whenever we talk about the topology of an abstract simplicial complex, we mean the topology of any geometric realization of the complex. By convention, we include the empty set in any simplicial complex.

For a graph G with vertex set V (always assumed to be nonempty), let I_G denote the independence complex of G. More precisely, I_G is the simplicial complex on the vertex set V with the property that a set $\sigma \subseteq V$ is a face of I_G if and only if there are no edges in G between the vertices in σ.

It is well-known that any simplicial complex is homotopy equivalent, even homeomorphic, to I_G for some graph G. Namely, any complex is homeomorphic to its barycentric subdivision, and it is easy to see that any such subdivision is the independence complex of a graph. In this note, we put restrictions on the lengths of the cycles of G. The following theorem is the main result and deals with the case that G is bipartite, which is equivalent to saying that G does not contain any cycles of odd length.

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Theorem 1 *A simplicial complex Δ is homotopy equivalent to I_G for some bipartite graph G if and only if Δ is homotopy equivalent to the suspension of some simplicial complex.*

Here, the suspension of a simplicial complex Γ is defined as

$$\mathrm{Susp}(\Gamma) = \{\sigma, \sigma \cup \{x\}, \sigma \cup \{y\} : \sigma \in \Gamma\},$$

where x and y are two new vertices not appearing in Γ. Writing $\mathrm{Susp}(\Gamma)$ as a union of two cones and applying the appropriate Mayer-Vietoris sequence [7, §25], one readily verifies that

$$\tilde{H}_i(\mathrm{Susp}(\Gamma); \mathbb{Z}) \cong \tilde{H}_{i-1}(\Gamma; \mathbb{Z})$$

for $i \geq 0$, where $\tilde{H}_i(\Gamma; \mathbb{Z})$ is the reduced simplicial homology in degree i of Γ. In particular, Theorem 1 implies the following fact: For every finite sequence $(A_2, \ldots, A_n)$ of finitely generated abelian groups, there is a bipartite graph G such that $\tilde{H}_i(I_G; \mathbb{Z})$ is isomorphic to A_i for $2 \leq i \leq n$. This answers a question by Engström [3, §1.4], who asked whether there exist triangle-free graphs G such that the homology of I_G contains torsion (nonzero elements of finite order). To the author's knowledge, there is no previously known method for constructing triangle-free graphs equipped with such torsion.

Barmak [1] recently extended Theorem 1 to triangle-free graphs, showing that the independence complex of a triangle-free graph is always homotopy equivalent to a suspension.

We also consider restrictions on the girth of a graph, i.e., the shortest length of a cycle in the graph. By convention, the girth of a forest is $+\infty$.

Theorem 2 *For every simplicial complex Γ and every $g \geq 3$, there is an integer $k \geq 0$ and a graph G with girth at least g such that $I_G \simeq \mathrm{Susp}^k(\Gamma)$.*

An interesting problem is to compute the minimum value of k in Theorem 2 for various Γ and g. For $g = 3$, we may always pick $k = 0$, because every simplicial complex is homotopy equivalent to some independence complex. For $g = 4$, Theorem 1 implies that we may pick $k = 1$; hence the minimum value of k is 0 or 1. For $g \geq 5$, the problem remains open. In fact, we do not even know whether there exists an integer $K(g)$ such that the minimum value of k in Theorem 2 is at most $K(g)$ for every simplicial complex Γ.

Another possibility is to restrict to planar graphs. By a recent result due to Skwarski [9], for every simplicial complex Γ, there is an integer $k \geq 0$ and a planar graph G such that $I_G \simeq \mathrm{Susp}^k(\Gamma)$.

2 Some useful topological facts

We will need a few facts from simplicial topology.

Lemma 2.1 *Let Γ be a contractible simplicial complex, and let Δ be a subcomplex of Γ. Then Γ/Δ is homotopy equivalent to the suspension of Δ.*

For a proof, see Jonsson [6, Lemma 3.18].

Lemma 2.2 *Let Γ be a simplicial complex, and let Δ be a contractible subcomplex of Γ. Then Γ/Δ and Γ are homotopy equivalent.*

For a proof, see Hatcher [4, Prop. 0.17].

An (order-preserving) *poset map* between two partially ordered sets $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ is a function $f : X \to Y$ such that $f(x) \leq_Q f(y)$ whenever $x \leq_P y$. For simplicity, we will write $f : P \to Q$. We identify a simplicial complex Δ with the partially ordered set $(\Delta, \subseteq)$.

