# Hard Squares on Grids With Diagonal Boundary Conditions

Jakob Jonsson[1]

July 18, 2008

**Abstract**

Let $m$ and $n$ be positive integers and let $S^{\mathrm{D}}_{m,n}$ be the graph obtained from the infinite square grid by identifying two vertices $(a,b)$ and $(c,d)$ whenever their difference $(c-a,d-b)$ is an integer linear combination of the two vectors $(-m,m)$ and $(n,n)$. We examine the reduced Euler characteristic $\tilde{\chi}_{m,n} := \tilde{\chi}(\Sigma^{\mathrm{D}}_{m,n})$, where $\Sigma^{\mathrm{D}}_{m,n}$ is the simplicial complex of independent sets in $S^{\mathrm{D}}_{m,n}$. We prove that the behavior of $\tilde{\chi}_{m,n}$ for fixed $m$ depends on the divisibility of $m$ by three. Specifically, if $m$ is not divisible by three, then $(\tilde{\chi}_{m,n} : n \geq 1)$ is periodic. If $m$ is divisible by three, then $\tilde{\chi}_{m,n} \sim -3D_{m/3} \cdot 4^{n/3}$ for large $n$, where $D_r$ equals the number of orbits under cyclic rotation of the set of all periodic sequences of length $2r$ with $r$ zeros and $r$ ones. The proof is based on previous results by the author and relates the Euler characteristic to certain periodic rhombus tilings of the plane. The results have an interpretation within the hard-square model of statistical mechanics, the Euler characteristic being equal to minus the Witten index.

## 1 Introduction

An *independent set* in a simple and loopless graph $G$ is a subset of the vertex set of $G$ such that no two vertices in the subset are adjacent. The family of independent sets of a graph forms a simplicial complex, the *independence complex* $\Sigma(G)$ of $G$ (not to be confused with the independence complex of the cycle matroid of $G$).

In this paper, we build on work from a previous paper [8], analyzing the independence complex of square grids with periodic boundary conditions. Such complexes show up in statistical mechanics in a model involving fermionic particles occupying vertices of a square grid subject to strong repulsive interactions. The latter means that a vertex in the grid cannot be occupied unless all neighbors of the vertex are unoccupied. Sometimes this is expressed as saying that the particles are "hard". We refer the interested reader to Baxter [2], Fendley, Schoutens, and van Eerten [6], and Huijse and Schoutens [7] for physical interpretations of the results of the present paper and its predecessor [8]. For more information and references about the hard-particle model on two-dimensional lattices, we refer to the references just listed and to Baxter [1], Fendley and Schoutens [5], and van Eerten [10].

Inspired by computational results due to Baxter [2], we focus on the "diagonal" grid $S^{\mathrm{D}}_{m,n}$ defined in the following manner:

Let $S$ be the infinite two-dimensional square grid; $S$ is an infinite graph with vertex set $\mathbb{Z}^2$ and with an edge between $(a_1,a_2)$ and $(b_1,b_2)$ if and only if $|a_1-b_1|+|a_2-b_2|=1$. Let $\mathbb{L}^{\mathrm{D}}_{m,n}$ be the additive subgroup of $\mathbb{Z}^2$ generated by $(-m,m)$ and $(n,n)$. $S^{\mathrm{D}}_{m,n}$ is the graph on the vertex set $V^{\mathrm{D}}_{m,n} := \mathbb{Z}^2/\mathbb{L}^{\mathrm{D}}_{m,n}$ induced by the canonical map $\varphi_{m,n} : \mathbb{Z}^2 \to \mathbb{Z}^2/\mathbb{L}^{\mathrm{D}}_{m,n}$; two vertices $w_1$ and $w_2$ are adjacent in $S^{\mathrm{D}}_{m,n}$ if and only if there are adjacent vertices $w'_1$ and $w'_2$ in $S$ such that $\varphi_{m,n}(w'_1) = w_1$ and $\varphi_{m,n}(w'_2) = w_2$.

[1]Research supported by European Graduate Program "Combinatorics, Geometry, and Computation", DFG-GRK 588/2.

To avoid misconceptions, we state already at this point that we label elements in $\mathbb{Z}^2$ according to the matrix convention; $(i, j)$ is the element in the $i$th row below row 0 and the $j$th column to the right of column 0.

One may consider other pairs of vectors than $(-m, m)$ and $(n, n)$. For any linearly independent vectors $u$ and $v$, we define $S_{u,v}$ to be the graph obtained from $S$ in the same manner as above, except that we form the quotient with respect to $\langle u, v\rangle$ instead of $\mathbb{L}^{\mathrm{D}}_{m,n}$, where $\langle u, v\rangle$ is the subgroup of $\mathbb{Z}^2$ generated by $u$ and $v$. Note that $S_{(-m,m),(n,n)} = S^{\mathrm{D}}_{m,n}$.

Let $\Delta$ be a family of subsets of a finite set. Borrowing terminology from statistical mechanics, we define the *partition function* $Z(\Delta; z)$ of $\Delta$ as

$$Z(\Delta; z) := \sum_{\sigma \in \Delta} z^{|\sigma|}.$$

Observe that the coefficient of $z^k$ in $Z(\Delta; z)$ is the number of sets in $\Delta$ of size $k$. In particular, if $\Delta$ is a simplicial complex, then $-Z(\Delta; -1)$ coincides with the reduced Euler characteristic of $\Delta$. $Z(\Delta; -1)$ is the Witten index of statistical physics [11]. Its absolute value provides a lower bound on the dimension of the space of zero-energy ground states, which is equivalent to saying that the absolute value of the Euler characteristic provides a lower bound on the dimension of the homology of $\Delta$.

Write $Z(\Delta) := Z(\Delta; -1)$ and $\Sigma^{\mathrm{D}}_{m,n} := \Sigma(S^{\mathrm{D}}_{m,n})$. The main object of this paper is to examine $Z(\Sigma^{\mathrm{D}}_{m,n})$. Our main results are as follows:

- If $m$ and $n$ are coprime, then

  $$Z(\Sigma^{\mathrm{D}}_{m,n}) = \begin{cases} 2 & \text{if 3 divides } mn \\ -1 & \text{if 3 does not divide } mn; \end{cases}$$

  see Theorem 7.1.

- If $m$ is not a multiple of three, then the generating function $\sum_{k\geq 1} Z(\Sigma^{\mathrm{D}}_{m,k})x^k$ is a weighted sum consisting of terms $x^n/(1-x^n)$ such that $\gcd(m, n) \neq 1$ and an "error term" $-\frac{x+2x^2}{1+x+x^2}$; see Theorem 7.2.

- If $m$ is a multiple of three, then $\sum_{k\geq 1} Z(\Sigma^{\mathrm{D}}_{m,k})x^k$ is a weighted sum consisting of terms $x^n/(1-x^n)$ such that $\gcd(m, n) \neq 1$, terms $x/(1-4x^3\cos^2(j\pi/3d))$ such that $0 \leq j \leq 3d-1$ and $3d|m$, and an "error term" $\frac{2x+4x^2}{1+x+x^2}$; see Theorem 7.3.

One may interpret these results in terms of the characteristic polynomial of a certain transfer matrix defined in Section 8:

- If $m$ is not a multiple of three, then the polynomial is a product consisting of the factor $t^2 + t + 1$ and a number of factors $t^n - 1$ such that $\gcd(m, n) \neq 1$; see Theorem 8.4.

- If $m$ is a multiple of three, then the polynomial can be expressed as a product consisting of the factor $1/(t^2 + t + 1)^2$, a number of factors $t^n - 1$ such that $\gcd(m, n) \neq 1$, and a number of factors $t^3 - 4\cos^2(j\pi/3d)$ such that $0 \leq j \leq 3d-1$ and $3d|m$; see Theorem 8.5.

Analyzing $Z(\Sigma^{\mathrm{D}}_{m,n})$ in further detail when $m$ and $n$ are both divisible by three, we also demonstrate that

$$Z(\Sigma^{\mathrm{D}}_{3\lambda r,3\mu r}) \sim \frac{9}{2} \cdot \frac{4^{\lambda r+\mu r}}{r\pi\sqrt{\lambda\mu}}$$

when $r$ grows large for any fixed positive integers $\lambda$ and $\mu$.

Our main tool in the proofs is a formula [8] relating the partition function (evaluated at $-1$) to certain rhombus tilings; see Section 2 for details about this formula and Sections 3 and 4 for more information about rhombus tilings. In Section 5, we examine $Z(\Sigma(S_{u,v}))$ for general vectors $u$ and $v$. We specialize to $u = (-m, m)$ in Section 6 and further to $u = (-m, m)$ and $v = (n, n)$ in Section 7. In Section 8, we discuss the transfer matrix view of the problem. Finally, Section 9 is devoted to asymptotic properties of $Z(\Sigma^{\mathrm{D}}_{m,n})$.

