# The exact chromatic number of the convex segment disjointness graph

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**Abstract**

Given a set of $n$ points in the plane in general and convex position, let $\Omega_n$ be the set of closed line segments joining pairs of elements in the point set. Let $D_n$ be the graph whose vertex set is $\Omega_n$, where two line segments are adjacent if and only if they are disjoint. In a more general setting, Araujo, Dumitrescu, Hurtado, Noy, and Urrutia introduced the problem of determining the chromatic number of $D_n$. Fabila-Monroy and Wood showed that a lower bound is given by $n - \lfloor\sqrt{2n + \frac{1}{4}} - \frac{1}{2}\rfloor$. The main result of the present note is that the chromatic number actually equals this lower bound. The proof is constructive.

## 1 Introduction

Let $n \geq 3$, and consider a convex $n$-gon $P_n$. Label the corners of $P_n$ with the integers $1, \ldots, n$ in clockwise order. Define

$$\Omega_n = \{ij : 1 \leq i < j \leq n\}.$$

To each element $ij$ in $\Omega_n$, we associate the closed line segment $\mathbf{s}_{i,j}$ between the corners in $P_n$ labelled $i$ and $j$. We define $D_n$ to be the graph on the vertex set $\Omega_n$ with the property that two vertices $ij$ and $k\ell$ are adjacent if and only if the corresponding line segments $\mathbf{s}_{i,j}$ and $\mathbf{s}_{k,\ell}$ are disjoint.

The chromatic number of $D_n$ has been studied by several authors [1, 2, 3]. The previous best known bounds are

$$n - \sqrt{2n + \tfrac{1}{4}} + \tfrac{1}{2} \leq \chi(D_n) \leq n - \sqrt{\tfrac{1}{2}n} - \tfrac{1}{2}(\ln n) + 4. \tag{1}$$

Dujmović and Wood [2] established the upper bound, and Fabila-Monroy and Wood [3] established the lower bound, improving earlier weaker bounds due to Araujo et al. [1]. The main result of this note is that the chromatic number is in fact given by the lower bound (rounded up to the nearest integer).

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**Theorem 1.1** *For $n \geq 3$, we have that*

$$\chi(D_n) = n - \left\lfloor \sqrt{2n + \tfrac{1}{4}} - \tfrac{1}{2} \right\rfloor.$$

An equivalent formulation is that

$$\chi(D_n) = n - k,$$

where $k$ is the unique integer satisfying $\binom{k+1}{2} \leq n < \binom{k+2}{2}$.

The proof of Theorem 1.1 is given by an explicit coloring, which we describe in Section 3.

# 2 Illustrating the graph with a diagram

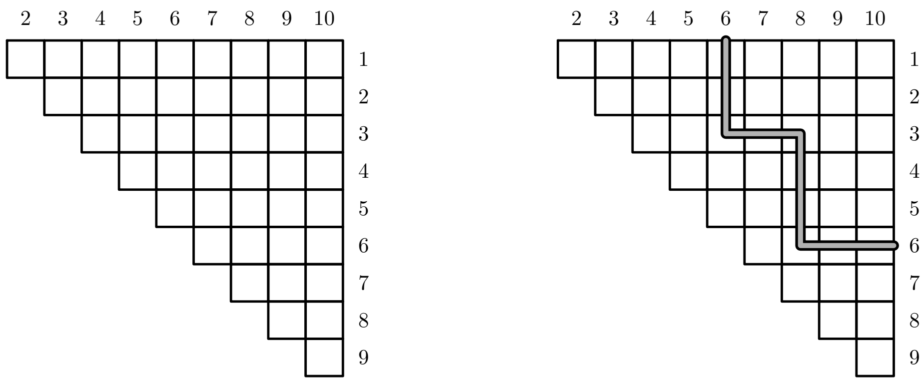


Figure 1: On the left the vertex set $\Omega_{10}$ of $D_{10}$ represented as a polyomino. On the right a maximal independent set in $D_{10}$ represented as a path in the corresponding polyomino.

A *polyomino* is a finite subset of $\mathbb{Z}^2$. For later convenience, we adopt the matrix convention for indexing rows and columns in $\mathbb{Z}^2$; row $a$ is just below row $a-1$, column $b$ is just to the right of column $b-1$, and $ab$ refers to the lattice point in row $a$ and column $b$. We identify the vertex $ab$ with the corresponding lattice point, which we represent as a unit square in figures. In this manner, we may represent $\Omega_n$ as a triangle-shaped polyomino as illustrated on the left in Figure 1.