A simplicial complex Δ is obtained from another simplicial complex Δ' via an *elementary collapse* if $\Delta' \setminus \Delta = \{\sigma, \tau\}$ and $\sigma \subsetneqq \tau$. This means that τ is the only face in Δ' properly containing σ. Elementary collapses are well-known to preserve homotopy type. See Cohen [2] for more information about elementary collapses. If we can obtain Δ from Δ' via a sequence of elementary collapses, then we say that we may *collapse* Δ' to Δ. If Δ' can be collapsed to a 0-simplex $\{\emptyset, \{v\}\}$, then Δ' is *collapsible.*

The following result is a special case of a technical lemma in discrete Morse theory [5, Lemma 4.1] [6, Lemma 4.2].

Lemma 2.3 *Let Δ be a simplicial complex, and let $f : \Delta \to Q$ be a poset map, where Q is an arbitrary finite poset. For each q in Q, suppose that there is a vertex $x = x(q)$ in Δ such that $\sigma \setminus \{x\} \in f^{-1}(q)$ if and only if $\sigma \cup \{x\} \in f^{-1}(q)$. Then Δ is collapsible and hence contractible.*

Proof. We give a self-contained proof, using double induction on the sizes of Δ and Q. Pick a maximal element q in Q. If $f^{-1}(q)$ is empty, then we may redefine f as a map from Δ to $Q \setminus \{q\}$. By induction on the size of Q, we have that Δ is collapsible.

Assume that $f^{-1}(q)$ is nonempty. By assumption, there is an element x such that $\sigma \setminus \{x\} \in f^{-1}(q)$ if and only if $\sigma \cup \{x\} \in f^{-1}(q)$. Pick a maximal set τ in $f^{-1}(q)$. Since f is a poset map, τ is a maximal face of Δ. Moreover, x belongs to τ, and $\tau \setminus \{x\}$ belongs to $f^{-1}(q)$. In addition, τ is the only maximal face of Δ that contains $\tau \setminus \{x\}$. Namely, by maximality of q, any face of Δ containing $\tau \setminus \{x\}$ is contained in $f^{-1}(q)$. If $y \notin \tau$ is such that $(\tau \setminus \{x\}) \cup \{y\}$ belongs to $f^{-1}(q)$, then $((\tau \setminus \{x\}) \cup \{y\}) \cup \{x\} = \tau \cup \{y\}$ also belongs to $f^{-1}(q)$, which contradicts the maximality of τ.

To conclude, either $\tau = \{x\}$ and hence $\Delta = \{\emptyset, \{x\}\}$, in which case Δ is collapsible by definition, or we may collapse Δ to $\Delta_0 = \Delta \setminus \{\tau \setminus \{x\}, \tau\}$. In the latter case, the restriction of f to Δ_0 is a poset map with properties as in the lemma; hence induction yields that Δ_0 is collapsible. As a consequence, Δ is collapsible as well. $\square$

3 Bipartite graphs

Let G be a bipartite graph with nonempty parts V and W. Let $\Gamma_{G,V} \subseteq 2^V$ be the simplicial complex defined in the following manner:

- A set $\sigma \subseteq V$ belongs to $\Gamma_{G,V}$ if and only if there is a vertex $w \in W$ such that $\sigma \cup \{w\}$ is an independent set in G.

Equivalently, $\sigma \subseteq V$ is not a face of $\Gamma_{G,V}$ if and only if σ covers W, meaning that each $w \in W$ is adjacent to some $v \in V$. Since $W \neq \emptyset$, we have that $\Gamma_{G,V}$ contains the empty set.

Theorem 3.1 *Let G be a bipartite graph with nonempty parts V and W. Then $I_G \simeq \mathrm{Susp}(\Gamma_{G,V})$.*

Proof. For each subset σ of V, let Δ_σ be the subfamily of I_G consisting of all faces τ such that $\tau \cap V = \sigma$. Write $\Gamma = \Gamma_{G,V}$. Define Σ to be the union of all Δ_σ such that $\sigma \in \Gamma$; note that Σ is a subcomplex of I_G. By definition of Γ, $\sigma \notin \Gamma$ if and only if $\Delta_\sigma = \{\sigma\}$. In particular,

$$I_G/\Sigma = 2^V/\Gamma \simeq \mathrm{Susp}(\Gamma);$$

for the homotopy equivalence, use Lemma 2.1. It remains to prove that Σ is contractible. This will imply the desired result, because then $I_G/\Sigma \simeq I_G$ by Lemma 2.2.