See Bousquet-Mélou, Linusson, and Nevo [3] for an analysis of variants of $S^{\mathrm{D}}_{m,n}$ with cylindric and open boundary conditions; the authors computed not only the Euler characteristic but also the homotopy type of the corresponding independence complexes. Thapper [9] examined a variant of $S_{(m,0),(0,n)}$, again with cylindric boundary conditions. In recent work [4], Engström used discrete Morse theory to compute upper bounds on the Euler characteristic for other two-dimensional grids. For numerical results, see Baxter [2] and van Eerten [10].

## 2 Relating the partition function to rhombus tilings

Write $\Sigma_{u,v} := \Sigma(S_{u,v})$. In a previous paper [8], we related $Z(\Sigma_{u,v})$ to certain rhombus tilings. The kind of rhombus tiling that we are interested in has the following properties:

- The entire plane is tiled.
- The intersection of two rhombi is either empty, a common corner, or a common side.
- The four corners of each rhombus belong to $\mathbb{Z}^2$.
- Each rhombus has side length $\sqrt{5}$, meaning that each side is parallel to and has the same length as $(1, 2)$, $(-1, 2)$, $(2, 1)$, or $(-2, 1)$.
- Each rhombus has area four or five.

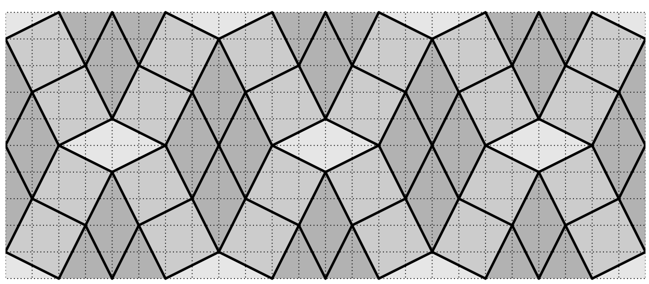

Figure 1: Portion of a rhombus tiling.

See Figure 1 for an example. One easily checks that a rhombus tiling with properties as above is uniquely determined by the set of rhombus corners in the tiling. From now on, we always identify a rhombus tiling with this set. We refer to a rhombus of area $k$ as a *$k$-rhombus*. There are two 4-rhombi and two 5-rhombi; see Figure 2. Note that the 5-rhombi are squares.

A rhombus tiling $\rho$ is *$(u, v)$-invariant* if $\rho$ is invariant under translation with the vectors $u$ and $v$. Let $R_{u,v}$ be the family of $(u, v)$-invariant rhombus tilings. We define $R^{+}_{u,v}$ as the

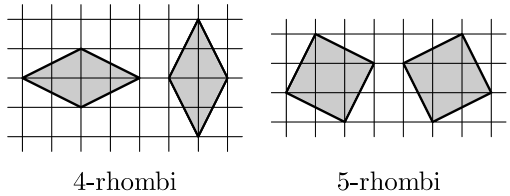

Figure 2: Six different rhombi.

subfamily of $R_{u,v}$ consisting of all rhombus tilings with an even number of cosets of $\langle u, v\rangle$; recall that we identify a given tiling with its set of corners. Write $R^-_{u,v} := R_{u,v} \setminus R^+_{u,v}$. For $d \in \mathbb{Z}$, define

$$\theta_d = \begin{cases} 2 & \text{if } 3|d; \\ -1 & \text{otherwise.} \end{cases} \tag{1}$$

The following theorem provides a concrete formula for $Z(\Sigma_{u,v})$ in terms of rhombus tilings.

**Theorem 2.1 (Jonsson [8])** *Let $u$ and $v$ be linearly independent vectors. Write $d := \gcd(u_1 - u_2, v_1 - v_2)$ and $d^* := \gcd(u_1 + u_2, v_1 + v_2)$. Then*

$$Z(\Sigma_{u,v}) = -(-1)^d \theta_d \theta_{d^*} + |R^+_{u,v}| - |R^-_{u,v}|,$$

*where $\theta_d$ is defined as in* (1).

# 3 Basic properties of rhombus tilings

The present section provides a summary of results about rhombus tilings from a previous paper [8]. For any element $x \in \mathbb{Z}^2$, define $\mathtt{e}(x) := x+(0,1)$ (east), $\mathtt{n}(x) := x+(-1,0)$ (north), $\mathtt{w}(x) := x + (0,-1)$ (west), and $\mathtt{s}(x) := x + (1,0)$ (south); recall our matrix convention for indexing elements in $\mathbb{Z}^2$. Given a set $\sigma$, we refer to an element $x$ as *blocked* in $\sigma$ if at least one of its neighbors $\mathtt{e}(x), \mathtt{n}(x), \mathtt{w}(x), \mathtt{s}(x)$ is contained in $\sigma$. Such a neighbor is said to *block* $x$.

**Proposition 3.1 (Jonsson [8])** *A nonempty set $\rho \subset \mathbb{Z}^2$ is a rhombus tiling if and only if all elements in $\rho$ are pairwise non-blocking and the following holds: For each $x \in \rho$ and each choice of signs $t, u \in \{+1, -1\}$, exactly one of the elements $\mathtt{s}^t\mathtt{e}^{2u}(x)$ and $\mathtt{s}^{2t}\mathtt{e}^u(x)$ belongs to $\rho$.*

Let $\rho$ be a rhombus tiling. For a given element $p \in \rho$, let $\alpha(p)$ be the one element among $\mathtt{s}^2\mathtt{e}(p)$ and $\mathtt{s}\mathtt{e}^2(p)$ that belongs to $\rho$; by Proposition 3.1, $\alpha(p)$ is well-defined. Furthermore, let $\beta(p)$ be the one element among $\mathtt{n}^2\mathtt{e}(p)$ and $\mathtt{n}\mathtt{e}^2(p)$ that belongs to $\rho$. See Figure 3 for an illustration. By symmetry, $\alpha$ and $\beta$ have well-defined inverses; hence $\alpha^r(p)$ and $\beta^r(p)$ are well-defined for all $r \in \mathbb{Z}$.

**Lemma 3.2 (Jonsson [8])** *For any rhombus tiling $\rho$, the functions $\alpha$ and $\beta$ satisfy the identity*

$$\alpha^r \circ \beta^s(p) = \beta^s \circ \alpha^r(p) = \alpha^r(p) + \beta^s(p) - p$$

*for all $p \in \rho$ and $r, s \in \mathbb{Z}$.*

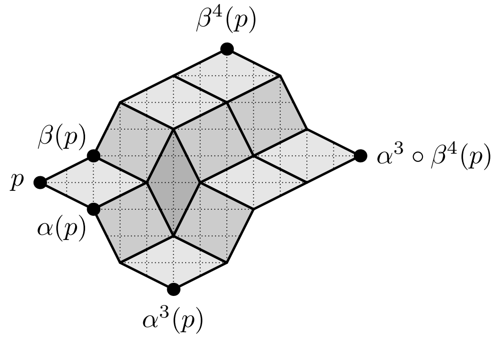


Figure 3: Illustration of the functions $\alpha$ and $\beta$. As predicted by Lemma 3.2, we have that $\alpha^3 \circ \beta^4(p) = \alpha^3(p) + \beta^4(p) - p$.

For $i \in \mathbb{Z}$, define $\delta_i(p) := \alpha^i(p) - \alpha^{i-1}(p)$ and $\epsilon_i(p) := \beta^i(p) - \beta^{i-1}(p)$; by symmetry, this is well-defined for $i \leq 0$.

**Lemma 3.3 (Jonsson [8])** *Let $\rho$ be a rhombus tiling and let $p, q \in \rho$. Then there are unique integers $r$ and $s$ such that $q = \alpha^r \circ \beta^s(p)$. Moreover, $\delta_i(q) = \delta_{i+r}(p)$ and $\epsilon_i(q) = \epsilon_{i+s}(p)$.*

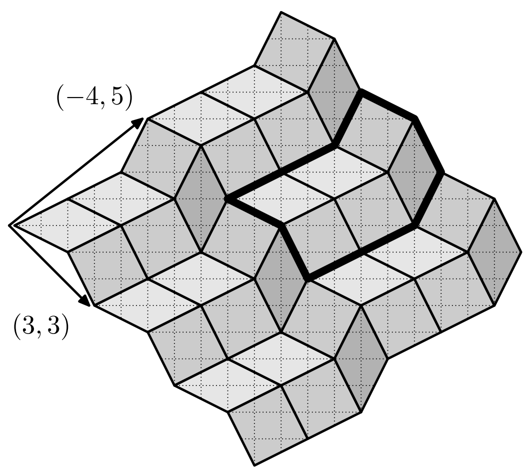


Figure 4: Portion of a periodic rhombus tiling with axes $(3, 3)$ and $(4, -5)$. The border of one "period" of the tiling is marked in bold.