Now, two distinct vertices $ab$ and $cd$ in $D_n$ are joined by an edge if and only if $a \leq c \leq b \leq d$ or $c \leq a \leq d \leq b$. In particular, for $ab$ and $cd$ to be nonadjacent, we must have that $cd$ lies in the nonshaded region in Figure 2. Specifically, $cd$ cannot be strictly southwest or strictly northeast of $ab$. Moreover, we must have that $\max\{a, c\} \leq \min\{b, d\}$.

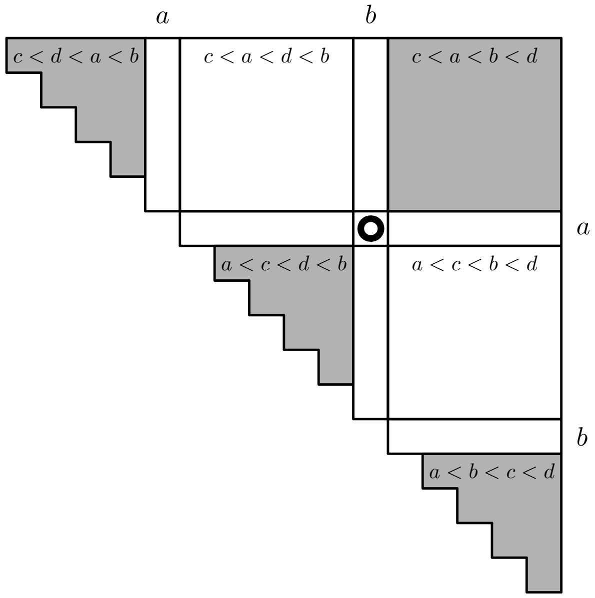


Figure 2: An element $(c, d)$ is adjacent to $(a, b)$ (marked with a thick circle) in the graph $D_n$ if and only if $(c, d)$ belongs to one of the shaded regions.

We conclude that any independent set $\sigma$ of $D_n$ is a subset of some rectangle of the form $[1, r] \times [r, n]$ (with the southwest corner $rr$ removed). Namely, choose $ab, cd \in \sigma$ such that $a$ is maximal and $d$ is minimal. Then $a' \leq a \leq d \leq b'$ for each $a'b' \in \sigma$. In fact, it is straightforward to show that each maximal independent set forms a path from $1r$ to $rn$ for some $r \in \{2, \dots, n-1\}$, where each step in the path is of the form $ij \to i(j+1)$ or $ij \to (i+1)j$. An example is given on the right in Figure 1. Conversely, every such path is a maximal independent set. We refer to such a path as a *thrackle path*; the corresponding set of line segments forms what is called a maximal thrackle [3].

To summarize, the chromatic number of $D_n$ is given by the minimum number of thrackle paths required to cover $\Omega_n$. In Figure 3, we show that it is possible to cover $\Omega_{15}$ with ten thrackle paths. As a consequence, $\chi(D_{15}) \leq 10$. Indeed, we have equality by the lower bound (1) due to Fabila-Monroy and Wood [3]. The given set of thrackle paths in Figure 3 is the one obtained from our proof of Theorem 1.1 given in Section 3. The paths are chosen in a greedy manner in the sense that we select each new path to be as far to the southwest as possible, while maintaining that each turn of the path should appear in an empty square.

Our proof requires that we analyze certain induced subgraphs of $D_n$. For $r \leq n$, let $D_{n,r}$ be the induced subgraph of $D_n$ obtained by removing all vertices

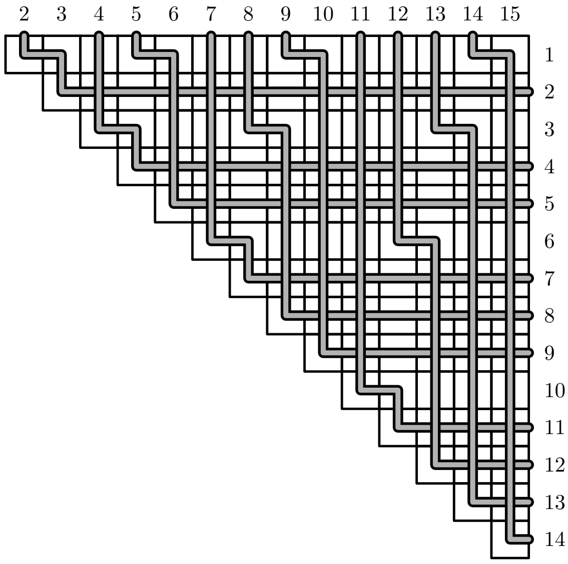


Figure 3: Ten thrackle paths covering $\Omega_{15}$. Note that the first three paths cover $\Omega_6$ and that the first six paths cover $\Omega_{10}$.