Now, note that we obtain a poset map $f : \Sigma \to \Gamma$ by defining $f^{-1}(\sigma) = \Delta_\sigma$. Look at an individual subfamily Δ_σ such that $\sigma \in \Gamma$. Let $w = w(\sigma) \in W$ be such that $\sigma \cup \{w\}$ is independent. We have that $\tau \setminus \{w\}$ belongs to Δ_σ if and only if $\tau \cup \{w\}$ belongs to Δ_σ. In particular, the conditions of Lemma 2.3 are satisfied, which implies that Σ is collapsible and hence contractible. $\square$

Theorem 3.2 *Let Γ be a simplicial complex. Then there is a bipartite graph G such that $I_G \simeq \mathrm{Susp}(\Gamma)$.*

Proof. Let V be the vertex set of Γ; add a vertex if Γ is the empty complex. We define the graph G in the following manner. Let the vertex set of G be the disjoint union of V and $M(\Gamma)$, where $M(\Gamma)$ is the set of maximal faces of Γ. Hence, starting with the vertex set V of Γ, we add a new vertex for each maximal face of Γ. The edges of G are all pairs $\{v, \mu\}$ such that $v \in V$, $\mu \in M(\Gamma)$ and $v \notin \mu$. For subsets $\sigma \subseteq V$ and $A \subseteq M(\Gamma)$, note that $\sigma \cup A$ is a face of I_G if and only if $\sigma \subseteq \mu$ for every $\mu \in A$.

By Theorem 3.1, it suffices to prove that $\Gamma = \Gamma_{G,V}$. To obtain this, consider a set $\sigma \subseteq V$. The set σ being a face of Γ is equivalent to saying that there is a maximal face $\mu \in M(\Gamma)$ such that $\sigma \subseteq \mu$. This in turn is equivalent to saying that there is a $\mu \in M(\Gamma)$ such that $\sigma \cup \{\mu\}$ is an independent set in G, which is equivalent to σ being a face of $\Gamma_{G,V}$. $\square$

Combining Theorems 3.1 and 3.2, we obtain Theorem 1.

Corollary 3.3 *Let $(A_0, A_1, A_2, \ldots)$ be a sequence of finitely generated Abelian groups such that $A_i = 0$ for all sufficiently large i. There is a bipartite graph G with nonempty parts such that $\tilde{H}_i(I_G; \mathbb{Z}) \cong A_i$ for $i \geq 0$ if and only if either A_0 is zero and A_1 is free or $A_0 \cong \mathbb{Z}$ and $A_i = 0$ for $i \geq 1$.*

Proof. Shifting the indices one step down, we get a characterization of all possible sequences of homology groups for a nonempty finite simplicial complex. By Theorem 1, we are done. $\square$

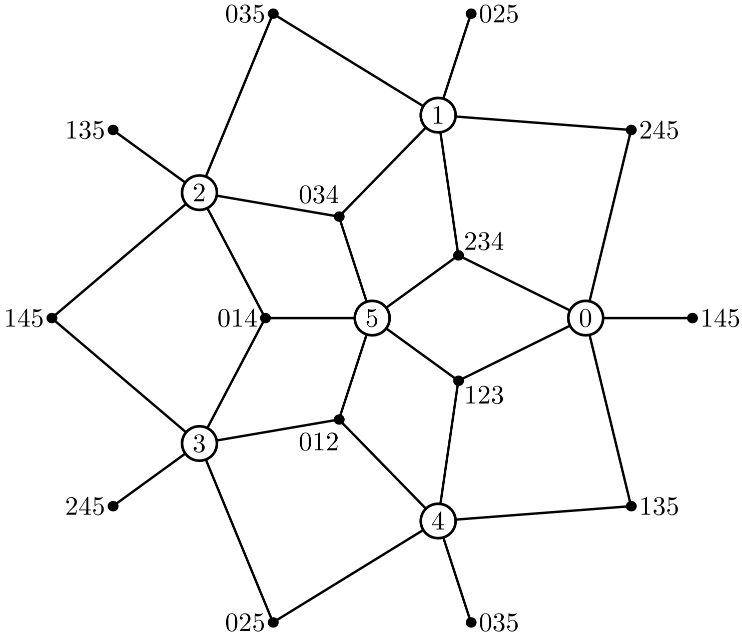

Figure 1: The bipartite graph G_1 obtained from the six-vertex triangulation of the projective plane $\mathbb{RP}^2$ via the procedure in the proof of Theorem 3.2. Vertices with the same label are the same.