Let $u$ and $v$ be linearly independent integer vectors. Let $\rho$ be a $(u, v)$-invariant rhombus tiling. By finiteness of $\mathbb{Z}^2/\langle u, v\rangle$ and Lemma 3.2, the sequences $(\delta_i(p) : i \in \mathbb{Z})$ and $(\epsilon_i(p) : i \in \mathbb{Z})$ are periodic. Let $K$ and $L$ be minimal such that $\delta_i = \delta_{i+K}(p)$ and $\epsilon_i(p) = \epsilon_{i+L}(p)$ for all $i \in \mathbb{Z}$. Define

$$\begin{aligned} x &:= \alpha^K(p) - p = \sum_{i=1}^{K} \delta_i(p); \\ y &:= \beta^L(p) - p = \sum_{i=1}^{L} \epsilon_i(p). \end{aligned}$$

$x$ and $y$ are the *axes* of $\rho$. One easily checks that $x$ and $y$ do not depend on $p$. See Figure 4 for an illustration.

**Theorem 3.4 (Jonsson [8])** *Let $x := (x_1, x_2)$ and $y := (-y_1, y_2)$ be vectors such that $x_1$, $x_2$, $y_1$, and $y_2$ are all positive. Then there are $(u, v)$-invariant rhombus tilings with axes $x'$ and $y'$ such that $x$ and $y$ are integer multiples of $x'$ and $y'$, respectively, if and only if the following hold:*

*(i) $x_1/2 \leq x_2 \leq 2x_1$ and $y_1/2 \leq y_2 \leq 2y_1$.*

*(ii) $x_1 + x_2$ and $y_1 + y_2$ are divisible by three.*

*(iii) $\langle x, y\rangle$ contains $\langle u, v\rangle$.*

*Assuming that the above conditions hold and defining*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \frac{1}{3} \cdot \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

*(i.e., $x = a \cdot (1, 2) + b \cdot (2, 1)$ and $y = c \cdot (-1, 2) + d \cdot (-2, 1)$), the number of such tilings is*

$$\binom{a+b}{a}\binom{c+d}{c} \cdot \left(\frac{ad + bc}{(a + b)(c + d)} + 4\right).$$

## 4 More on periodic rhombus tilings

By convention, all rhombus tilings in this section are "doubly periodic," meaning that they are $(u, v)$-invariant for some linearly independent vectors $u$ and $v$. Some arguments would go through without this assumption, but we are only interested in the doubly periodic situation.

For any nonzero vector $u$, let $R_u$ be the family of rhombus tilings that are $(u, v)$-invariant for some vector $v$.

**Lemma 4.1** *Let $u$ be a vector and let $\rho \in R_u$. Let $p$ and $q$ be elements in $\rho$ such that $q - p = u$. Write $q = \alpha^r \circ \beta^s(p)$ and define $x := \alpha^r(p) - p$ and $y := \beta^s(p) - p$. Then $\rho$ is $(x, y)$-invariant. In particular, $x$ and $y$ are multiples of the two axes of $\rho$.*

*P*roof. Since $\rho \in R_u$, we have that

$$\delta_i(p) = \delta_i(q) = \delta_{i+r}(p);$$

the second equality is by Lemma 3.3. Using the same argument, we obtain that $\epsilon_j(p) = \epsilon_{j+s}(p)$. As a consequence, the lemma follows. □

Note that $x$ will be zero if $r$ is zero; the analogous property holds for $y$ in terms of $s$.

**Lemma 4.2** *Let $p, q \in \mathbb{Z}^2$ and let $t_1, t_2 \in \{-1, 1\}$. Then there are only finitely many nonnegative integer solutions $(a, b, c, d)$ to the equation*

$$q - p = at_1(1, 2) + bt_1(2, 1) + ct_2(-1, 2) + dt_2(-2, 1). \tag{2}$$

*Proof.* By symmetry, we may assume that $t_1 = t_2 = 1$. Any solution $(a, b, c, d)$ to (2) must satisfy $q_2 - p_2 = 2a + b + 2c + d$, where $p_2$ and $q_2$ are the second coordinates of $p$ and $q$, respectively. Clearly, there are only finitely many such nonnegative integer solutions $(a, b, c, d)$. □

**Lemma 4.3** *Let $u$ be a nonzero vector. Then the following hold:*

(1) *We have that $R_u$ is finite if and only if either of the following does not hold:*

  (i) $|u_1|/2 \leq |u_2| \leq 2|u_1|$.

  (ii) $|u_1| + |u_2|$ *is divisible by three.*

(2) *There are only finitely many rhombus tilings $\rho$ in $R_u$ such that neither of the axes of $\rho$ is parallel to $u$.*

*Proof.* (1) Let $\rho$ be a rhombus tiling in $R_u$. The origin $(0, 0)$ or one of its four neighbors must be present in $\rho$; let $p$ be this element. Let $q := p + u$ and write $q = \alpha^r \circ \beta^s(p)$. By Lemma 4.2, there are only finitely many possiblities for $(r, s)$.

First, suppose that either (i) or (ii) does not hold. By Theorem 3.4 and Lemma 4.1, we have that $r$ and $s$ are nonzero. Using symmetry, we may assume that $r, s > 0$. Clearly, there are again only finitely many possibilities for $\delta_1, \dots, \delta_r, \epsilon_1, \dots, \epsilon_s$. By Lemma 4.1, $\rho$ is uniquely determined by $p$ and these vectors. We just deduced that there are only finitely many choices for all these objects; hence we are done.

Next, assume to the contrary that (i) and (ii) do hold. Theorem 3.4 yields that there are infinitely many doubly periodic rhombus tilings such that $u$ is a multiple of one of the axes. Namely, there are infinitely many choices for the other axis. We conclude that $R_u$ is infinite.

(2) With notation as above, no axis being parallel to $u$ means that $r$ and $s$ are both nonzero. Finiteness follows immediately from the proof of (1); there are only finitely many pairs $(r, s)$ to consider. □

For vectors $u$ and $v$, let $R_u(v)$ be the family of rhombus tilings that are $(u, n \cdot v)$-invariant for some positive integer $n$.

**Lemma 4.4** *Let $u$ and $v$ be linearly independent vectors. Then $R_u(v)$ coincides with $R_u$.*

*Proof.* Let $\rho$ be a rhombus tiling in $R_u$ and let $x$ and $y$ be the axes of $\rho$. It is clear that there are (unique) rational numbers $a$ and $b$ such that $ax + by = v$. Choose the integer $\lambda$ such that $\lambda a$ and $\lambda b$ are integers. It follows that $\lambda ax + \lambda by = \lambda v$. □

# 5 General results

Let $u$ and $v$ be linearly independent vectors. By Lemma 4.4, $R_u = R_u(v)$. Let $R_u^-(v)$ be the family of tilings $\rho$ in $R_u$ such that $\rho \in R_{u,n\cdot v}^-$ for some integer $n$. Write $R_u^+(v) := R_u \setminus R_u^-(v)$. For a tiling $\rho$ in $R_u$, let $\nu := \nu(\rho)$ be the smallest positive integer such that $\rho$ is invariant under translation with the vector $\nu \cdot v$. One readily verifies that $\rho \in R_{u,n\cdot v}^-$ if and only if $\rho \in R_u^-(v)$ and $n/\nu(\rho)$ is an odd integer. Furthermore, $\rho \in R_{u,n\cdot v}^+$ if and only if either $\rho \in R_u^+$ and $n/\nu(\rho)$ is an integer or $\rho \in R_u^-$ and $n/\nu(\rho)$ is an even integer.

For each positive integer $n$, let $\psi_{u,v}^+(n)$ be the number of tilings $\rho \in R_u^+(v)$ such that $\nu(\rho) = n$. Define $\psi_{u,v}^-(n)$ analogously for tilings in $R_u^-(v)$. It is clear that $\psi_{u,v}^+(n)$ and $\psi_{u,v}^-(n)$

are integer multiples of $n$, because a given tiling $\rho$ such that $\nu(\rho) = n$ must have the property that the tilings $\rho, \rho + v, \rho + 2 \cdot v, \ldots, \rho + (n-1) \cdot v$ are all distinct; otherwise, $\nu(\rho)$ would be a proper divisor of $n$.