$ij$ such that $j \le r$; the vertex set of $D_{n,r}$ is

$$\begin{aligned}\Omega_{n,r} &= \Omega_n \setminus \Omega_r \\ &= [1,r] \times [r+1,n] \ \cup\ \{(i,j) : r+1 \le i < j \le n\},\end{aligned}$$

Note that $D_{n,1} = D_n$.

# 3 Proof of the main result

To prove Theorem 1.1, it suffices to prove that there is a coloring of $D_n$ with $f(n) = n - \left\lfloor \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rfloor$ colors. By the discussion in Section 2, this is equivalent to covering $\Omega_n$ with $f(n)$ thrackle paths.

**Lemma 3.1** *For $r \ge 1$ and $n \ge 2r+1$, we have that $\chi(D_{n,r}) \le \chi(D_{n-r,r+1}) + r$.*

*Proof.* For $r+1 \le j \le 2r$, let

$$A_j = [1, 2r-j+1] \times \{j\} \ \cup\ [2r-j+1, j] \times \{j+1\} \ \cup\ \{j\} \times [j+2, n].$$

Each $A_j$ is a thrackle path, and the union $U = A_{r+1} \cup \cdots \cup A_{2r}$ is equal to the complement in $\Omega_{n,r}$ of the set

$$W = [1,r] \times [2r+2, n] \ \cup\ \{(i,j) : 2r+1 \le i < j \le n\}.$$

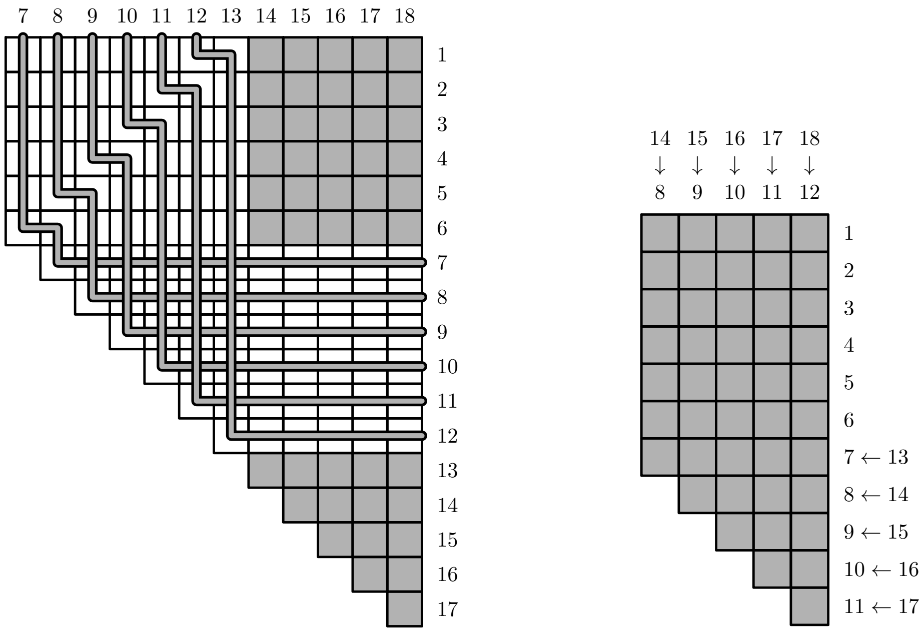


Figure 4: Six thrackle paths leaving only the shaded part $W$ of $\Omega_{18,6}$ uncovered. After applying the relabeling $i \mapsto i-6$ to the higher-valued vertices as illustrated on the right, we may identify $W$ with $\Omega_{18-6,6+1} = \Omega_{12,7}$.

See Figure 4 for an illustration.

Applying the relabelling $i \mapsto i - r$ for $2r + 1 \leq i \leq n$, we obtain that the induced subgraph of $D_{n,r}$ on the vertex set $W$ is isomorphic to the induced subgraph on the vertex set

$$\begin{aligned} & [1, r] \times [r+2, n-r] \ \cup\ \{(i,j) : r+1 \leq i < j \leq n-r\} \\ = \ & [1, r+1] \times [r+2, n-r] \ \cup\ \{(i,j) : r+2 \leq i < j \leq n-r\}, \end{aligned}$$

which is $\Omega_{n-r,r+1}$.