The most important consequence is that there are bipartite graphs G such that the homology of I_G contains arbitrarily complicated torsion. The smallest simplicial complex with torsion in its homology is the six-vertex triangulation of the projective plane $\mathbb{RP}^2$; the maximal faces of this complex are

$$\{012, 123, 234, 034, 014, 025, 245, 145, 135, 035\}.$$

Using the proof of Theorem 3.2, we obtain that $I_{G_1} \simeq \mathrm{Susp}(\mathbb{RP}^2)$, where G_1 is the bipartite graph in Figure 1. In particular, $\tilde{H}_2(I_{G_1}; \mathbb{Z}) \cong \mathbb{Z}_2$. Note that G_1 consists of 16 vertices and 30 edges. We are not aware of any bipartite graph G with fewer vertices such that the homology of I_G contains torsion. It might be worth mentioning that the f-vector of I_{G_1} is

$$(1, 16, 90, 230, 310, 288, 217, 120, 45, 10, 1)$$

One may compare to the graph G_2 in Figure 2. This graph has eleven vertices and 25 edges, and the associated independence complex I_{G_2} is a triangulation

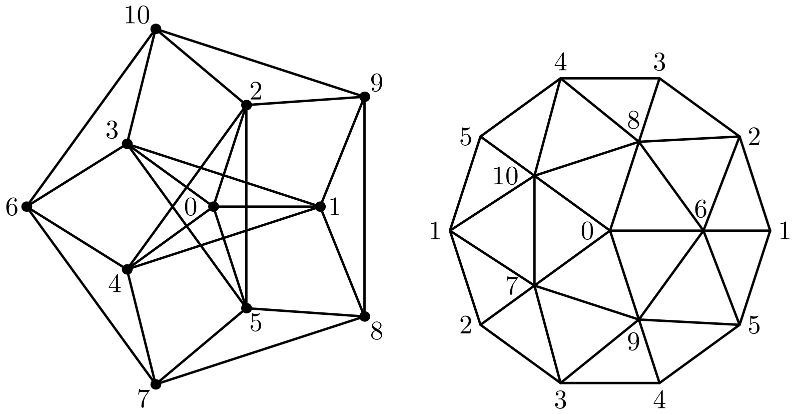

Figure 2: On the left a graph G_2 on eleven vertices such that the independence complex I_{G_2} is a triangulation of the projective plane $\mathbb{RP}^2$. On the right a geometric realization of I_{G_2}.

of $\mathbb{RP}^2$. In particular, $\tilde{H}_1(I_{G_2}) \cong \mathbb{Z}_2$. Note that G_2 contains triangles and also cycles of length five. The f-vector of I_{G_2} is $(1, 11, 30, 20)$.

4 On graphs with a certain girth

For a graph G, let $g(G)$ denote the girth of G, i.e., the shortest length of a cycle in G. We have not been able to find a characterization of possible homology groups of I_G for graphs G with a certain girth, but we have some partial results.

Lemma 4.1 *Let G be a graph such that there are κ distinct cycles of length $g(G)$ in G. Then there is a graph G' satisfying*

$$I_{G'} \simeq \mathrm{Susp}(I_G)$$

such that $g(G') \geq g(G)$ and such that there are at most $\kappa - 1$ distinct cycles of length $g(G)$ in G'.

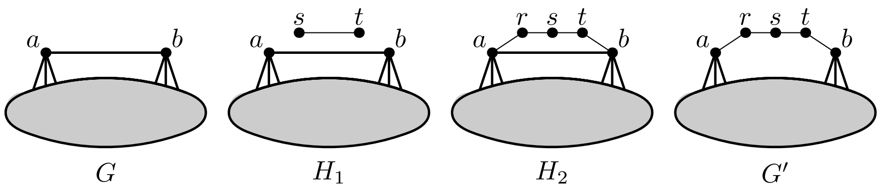

Figure 3: The graphs G, H_1, H_2, and G' in the proof of Lemma 4.1.