**Theorem 5.1** *Let $u := (u_1, u_2)$ and $v := (v_1, v_2)$ be two linearly independent vectors. Write $d_k := \gcd(u_1 - u_2, k(v_1 - v_2))$ and $d_k^* := \gcd(u_1 + u_2, k(v_1 + v_2))$, and define $\gamma_k := -(-1)^{d_k} \theta_{d_k} \theta_{d_k^*}$. Then*

$$\sum_{k \geq 1} Z(\Sigma_{u,k \cdot v}) \cdot x^k = \sum_{n \geq 1} \frac{\psi^+_{u,v}(n) x^n}{1 - x^n} - \sum_{n \geq 1} \frac{\psi^-_{u,v}(n) x^n}{1 + x^n} + \frac{1}{1 - x^6} \sum_{k=1}^{6} \gamma_k x^k.$$

*Proof.* By Theorem 2.1, we have that

$$Z(\Sigma_{u,k \cdot v}) = \gamma_k + |R^+_{u,k \cdot v}| - |R^-_{u,k \cdot v}|.$$

One easily checks that

$$\begin{aligned} \sum_{k \geq 1} \left( |R^+_{u,k \cdot v}| - |R^-_{u,k \cdot v}| \right) x^k &= \sum_{\rho \in R^+_u(v)} \frac{x^{\nu(\rho)}}{1 - x^{\nu(\rho)}} - \sum_{\rho \in R^-_u(v)} \frac{x^{\nu(\rho)}}{1 + x^{\nu(\rho)}} \\ &= \sum_{n \geq 1} \frac{\psi^+_{u,v}(n) x^n}{1 - x^n} - \sum_{n \geq 1} \frac{\psi^-_{u,v}(n) x^n}{1 + x^n}. \end{aligned}$$

Now, note that $(\gamma_k, k \geq 1)$ is periodic with period dividing six. As a consequence,

$$\sum_{k \geq 1} \gamma_k x^k = \frac{1}{1 - x^6} \sum_{k=1}^{6} \gamma_k x^k,$$

which concludes the proof. □

As an immediate consequence, we have the following result:

**Corollary 5.2** *If either* (i) *or* (ii) *in Lemma 4.3 is true, meaning that $R_u$ is finite, then the sequence $(Z(\Sigma_{u,k \cdot v}), k \geq 1)$ is periodic.*

**Proposition 5.3** *Let $u := (u_1, u_2)$ and $v := (v_1, v_2)$ be linearly independent vectors such that $u_1 + u_2$ and $v_1 + v_2$ are both even. Let $\rho$ be a $(u,v)$-invariant rhombus tiling. Then $\rho$ contains an even number of cosets of $\langle u, v \rangle$. In particular, $\psi^-_{u,v}(n)$ is zero for all $n$.*

*Proof.* Let $x := a(1,2) + b(2,1)$ and $y := c(-1,2) + d(-2,1)$ be the axes of $\rho$. Modulo $\langle x, y \rangle$, there are $ac + bd$ 4-rhombi and $ad + bc$ 5-rhombi in $\rho$. Let $\kappa_4$ and $\kappa_5$ be the number of 4-rhombi and 5-rhombi, respectively, modulo $\langle u, v \rangle$. It is straightforward to prove that $4\kappa_4 + 5\kappa_5 = \det(u,v) = u_1 v_2 - u_2 v_1$, and the right-hand side is even by assumption. In particular, $\kappa_5$ is even. It remains to prove that $\kappa_4$ is even. For this, let $R$ be the $2 \times 2$ integer matrix satisfying $(x, y) \cdot R = (u, v)$, i.e.,

$$\begin{pmatrix} a + 2b & -c - 2d \\ 2a + b & 2c + d \end{pmatrix} \cdot R = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}.$$

It is clear that $\kappa_4 = (ac+bd) \cdot |\det R|$. If $ac+bd$ is even, then we are done; hence assume that $ac+bd$ is odd. Now, computing modulo two and multiplying the vector $(1,1)$ to both sides, we obtain

$$(a+b, c+d) \cdot R \equiv (u_1+u_2, v_1+v_2) \equiv (0,0) \pmod{2}.$$

Since $ac+bd$ is odd, the vector $(a+b, c+d)$ is nonzero modulo two. As a consequence, $R$ is singular modulo two, which implies that $\det R$ is even; hence we are done. □

## 6 The case $u = (-m, m)$ and general $v$

We examine the important special case that $u$ is equal to $(-m, m)$ for some $m \geq 1$. In this section, we consider general $v$; see Section 7 for the case $v = (1,1)$.

Let $Q_u$ be the subfamily of $R_u$ consisting of all tilings with one axis parallel to $u$. Define $Q_{u,v} = R_{u,v} \cap Q_u$. One easily checks that all rhombus tilings in $Q_{u,v}$ contain an even number of cosets of $\langle u, v\rangle$; this is because the period of the sequence $(\epsilon_i(p) : i \in \mathbb{Z})$ in Lemma 3.3 is even, the second axis being of the form $(-s, s)$ for some $s$.

Let $\mu$ be the Möbius function;

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes;} \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$B_r := \frac{1}{2r} \sum_{d|r} \mu_{r/d} \cdot \binom{2d}{d}.$$

$B_r$ is the number of cyclic binary sequences of length $2r$ with $r$ zeros and $r$ ones and with fundamental period $2r$. Here, "cyclic" means that we identify two sequences whenever they are identical up to rotation. The fundamental period of a sequence is the smallest $s$ such that the sequence is invariant under $s$-step rotation. The first few values of $B_r$, starting with $B_1$, are

$$1, 1, 3, 8, 25, 75, 245, 800, 2700, 9225, 32065, 112632, 400023, 1432613.$$

For example, for $r = 3$, $B_r$ counts the sequences $111000, 110100$, and $110010$. There is a fourth sequence $101010$ that should not be counted, because this sequence has fundamental period 2.

**Lemma 6.1** *Let $m$ be a multiple of three and let $u = (-m, m)$. Let $v = (v_1, v_2)$ be an integer vector such that $u$ and $v$ are linearly independent and such that $v_1, v_2 > 0$. Then the following hold:*

(i) *If $v_1 + v_2$ is not divisible by three, then $Q_{u,v}$ is empty.*

(ii) *Define $\xi_d := \cos(2\pi/d) + i\sin(2\pi/d)$. If $v = r(1,2) + s(2,1)$, where $r, s \geq 0$ and $r+s \geq 1$, then*

$$|Q_{u,v}| = \sum_{3d|m} 3B_d \sum_{j=0}^{3d-1} \xi_{3d}^{-jr} (1 + \xi_{3d}^j)^{r+s}.$$

*Proof.* For a rhombus tiling to belong to $Q_{u,v}$, we have that $u$ must be a multiple of the second axis $y'$ and $v+\lambda y'$ must be a multiple of the first axis for some integer $\lambda$; apply Theorem 3.4. Since the coordinate sum of $v+\lambda y'$ is divisible by three by the same theorem and since the coordinate sum of $y'$ is zero, we obtain that the coordinate sum of $v$ is divisible by three. As a consequence, we have proved (i).