By construction, we can color the region $U$ with $r$ colors and the region $W$ with $\chi(D_{n-r,r+1})$ colors, which concludes the proof. $\square$

**Lemma 3.2** *For $1 \leq k \leq n \leq 2k$, we have that $\chi(D_{n,k}) = n - k$.*

*Proof.* The elements $(1, n), (2, n-1), \ldots, (n-k, k+1)$ form a clique; hence $\chi(D_{n,k}) \geq n - k$.

It remains to show that $\chi(D_{n,k}) \leq n - k$. The construction is very similar to that in the proof of Lemma 3.1. For $k + 1 \leq j \leq n - 1$, again let

$$A_j = [1, 2k - j + 1] \times \{j\} \ \cup\ [2k - j + 1, j] \times \{j + 1\} \ \cup\ \{j\} \times [j + 2, n].$$

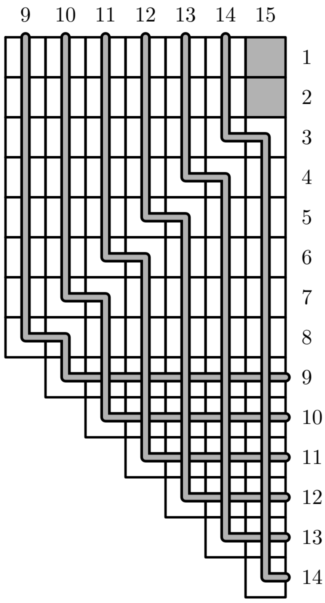


Figure 5: Six thrackle paths almost covering $\Omega_{15,8}$; we need a seventh path to cover the two shaded squares $(1, 15)$ and $(2, 15)$ in the upper right corner.

Moreover, define

$$A_n = [1, n-1] \times \{n\}.$$

Each $A_j$ forms an independent set in $D_{n,k}$, and the union $A_{k+1} \cup A_{k+2} \cup \cdots \cup A_n$ equals the vertex set of $D_{n,k}$. See Figure 5 for an illustration. As a consequence, we indeed have that $\chi(D_{n,k}) \leq n - k$. $\square$

**Theorem 3.3** *For $n \geq r$, we have that*

$$\chi(D_{n,r}) = n - \left\lfloor \sqrt{2n + r(r-1) + \tfrac{1}{4}} - \tfrac{1}{2} \right\rfloor.$$

*Proof.* Define $d_{r,r} = 0$ and

$$d_{r,m} = d_{r,m-1} + (m-1) = \frac{(r+m-1)(m-r)}{2}$$

for $m > r$. By Lemma 3.1, we have that

$$\chi(D_{n,r}) \leq \chi(D_{n-d_{r,m},m}) + d_{r,m}$$

whenever $n - d_{r,m} \geq m$. Let $k \geq r$ be maximal such that $n - d_{r,k} \geq k$; thus

$$n - d_{r,k+1} = n - d_{r,k} - k < k + 1,$$

which means that $k \leq n - d_{r,k} \leq 2k$. Lemma 3.2 yields that

$$\chi(D_{n,r}) \leq \chi(D_{n-d_{r,k},k}) + d_{r,k} = n - d_{r,k} - k + d_{r,k} = n - k.$$

Now,

$$n \geq d_{r,k} + k = \frac{(r+k-1)(k-r)}{2} + k = \frac{k^2 + k - r(r-1)}{2},$$

which yields that

$$k \leq \sqrt{2n + r(r-1) + \tfrac{1}{4}} - \tfrac{1}{2}.$$

By maximality of $k$, we obtain that

$$\chi(D_{n,r}) \leq n - \left\lfloor \sqrt{2n + r(r-1) + \tfrac{1}{4}} - \tfrac{1}{2} \right\rfloor.$$

To show that we indeed have equality, note that

$$\begin{aligned} \chi(D_{n,r}) &\geq \chi(D_{n+d_{1,r}}, 1) - d_{1,r} \\ &\geq n + d_{1,r} - \sqrt{2(n + d_{1,r}) + \tfrac{1}{4}} + \tfrac{1}{2} - d_{1,r} \\ &= n - \sqrt{2n + r(r-1) + \tfrac{1}{4}} + \tfrac{1}{2}. \end{aligned}$$

Here, we apply the lower bound in (1) due to Fabila-Monroy and Wood [3]. □

# References


[1] G. Araujo, A. Dumitrescu, F. Hurtado, M. Noy, and J. Urrutia, On the chromatic number of some geometric type Kneser graphs *Comput. Geom. Theory Appl.* **32** (2005), no 1, 59–69.

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[3] R. Fabila-Monroy and D. R. Wood, The chromatic number of the convex segment disjointness graph, *XIV Spanish Meeting on Computational Geometry*, Alcalá de Henares, Spain, June 27-30, 2011.