Proof. Let a and b be any adjacent vertices in G such that there is a cycle of length $g(G)$ in G containing the edge $ab = \{a, b\}$. Form a new graph H_1 from G by adding the vertices s and t and the edge st. Form another new graph H_2 from G by adding three new vertices r, s and t and the four edges ar, rs, st, tb. Finally, let G' be the graph obtained from H_1 by removing the edge ab.

The process of replacing the edge ab with the sequence of edges (ar, rs, st, tb) corresponds to replacing every cycle in G containing ab with a new cycle containing three more vertices. Since there is at least one such cycle of length g in G, and since there are no cycles of length less than g, the number of cycles of length g in G' is strictly less than κ.

Note that

$$I_{H_2} \simeq I_{G'}.$$

Namely, we obtain a collapse from $I_{G'}$ to I_{H_2} by collapsing all pairs of the form $(\sigma \cup \{a, b\}, \sigma \cup \{a, b, s\})$, where $\sigma \cap \{a, b, r, s, t\} = \emptyset$. Moreover, by construction we have that

$$\mathrm{Susp}(I_G) = I_{H_1}.$$

In particular, it remains to prove that

$$I_{H_1} \simeq I_{H_2}. \tag{1}$$

First, consider H_1. Let Δ_1 be the subcomplex of I_{H_1} generated by all faces containing t but not containing a. It is clear that I_{H_1}/Δ_1 consists of all faces $\sigma \in I_{H_1}$ such that $\sigma \cap \{a, b, r, s, t\} \in \{\{a\}, \{a, s\}, \{a, t\}, \{s\}, \{b, s\}\}$.

Next, consider H_2. Let Δ_2 be the subcomplex of I_{H_2} generated by all faces containing r. This time, I_{H_2}/Δ_2 consists of all faces $\sigma \in I_{H_2}$ such that $\sigma \cap \{a, b, r, s, t\} \in \{\{a\}, \{a, s\}, \{a, t\}, \{s\}, \{b, s\}\}$.

Since $H_1 \setminus \{s, t\}$ and $H_2 \setminus \{r, s, t\}$ are identical, and since there are no edges from $\{r, s, t\}$ to any vertex outside $\{a, b, r, s, t\}$, the conclusion is that

$$I_{H_1}/\Delta_1 = I_{H_2}/\Delta_2.$$

Since Δ_1 and Δ_2 are cones, they are both collapsible. By Lemma 2.2, this implies (1). □

Theorem 4.2 *Let $g \geq 3$. For every simplicial complex Γ, there is an integer $k \geq 0$ and a graph G such that $g(G) \geq g$ and $I_G \simeq \mathrm{Susp}^k(\Gamma)$.*

Proof. Taking the barycentric subdivision of Γ, we obtain that the theorem is true for $g = 3$; we may pick $k = 0$. Assume that $g > 3$, and assume by induction that there is a graph G such that $g(G) \geq g - 1$ and

$$I_G \simeq \mathrm{Susp}^{k_0}(\Gamma)$$

for some $k_0 \geq 0$. As in Lemma 4.1, let κ be the number of cycles of length $g - 1$ in G. If $\kappa = 0$, then $g(G) \geq g$, and we are done. Otherwise, use Lemma 4.1 to deduce that there is a graph $G^{(1)}$ satisfying

$$I_{G^{(1)}} \simeq \mathrm{Susp}(I_G)$$

such that $g(G^{(1)}) \geq g-1$ and such that there are at most $\kappa-1$ distinct cycles of length $g-1$ in $G^{(1)}$. Applying the same lemma to $G^{(1)}$, we obtain a graph $G^{(2)}$ satisfying

$$I_{G^{(2)}} \simeq \mathrm{Susp}(I_{G^{(1)}}) \simeq \mathrm{Susp}^2(I_G)$$

with the same properties as $G^{(1)}$, except that the number of distinct cycles of length $g-1$ is strictly smaller. Repeating this procedure, we finally arrive at a graph $G^{(r)}$ with no cycles of length less than g such that

$$I_{G^{(r)}} \simeq \mathrm{Susp}^r(I_G) \cong \mathrm{Susp}^{k_0+r}(\Gamma).$$

$\square$

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