To prove (ii), let $e$ be a positive integer dividing $m/3$, let $d$ be a positive integer dividing $e$, and let $\lambda$ be any integer. Define $Q_{u,v}^{e,d,\lambda}$ as the subset of $Q_{u,v}$ consisting of all rhombus tilings such that

$$x := v + \lambda(-3e, 3e) = (r+3e\lambda)(1,2) + (s-3e\lambda)(2,1)$$

is a multiple of the first axis and $y := (-3d, 3d)$ is a multiple of the second axis. Clearly, $s - 3e\lambda \geq 0$ and $r + 3e\lambda \geq 0$. Let $Q_{u,v}^{e,\lambda}$ be the set of tilings such that the first axis has properties as before and the second axis is equal to $(-3e, 3e)$. Note that $Q_{u,v}$ is the disjoint union of $Q_{u,v}^{e,\lambda}$ over all $e$ dividing $m/3$ and $-\lfloor r/3e \rfloor \leq \lambda \leq \lfloor s/3e \rfloor$; apply the proof of (i). Möbius inversion yields that

$$|Q_{u,v}^{e,\lambda}| = \sum_{d|e} \mu_{e/d} |Q_{u,v}^{e,d,\lambda}|. \tag{3}$$

With notation as in Theorem 3.4 applied to $Q_{u,v}^{e,d,\lambda}$, we have that $a = r + 3e\lambda$, $b = s - 3e\lambda$, and $c = d$. Applying the theorem, we conclude that

$$\begin{aligned} |Q_{u,v}^{e,d,\lambda}| &= \binom{r+s}{r+3e\lambda}\binom{2d}{d} \cdot \left( \frac{(r+3e\lambda)d + (s-3e\lambda)d}{(r+s)(2d)} + 4 \right) \\ &= \frac{9}{2}\binom{r+s}{r+3e\lambda}\binom{2d}{d}. \end{aligned}$$

(3) yields that

$$|Q_{u,v}^{e,\lambda}| = \frac{9}{2}\binom{r+s}{r+3e\lambda} \sum_{d|e} \mu_{e/d} \binom{2d}{d} = 9eB_e \binom{r+s}{r+3e\lambda}.$$

Let $Q_{u,v}^{e} = \sum_\lambda |Q_{u,v}^{e,\lambda}|$. Observing that $\sum_{j=0}^{3e-1} \xi_{3e}^{ji}$ equals $3e$ if $i$ is a multiple of $3e$ and zero otherwise, we obtain that

$$\begin{aligned} |Q_{u,v}^{e}| &= \sum_\lambda 9eB_e \binom{r+s}{r+3e\lambda} = \sum_{i=0}^{r+s} 3B_e \binom{r+s}{i} \sum_{j=0}^{3e-1} \xi_{3e}^{j(i-r)} \\ &= \sum_{j=0}^{3e-1} 3B_e \xi_{3e}^{-jr} (1+\xi_{3e}^{j})^{r+s}. \end{aligned}$$

To settle the proof of (ii), sum over all $e$ dividing $m/3$. $\square$

Define

$$D_r := \sum_{d|r} B_d.$$

$D_r$ is the number of cyclic binary sequences of length $2r$ and with $r$ zeros and $r$ ones. This time there is no requirement on the fundamental period. The first few values, starting with $D_1$, are

$$1, 2, 4, 10, 26, 80, 246, 810, 2704, 9252, 32066, 112720, 400024, 1432860.$$

For a variable $r$ and two nonzero sequences $(a_r : r \geq 1)$ and $(b_r : r \geq 1)$, we define "$a_r \sim b_r$ for large $r$" to mean that

$$\lim_{r\to\infty} \frac{a_r}{b_r} = 1.$$

**Corollary 6.2** *With notation and assumptions as in* Lemma 6.1 (ii)*, we have that*

$$|Q_{u,k\cdot v}| \sim 2^{k(r+s)} \cdot 3D_{m/3}$$

*for large $k$.*

*P*roof. Note that $-1 \leq \cos(j\pi/3d) < 1$ and hence $|1+\xi_{3d}^j| < 2$ whenever $1 \leq j \leq 3d-1$. As a consequence, the term $1/(1-2^{r+s}x)$ obtained by setting $j=0$ is the one dominating term in the formula in Lemma 6.1 (ii). The coefficient of this term is clearly $\sum_{3d|m} 3B_d = 3D_{m/3}$. □

For each positive integer $n$, let $\hat{\psi}^+_{u,v}(n)$ be the number of tilings $\rho \in R_u^+(v) \setminus Q_u$ such that $\nu(\rho) = n$. Define $\hat{\psi}^-_{u,v}(n)$ analogously for tilings in $R_u^-(v) \setminus Q_u$. By Proposition 5.3, $R_u^-(v)$ is empty whenever the coordinate sum of $v$ is even.

**Theorem 6.3** *Let $m$ be any positive integer and let $u = (-m, m)$. Let $v = (v_1, v_2)$ be an integer vector such that $u$ and $v$ are linearly independent and such that $v_1, v_2 > 0$. Write $v = r(1,2) + s(2,1)$. Define $d_k := \gcd(2m, k(-r+s))$ and $\gamma_k := -2\cdot(-1)^{3k(r+s)}\theta_{d_k}$. Then*

$$\sum_{k\geq 1} Z(\Sigma_{u,k\cdot v})x^k = \sum_{n\geq 1} \frac{\hat{\psi}^+_{u,v}(n)x^n}{1-x^n} - \sum_{n\geq 1} \frac{\hat{\psi}^-_{u,v}(n)x^n}{1+x^n} + \frac{1}{1-x^6}\sum_{k=1}^{6} \gamma_k x^k + W,$$

*where almost all terms in the first two sums on the right-hand side are zero and $W$ is defined as follows:*

- *If $m$ is not a multiple of three, then $W = 0$.*
- *If $m$ is a multiple of three and $v_1 + v_2$ is not, then*

$$W = \sum_{3d|m} 3B_d \sum_{j=0}^{3d-1} \left( \frac{1}{1 - x^3\xi_{3d}^{-3jr}(1+\xi_{3d}^j)^{3(r+s)}} - 1 \right).$$

  *Note that $r$ and $s$ are not integers, but $3r$, $3s$, and $r-s$ are.*

- *If both $m$ and $v_1 + v_2$ are multiples of three, then*

$$W = \sum_{3d|m} 3B_d \sum_{j=0}^{3d-1} \left( \frac{1}{1 - x\xi_{3d}^{-jr}(1+\xi_{3d}^j)^{r+s}} - 1 \right).$$

*When $m$ is a multiple of three, we have that*

$$Z(\Sigma_{u,k\cdot v}) \sim 2^{k(r+s)} \cdot 3D_{m/3}$$

*for large $k$, the restriction being that $k$ is a multiple of three if $v_1 + v_2$ is not a multiple of three.*

*Proof.* Note that $\gcd(u_1 - u_2, k(v_1 - v_2)) = \gcd(-2m, k(-r+s)) = d_k$ and $d_k^* := \gcd(u_1 + u_2, k(v_1+v_2)) = 3k(r+s)$, which implies that

$$-(-1)^{d_k}\theta_{d_k}\theta_{d_k^*} = -(-1)^{k(r+s)}\theta_{d_k}\cdot 2.$$

By Theorems 2.1 and 5.1 and Lemmas 4.3 and 6.1, we are done. For the final statement, apply Corollary 6.2. □

# 7 The special case $u = (-m, m)$ and $v = (1, 1)$

Throughout this section, we fix $u := (-m, m)$ and let $v$ be a multiple of $(1, 1)$. Our first result is analogous to a previous result [8] stating that there are no $((m, 0), (0, n))$-invariant rhombus tilings whenever $\gcd(m, n) = 1$.

**Theorem 7.1** *Let $m$ and $n$ be positive integers such that $\gcd(m, n) = 1$ and let $u = (-m, m)$ and $v = (n, n)$. Then there are no $(u, v)$-invariant rhombus tilings. In particular,*

$$Z(\Sigma_{u,v}) = \begin{cases} 2 & \text{if } 3 \text{ divides } mn \\ -1 & \text{if } 3 \text{ does not divide } mn. \end{cases}$$

*Proof.* Suppose that $\rho$ is a $(u, v)$-invariant rhombus tiling. Let $x = (x_1, x_2)$ and $y = (-y_1, y_2)$ be the axes of $\rho$. Since $\langle x, y\rangle$ contains $\langle u, v\rangle$ by Theorem 3.4 (iii), we have that there is a $2 \times 2$ integer matrix $A$ such that

$$\begin{pmatrix} x_1 & -y_1 \\ x_2 & y_2 \end{pmatrix} \cdot A = \begin{pmatrix} -m & n \\ m & n \end{pmatrix}.$$

However, we also have that the determinant $x_1y_2 + x_2y_1$ divides $2mn$. Define $m_0$ and $n_0$ such that $x_1y_2 + x_2y_1 = \epsilon m_0 n_0$, where $m_0 | m$, $n_0 | n$, and $\epsilon \in \{1, 2\}$. Since

$$A = \frac{1}{\epsilon m_0 n_0}\begin{pmatrix} m(y_1 - y_2) & n(y_1 + y_2) \\ m(x_1 + x_2) & n(x_1 - x_2) \end{pmatrix},$$

it follows that $m_0$ divides $x_1 - x_2$ and $y_1 + y_2$ and that $n_0$ divides $x_1 + x_2$ and $y_1 - y_2$. As a consequence,

$$A' := \begin{pmatrix} (y_1 - y_2)/n_0 & (y_1 + y_2)/m_0 \\ (x_1 + x_2)/n_0 & (x_1 - x_2)/m_0 \end{pmatrix}$$

is an integer matrix with determinant $-2\epsilon$. Now, by Theorem 3.4 (i), we have that $|x_1 - x_2| \le \frac{1}{3}(x_1 + x_2)$ and $|y_1 - y_2| \le \frac{1}{3}(y_1 + y_2)$, which yields that

$$\begin{aligned} 4 \ge |\det A'| &= \left(\frac{x_1 + x_2}{n_0}\cdot\frac{y_1 + y_2}{m_0} - \frac{x_1 - x_2}{m_0}\cdot\frac{y_1 - y_2}{n_0}\right) \\ &\ge \left(\frac{3|x_1 - x_2|}{n_0}\cdot\frac{3|y_1 - y_2|}{m_0} - \frac{|x_1 - x_2|}{m_0}\cdot\frac{|y_1 - y_2|}{n_0}\right) \\ &= 8\cdot\frac{|x_1 - x_2|}{m_0}\cdot\frac{|y_1 - y_2|}{n_0}. \end{aligned}$$

Since the last expression is a product of integers, it follows that either $x_1 = x_2$ or $y_1 = y_2$. As a consequence, $|\det A'| = \frac{x_1+x_2}{n_0}\cdot\frac{y_1+y_2}{m_0}$. However, $x_1 + x_2$ and $y_1 + y_2$ are both divisible by three by Theorem 3.4 (ii). Since $\det A'$ is not divisible by three, we must have that both $m_0$ and $n_0$ are divisible by three. This is a contradiction.

For the last statement in the theorem, apply Theorem 2.1. $\square$

**Theorem 7.2** *Let $m$ be an integer that is not divisible by three and let $u = (-m, m)$ and $v = (1, 1)$. Then*

$$\sum_{k\geq 1} Z(\Sigma_{u,k\cdot v})x^k = \sum_{n\geq 1} \frac{\psi^+_{u,v}(n)x^n}{1-x^n} - \frac{x+2x^2}{1+x+x^2}.$$

*Here, $\psi^+_{u,v}(n) = 0$ for sufficiently large $n$ and also whenever $m$ and $n$ are coprime.*

*Proof.* By Lemma 4.3, Theorem 5.1, and Proposition 5.3, it suffices to prove that

$$\frac{1}{1-x^6}\sum_{k=1}^{6} \gamma_k x^k = -\frac{x+2x^2}{1+x+x^2}.$$

Now, $\gcd(u_1 - u_2, k(v_1 - v_2)) = 2m$ and $\gcd(u_1 + u_2, k(v_1 + v_2)) = 2k$; hence $\gamma_1 = \gamma_2 = \gamma_4 = \gamma_5 = -1$ and $\gamma_3 = \gamma_6 = 2$. A straightforward computation yields the identity. For the final statement, apply Theorem 7.1. $\square$

**Theorem 7.3** *Let $m$ be a multiple of three and let $u = (-m, m)$ and $v = (1, 1)$. Then*

$$\begin{aligned}\sum_{k\geq 1} Z(\Sigma_{u,k\cdot v})x^k &= \sum_{n\geq 1} \frac{\hat{\psi}^+_{u,v}(n)x^n}{1-x^n} + \frac{2x+4x^2}{1+x+x^2} \\ &+ \sum_{3d|m} 3B_d \sum_{j=0}^{3d-1}\left(\frac{1}{1-4x^3\cos^2(j\pi/3d)} - 1\right).\end{aligned}$$

*Here, $\hat{\psi}^+_{u,v}(n) = 0$ for sufficiently large $n$ and also whenever $m$ and $n$ are coprime. As a consequence, when $k$ is a multiple of three, we have that*

$$Z(\Sigma_{u,k\cdot v}) \sim 4^{k/3} \cdot 3D_{m/3}.$$

*Proof.* The theorem is an immediate consequence of Theorem 6.3 and Proposition 5.3, except that we need to prove that

$$\frac{1}{1-x^6}\sum_{k=1}^{6} \gamma_k x^k = \frac{2x+4x^2}{1+x+x^2}.$$

Now, $\gcd(u_1 - u_2, k(v_1 - v_2)) = 2m$ and $\gcd(u_1 + u_2, k(v_1 + v_2)) = 2k$; hence $\gamma_1 = \gamma_2 = \gamma_4 = \gamma_5 = 2$ and $\gamma_3 = \gamma_6 = -4$. As in the proof of Theorem 7.2, we easily deduce the desired result. $\square$

# 8 Transfer matrices

Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two linearly independent integer vectors such that $|u_1v_2 - u_2v_1|$ is greater than or equal to each of $|u_1|$ and $|u_2|$. Let $\ell_u$ be a line parallel to $u$ not passing through any integer points and let $p_0$ be any point on $\ell_u$. Define $\hat{I}_{u,v}$ to be the subset of $\mathbb{Z}^2$ consisting of all points between $\ell_u$ and $\ell_u + v$.

**Lemma 8.1** *Let $k$ and $k'$ be integers. Then, with assumptions as above, two elements $x \in \hat{I}_{u,v} + k \cdot v$ and $x' \in \hat{I}_{u,v} + k' \cdot v$ cannot be adjacent unless $|k - k'| \leq 1$.*

*P*roof. Write $x = p_0 + \mu \cdot u + (k + \lambda) \cdot v$ and $x' = p_0 + \mu' \cdot u + (k' + \lambda') \cdot v$, where $\mu, \mu' \in \mathbb{R}$ and $0 < \lambda, \lambda' < 1$. We obtain that

$$x - x' = (\mu - \mu') \cdot u + (\lambda - \lambda' + k - k') \cdot v.$$

Assume that $x - x' \in \{(\pm 1, 0), (0, \pm 1)\}$; by symmetry, we may assume that $x - x' = (\pm 1, 0)$. Multiplying $(u_2, -u_1)$ to both sides, we obtain that

$$|u_2| = |\lambda - \lambda' + k - k'| \cdot |u_1 v_2 - u_2 v_1| \geq |\lambda - \lambda' + k - k'| \cdot |u_2|;$$

the inequality is by assumption. It follows that $|\lambda - \lambda' + k - k'| \leq 1$, which implies that $|k - k'| < 2$ and hence $|k - k'| \leq 1$. $\Box$

Note that $\hat{I}_{u,v}$ is closed under translation with the vector $u$; we define $I_{u,v} := \hat{I}_{u,v}/\langle u \rangle$. It is clear that the map $x \mapsto x + \langle v \rangle$ is a bijection from $I_{u,v}$ to $V_{u,v} = \mathbb{Z}^2/\langle u, v \rangle$. In particular, $I_{u,v}$ is a finite set of size $|u_1 v_2 - u_2 v_1|$.

Let us introduce a *transfer matrix* $T_u^v(z)$. It should be noted that this matrix depends on the choice of the line $\ell_u$. The rows and columns of $T_u^v(z)$ are indexed by all independent subsets of $I_{u,v}$ in $S/\langle u \rangle$. The element on position $(\sigma, \tau)$ in $T_u^v(z)$ is defined to be

$$t_{\sigma,\tau}(z) := \begin{cases} z^{|\sigma|} & \text{if } \sigma \cup (\tau + v) \text{ is an independent set;} \\ 0 & \text{otherwise.} \end{cases}$$

Write $T_u^v := T_u^v(-1)$.

**Proposition 8.2** *Let notation and assumptions be as above. For each $k \geq 1$, we have that*

$$\mathrm{Tr}((T_u^v)^k) = Z(\Sigma_{u,k \cdot v}).$$

*P*roof. We have that

$$\mathrm{Tr}((T_u^v)^k) = \sum_{\sigma_0, \ldots, \sigma_{k-1}} (-1)^{|\sigma_0| + \cdots + |\sigma_{k-1}|},$$

where the sum ranges over all $k$-tuples $(\sigma_0, \ldots, \sigma_{k-1})$ such that $\sigma_i \cup (\sigma_{i+1} + v)$ is an independent set for $0 \leq i \leq k-1$; $\sigma_k = \sigma_0$. We refer to a $k$-tuple $(\sigma_0, \ldots, \sigma_{k-1})$ with this property as *compatible*. Let $\Gamma_{u,v}^k$ be the family of compatible $k$-tuples. It suffices to prove that the map $\varphi : \Gamma_{u,v}^k \to \Sigma_{u,k \cdot v}$ defined by

$$\varphi(\sigma_0, \ldots, \sigma_{k-1}) = \bigcup_{i=0}^{k-1} (\sigma_i + i \cdot v) + \langle k \cdot v \rangle$$

is a size-preserving bijection.

First of all, we note that the $k$ sets $\sigma_i + i \cdot v$ are pairwise disjoint; this is by the way we constructed $I_{u,v}$. In particular, $\varphi$ is injective. Moreover, the image of $\varphi$ lies in $\Sigma_{u,k \cdot v}$. Namely, let $(\sigma_0, \ldots, \sigma_{k-1})$ be compatible and suppose that $\varphi(\sigma_0, \ldots, \sigma_{k-1})$ contains two adjacent elements. By Lemma 8.1, these two elements must appear in $\sigma_i + i \cdot v$ and $\sigma_j + j \cdot v$ for some $i$ and $j$ such that $i = j$ or $i = j \pm 1$ (again, $\sigma_k = \sigma_0$). However, this is not possible, because $(\sigma_0, \ldots, \sigma_{k-1})$ is compatible. Finally, to see that $\varphi$ is onto, let $\sigma$ be a member of

$\Sigma_{u,k\cdot v}$. For $0 \leq i \leq k-1$, define $\sigma_i := (\sigma - i \cdot v) \cap I_{u,v}$. Each individual $\sigma_i$ is clearly independent. Moreover,

$$\sigma_i \cup (\sigma_{i+1} + v) = (\sigma - i \cdot v) \cap (I_{u,v} \cup (I_{u,v} + i \cdot v)),$$

which is again independent. □

Let $P_u^v(t)$ be the characteristic polynomial of $T_u^v$; hence $P_u^v(t) := \det(tI - T_u^v)$.

**Theorem 8.3** *With notation and assumptions as in* Theorem 6.3*, we have that*

$$P_u^v(t) = t^{a_{u,v}} \cdot g(t) \cdot \prod_{n\geq 1} (t^n - 1)^{\hat{\psi}^+_{u,v}(n)/n} \cdot \prod_{n\geq 1} (t^n + 1)^{\hat{\psi}^-_{u,v}(n)/n} \cdot h(t),$$

*where $a_{u,v}$ is a nonnegative integer, $g$ is a rational function satisfying*

$$\frac{tg'(t)}{g(t)} - \deg g = \frac{1}{t^6 - 1} \sum_{k=1}^{6} \gamma_{6-k} t^k,$$

*and $h(t)$ is defined as follows:*

- *If $m$ is not a multiple of three, then $h(t) = 1$.*
- *If $m$ is a multiple of three and $v_1 + v_2$ is not, then*

$$h(t) = \prod_{3d|m} \prod_{j=0}^{3d-1} (t^3 - \xi_{3d}^{-3jr} (1 + \xi_{3d}^j)^{3(r+s)})^{B_d}.$$

- *If both $m$ and $v_1 + v_2$ are multiples of three, then*

$$h(t) = \prod_{3d|m} \prod_{j=0}^{3d-1} (t - \xi_{3d}^{-jr} (1 + \xi_{3d}^j)^{r+s})^{3B_d}.$$

*P*roof. First, note that

$$|u_1 v_2 - u_2 v_1| = m(3r + 3s) > m = |u_1| = |u_2|,$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. In particular, Proposition 8.2 applies.

For any matrix $M$, it is well-known and easy to prove that

$$\sum_{n\geq 1} \frac{\mathrm{Tr}(M^n)}{t^n} = \frac{tP'(t)}{P(t)} - \deg P = t\frac{d}{dt} \log(P(t)) - \deg P,$$

where $P(t) = \det(tI - M)$. As a consequence, the theorem follows immediately from Theorem 6.3 and Proposition 8.2. □

**Theorem 8.4** *Let $m$ be an integer that is not divisible by three and let $u = (-m, m)$ and $v = (1, 1)$. Then*

$$P_u^v(t) = t^{a_{u,v}} \cdot (t^2 + t + 1) \cdot \prod_{n\geq 1} (t^n - 1)^{\psi^+_{u,v}(n)/n},$$

*where $a_{u,v}$ is a nonnegative integer. Here, $\psi^+_{u,v}(n) = 0$ for sufficiently large $n$ and also whenever $m$ and $n$ are coprime.*

*Proof.* Defining $g(t) = t^2 + t + 1$, we obtain that

$$\frac{tg'(t)}{g(t)} - \deg g = \frac{2t^2 + t}{t^2 + t + 1} - 2 = -\frac{t^2 + t - 2}{t^3 - 1} = -\frac{t + 2}{t^2 + t + 1} = -\frac{x + 2x^2}{1 + x + x^2},$$

where $x = 1/t$. Using the same techniques as in the proof of Theorem 8.3, we are done by Theorem 7.2. $\square$

**Theorem 8.5** *Let $m$ be a multiple of three and let $u = (-m, m)$ and $v = (1, 1)$. With notation as in* Theorem 7.3*, we have that*

$$P_u^v(t) = \frac{t^{a_{u,v}}}{(t^2 + t + 1)^2} \prod_{n \geq 1} (t^n - 1)^{\hat{\psi}^+_{u,v}(n)/n} \cdot \prod_{3d|m} \prod_{j=0}^{3d-1} (t^3 - 4\cos^2(j\pi/3d))^{B_d},$$

*where $a_{u,v}$ is a nonnegative integer. Here, $\psi^+_{u,v}(n) = 0$ for sufficiently large $n$ and also whenever $m$ and $n$ are coprime.*

*Proof.* With $g(t) = (t^2 + t + 1)^{-2}$, we obtain that

$$\frac{tg'(t)}{g(t)} - \deg g = \frac{-4t^2 - 2t}{t^2 + t + 1} + 4 = \frac{2t + 4}{t^2 + t + 1} = \frac{2x + 4x^2}{1 + x + x^2},$$

where $x = 1/t$. As a consequence, we are done by Theorems 7.3 and 8.3. $\square$

# 9 Estimating $Z(\Sigma^{\mathrm{D}}_{3r,3k})$

By Theorem 7.3, we have that

$$Z(\Sigma^{\mathrm{D}}_{3r,3k}) \sim 4^k \cdot 3D_r$$

for large $k$ for each fixed $r$; recall that $\Sigma^{\mathrm{D}}_{m,n} = \Sigma_{(-m,m),(n,n)}$.

Instead of keeping $r$ fixed as $k$ tends to infinity, let us consider the case that both values grow. Specifically, we fix two integer constants $\lambda$ and $\mu$ and examine $Z(\Sigma^{\mathrm{D}}_{3\lambda r,3\mu r})$ for large $r$. Write $u := (-3\lambda r, 3\lambda r)$ and $v := (3\mu r, 3\mu r)$. The purpose of the present section is to prove that

$$Z(\Sigma^{\mathrm{D}}_{3\mu r,3\lambda r}) \sim \frac{9}{2}\binom{2\lambda r}{\lambda r}\binom{2\mu r}{\mu r}$$

for large $r$.

**Lemma 9.1** *Let $u = (-3r, 3r)$ and $v = (3k, 3k)$, where $r, k \geq 1$. Then the size of the set $R_{u,v} \setminus Q_{u,v}$ is less than $2^{3r}$.*

*Proof.* Write $v_1 := (1, 1)$. Choosing $\ell_u$ just to the left of the origin, we obtain that the set $I_{u,v_1}$ introduced in Section 8 equals the set

$$\{(-i, i) : 0 \leq i \leq 3r - 1\} \cup \{(-i, i + 1) : 0 \leq i \leq 3r - 1\};$$

let $A$ be the first set and let $B$ be the second set in this union.

Now, consider the transfer matrix $T_u^{v_1}$ defined in Section 8. For each subset $\sigma$ of $A$, let $\Delta(\sigma)$ be the set of column indices $\tau$ in $T_u^{v_1}$ such that $\tau \cap A = \sigma$. One easily checks that the only elements in $I_{u,v_1}$ that are adjacent to elements in $I_{u,v_1} - v_1$ are the elements in $A$. In particular, all columns with indices in a given $\Delta(\sigma)$ are identical. This implies that the rank of $T_u^{v_1}$ is at most $2^{|A|} = 2^{3r}$, which in turn implies that there are at most $2^{3r}$ nonzero roots of the characteristic polynomial of $T_u^{v_1}$. By Theorem 8.5 and the fact that $|R_{u,v} \setminus Q_{u,v}| = \sum_{n|3k} \hat{\psi}^+_{u,v_1}(n)$ (apply Proposition 5.3), it follows that

$$|R_{u,v} \setminus Q_{u,v}| - 4 + \sum_{d|r} \sum_{j=0}^{3d-1} 3B_d \leq 2^{3r}.$$

The double sum is clearly greater than four, which concludes the proof. □

The transfer matrix being reducible to a matrix of size $2^{3r}$ is a well-known fact; see Baxter [2].

**Lemma 9.2** *For all positive integers $t$ and $k$, we have that*

$$\sum_{j=0}^{t-1} \cos^{2k}(j\pi/t) = t\binom{2k}{k}4^{-k} + \epsilon_t,$$

*where $-1 \leq \epsilon_t \leq 1$.*

*P*roof. By a standard approximation argument, we obtain that

$$\sum_{j=0}^{t-1} \cos^{2k}(j\pi/t) \geq \frac{t}{\pi}\int_0^{\pi} \cos^{2k}(x)dx - 1.$$

Writing $\cos^{2k}(x) = \cos^{2k-2}(x) - (\cos^{2k-2}(x)\sin(x)) \cdot \sin(x)$, applying partial integration, and using induction on $k$, one easily proves that

$$\int_0^{\pi} \cos^{2k}(x)dx = \pi\binom{2k}{k}4^{-k}.$$

Another standard approximation argument yields that

$$\sum_{j=0}^{t-1} \cos^{2k}(j\pi/t) \leq \frac{t}{\pi}\int_0^{\pi} \cos^{2k}(x)dx + 1;$$

hence we are done. □

**Lemma 9.3** *We have that*

$$2rB_r \sim \binom{2r}{r} \sim \frac{4^r}{\sqrt{r\pi}}$$

*for large $r$.*

*P*roof. For the second approximation, apply Stirling's formula: $r! \sim \sqrt{2\pi} \cdot r^{r+1/2}e^{-r}$. Now, with $s = \lfloor r/2 \rfloor$, we obtain that

$$\binom{2r}{r} - \sum_{d=1}^{s}\binom{2d}{d} \leq 2rB_r \leq \binom{2r}{r} + \sum_{d=1}^{s}\binom{2d}{d}.$$

Since

$$\sum_{d=1}^{s} \binom{2d}{d} \leq \sum_{d=1}^{s} 4^d \leq C \cdot 2^r$$

for some constant $C$, the first approximation follows. $\square$

**Lemma 9.4** *We have that*

$$2r \sum_{d|r} B_d \sim \sum_{d|r} 2dB_d \sim \binom{2r}{r} \sim \frac{4^r}{\sqrt{r\pi}}$$

*for large $r$.*

*P*roof. By Lemma 9.3, we have that

$$\sum_{d|r} 2dB_d = 2rB_r + \sum_{d|r, d\neq r} (1+\delta_d) \cdot \frac{4^d}{\sqrt{d\pi}},$$

where $\delta_d \to 0$ when $d \to \infty$. Now,

$$\sum_{d|r, d\neq r} (1+\delta_d) \cdot \frac{4^d}{\sqrt{d\pi}} \leq \sum_{d=1}^{\lfloor r/2 \rfloor} C \cdot \frac{4^d}{\sqrt{d\pi}} \leq \sum_{d=1}^{\lfloor r/2 \rfloor} C \cdot 4^d \leq C' \cdot 2^r,$$

where $C$ and $C'$ are constants not depending on $r$. This yields the second approximation. $2r \sum_{d|r} B_d$ is treated similarly. $\square$

**Theorem 9.5** *Let $\lambda$ and $\mu$ be any positive integers. Then*

$$Z(\Sigma^{\mathrm{D}}_{3\lambda r, 3\mu r}) \sim \frac{9}{2} \binom{2\lambda r}{\lambda r} \binom{2\mu r}{\mu r} \sim \frac{9}{2} \cdot \frac{4^{\lambda r + \mu r}}{r\pi\sqrt{\lambda\mu}}$$

*for large $r$.*

*P*roof. By symmetry, we may assume that $\mu \geq \lambda$. Let $u := (-3\lambda r, 3\lambda r)$ and $v := (3\mu r, 3\mu r)$. By Theorem 2.1, Proposition 5.3, and Lemma 9.1, we have that

$$Z(\Sigma^{\mathrm{D}}_{3\lambda r, 3\mu r}) = |R_{u,v} \setminus Q_{u,v}| + |Q_{u,v}| - 4 = |Q_{u,v}| + O(4^{3\lambda r/2}).$$

Since $\mu \geq \lambda$, the ordo term is much smaller than the right-hand side in the theorem; hence it suffices to prove that

$$|Q_{u,v}| \sim \frac{9}{2} \cdot \frac{4^{\lambda r + \mu r}}{r\pi\sqrt{\lambda\mu}}.$$

Now, Lemma 6.1 yields that

$$|Q_{u,v}| = \sum_{d|\lambda r} 3B_d \sum_{j=0}^{3d-1} 4^{\mu r} \cos^{2\mu r}(j\pi/3d).$$

Applying Lemmas 9.2, 9.3, and 9.4, we obtain that

$$\begin{aligned}
|Q_{u,v}| &= \sum_{d|\lambda r} 3B_d \cdot \left(3d\binom{2\mu r}{\mu r} + \epsilon_{3d}4^{\mu r}\right) \\
&= \frac{9}{2}\binom{2\mu r}{\mu r} \cdot \sum_{d|\lambda r} 2dB_d + 3 \cdot 4^{\mu r} \cdot \sum_{d|\lambda r} B_d\epsilon_{3d} \\
&\sim \frac{9}{2} \cdot \frac{4^{\mu r}}{\sqrt{\mu r\pi}} \cdot \frac{4^{\lambda r}}{\sqrt{\lambda r\pi}} + 3 \cdot 4^{\mu r} \cdot \sum_{d|\lambda r} B_d\epsilon_{3d} \\
&= \frac{9}{2} \cdot \frac{4^{\lambda r+\mu r}}{r\pi\sqrt{\lambda\mu}}\left(1 + \frac{2\pi\sqrt{\lambda\mu}}{3} \cdot \frac{r}{4^{\lambda r}} \cdot \sum_{d|\lambda r} B_d\epsilon_{3d}\right).
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{2\pi\sqrt{\lambda\mu}}{3} \cdot \frac{r}{4^{\lambda r}} \cdot \left|\sum_{d|\lambda r} B_d\epsilon_{3d}\right| \le \frac{2\pi\sqrt{\lambda\mu}}{3} \cdot \frac{r}{4^{\lambda r}} \cdot \sum_{d|\lambda r} B_d \\
\sim\ &\frac{2\pi\sqrt{\lambda\mu}}{3} \cdot \frac{r}{4^{\lambda r}} \cdot \frac{1}{2\lambda r} \cdot \frac{4^{\lambda r}}{\sqrt{\pi\lambda r}} = \frac{\sqrt{\pi\mu}}{3\lambda} \cdot \frac{1}{\sqrt{r}}
\end{aligned}$$

by Lemma 9.4. Since this is $o(1)$, we are done. □

## Acknowledgments

The greater part of this work was carried out at the Technische Universität Berlin. The problem of examining $Z(\Sigma^{\mathrm{D}}_{m,n})$ was suggested to the author by Paul Fendley. I also had fruitful discussions on the subject with Kareljan Schoutens.

## References

[1] R. J. Baxter, Hard Hexagons: Exact Solution, *J. Physics A* **13** (1980), 1023–1030.

[2] R. J. Baxter, Hard squares, Manuscript, September 7, 2004.

[3] M. Bousquet-Mélou, S. Linusson and E. Nevo, On the independence complex of square grids, *J. Algebraic Combin.*, to appear.

[4] A. Engström, *Upper bounds on the Witten index for supersymmetric lattice models by discrete Morse theory*, preprint (2007), 9 pages.

[5] P. Fendley and K. Schoutens, Exact results for strongly-correlated fermions in 2+1 dimensions, *Phys. Rev. Lett.* **95** (2005), 046403.

[6] P. Fendley, K. Schoutens, and H. van Eerten, Hard squares with negative activity, *J. Phys. A: Math. Gen.* **38** (2005), no. 2, 315–322.

[7] L. Huijse and K. Schoutens, Superfrustration of charge degrees of freedom, *Proceedings of the XXIII IUPAP International Conference on Statistical Physics*, Genova, Italy, 2007. arXiv:0709.4120v1

[8] J. Jonsson, Hard squares with negative activity and rhombus tilings of the plane, *Electronic J. Combin.* **13** (2006), no. 1, R67.

[9] J. Thapper, *Combinatorial Considerations on Two Models from Statistical Mechanics*, Lic Thesis, Linköping University, 2007.

[10] H. van Eerten, Extensive ground state entropy in supersymmetric lattice models, *J. Math. Phys.* **46** (2005), 123302.

[11] E. Witten, Constraints on supersymmetry breaking, *Nucl. Phys.* **B202** (1982), 253–316.