Introduction

This thesis deals with problems concerning Euler trails and directed spanning trees in digraphs. The problems as well as their solutions are technical in nature. Therefore, we have decided that this Introduction should be devoted to the special case of de Bruijn sequences. In Chapter 1 we present a more detailed summary of the thesis as well as formal definitions of some graph-theoretical concepts used already here.

The choice of subject for the Introduction is convenient as the origin of the author's interest in the area was the following problem:

*Count the number $b_n$ of binary cyclic sequences of length $2^n$ having the property that each binary sequence of length $n$ appears exactly once as a subsequence.*

**Example 1** For $n = 3$, there are two such binary cyclic sequences of length $2^3 = 8$:

$$S_1 = (00010111) \text{ and } S_2 = (11101000)$$

(rotations of the same sequence are considered as equal). For example,

$$(000), (001), (010), (011), (101), (011), (110), (100)$$

are the eight subsequences in $S_1$ of length 3 (we continue from the beginning when we reach the end of the cyclic sequence $S_1$).

As the example indicates, $b_3 = 2$. Furthermore, one easily deduces that $b_2 = 1$ and $b_4 = 16$, and a computer tells us that $b_5 = 2048$. Hence it seems to be reasonable to suggest that

$$b_n = 2^{2^{n-1} - n}.$$  

This is indeed true, and it was proved by N.G. de Bruijn [Br] in 1946. For this reason, sequences with the property just described are called *de Bruijn sequences*.

The proof of de Bruijn is based on a graph-theoretic interpretation of the problem; with this interpretation the above formula for $b_n$ becomes a corollary to a more general result by de Bruijn about Euler trails in digraphs. Any de Bruijn sequence can be identified with an Euler trail in a certain digraph $G_n$; an Euler trail is a cyclic permutation of the arcs such that the head of an arc is equal to the tail of the image of the arc. The vertex set of $G_n$ is the set $V = \{0, \ldots, 2^{n-1} - 1\}$, while the arc set is the set $A = \{0, \ldots, 2^n - 1\}$. Note that every $a \in A$ can be written as $2v + p$ for some $v \in V$ and $p \in \{0, 1\}$. The
tail of this arc is the vertex $v$, while the head is the vertex $(2v + p) \mod |V|$. $G_4$ is illustrated in Figure 1 with vertices written as binary numbers.

The bijective correspondence between de Bruijn sequences and Euler trails is described as follows. Let $(s_0, \ldots, s_{|A|-1})$ be a de Bruijn sequence. The Euler trail in $G_n$ associated with this sequence is the cyclic permutation of the arc set $A$ obtained by letting the $i$th element in the permutation be the number with binary representation $s_is_{i+1} \ldots s_{i+n-1}$ (the indexes are computed modulo $|A|$).

**Example 2** The sequence $(1001111010100000)$ corresponds to the Euler trail

$$
\pi = (9, 3, 7, 15, 14, 13, 10, 5, 11, 6, 12, 8, 0, 1, 2, 4);
$$

the first four digits 1001 in the sequence form the binary representation of 9, the four digits 0011 on positions 2-5 form the binary representation of 3, and so on. A complete table containing all de Bruijn sequences of length 16 and the corresponding Euler trails is listed below.

<table>
<thead>
<tr>
<th>de Bruijn sequence</th>
<th>Euler trail</th>
</tr>
</thead>
<tbody>
<tr>
<td>(000010110100111)</td>
<td>(0, 1, 2, 5, 11, 6, 13, 10, 4, 9, 3, 7, 15, 14, 12, 8)</td>
</tr>
<tr>
<td>(000010011011011)</td>
<td>(0, 1, 2, 4, 9, 3, 6, 13, 10, 5, 11, 7, 15, 14, 12, 8)</td>
</tr>
<tr>
<td>(000011101001101)</td>
<td>(0, 1, 3, 7, 15, 14, 12, 9, 2, 5, 11, 6, 13, 10, 4, 8)</td>
</tr>
<tr>
<td>(000011010111001)</td>
<td>(0, 1, 3, 6, 12, 9, 2, 5, 11, 7, 15, 14, 12, 9, 2, 4, 8)</td>
</tr>
<tr>
<td>(000010011110101)</td>
<td>(0, 1, 2, 4, 9, 3, 7, 15, 14, 13, 10, 5, 11, 6, 12, 8)</td>
</tr>
<tr>
<td>(000010111010011)</td>
<td>(0, 1, 2, 5, 11, 7, 15, 14, 13, 10, 4, 9, 3, 6, 12, 8)</td>
</tr>
<tr>
<td>(000011101011100)</td>
<td>(0, 1, 3, 7, 15, 14, 13, 10, 5, 11, 6, 12, 9, 2, 4, 8)</td>
</tr>
<tr>
<td>(000011001101110)</td>
<td>(0, 1, 3, 6, 12, 9, 2, 5, 11, 7, 15, 14, 13, 10, 4, 8)</td>
</tr>
<tr>
<td>(000010011010011)</td>
<td>(0, 1, 3, 7, 15, 14, 13, 10, 4, 9, 2, 5, 11, 6, 12, 8)</td>
</tr>
<tr>
<td>(000011110100011)</td>
<td>(0, 1, 3, 7, 15, 14, 13, 11, 6, 12, 9, 2, 5, 10, 4, 8)</td>
</tr>
<tr>
<td>(000011010001111)</td>
<td>(0, 1, 2, 5, 10, 4, 9, 3, 7, 15, 14, 13, 11, 6, 12, 8)</td>
</tr>
<tr>
<td>(000010110011110)</td>
<td>(0, 1, 2, 5, 11, 6, 12, 9, 3, 7, 15, 14, 13, 10, 4, 8)</td>
</tr>
<tr>
<td>(000011001001011)</td>
<td>(0, 1, 3, 6, 13, 10, 4, 9, 2, 5, 11, 7, 15, 14, 12, 8)</td>
</tr>
<tr>
<td>(110111100101000)</td>
<td>(0, 1, 3, 6, 13, 11, 7, 15, 14, 12, 9, 2, 5, 10, 4, 8)</td>
</tr>
<tr>
<td>(000010100011011)</td>
<td>(0, 1, 2, 5, 10, 4, 9, 3, 6, 13, 11, 7, 15, 14, 12, 8)</td>
</tr>
<tr>
<td>(101111001110100)</td>
<td>(0, 1, 2, 5, 11, 7, 15, 14, 12, 9, 3, 6, 13, 10, 4, 8)</td>
</tr>
</tbody>
</table>

Let $\pi$ be an Euler trail corresponding to a de Bruijn sequence in $G_n$. Note that $\pi(a)$ is equal to either $2a$ or $2a + 1 \mod |A|$ for all arcs $a$. Moreover, if $\pi(a) = 2a$, then $\pi(a + |V|) = 2a + 1$ and vice versa $(0 \leq a < |V|)$. In particular, any de
Bruijn sequence is uniquely determined by the values of \(\pi(0), \pi(1), \ldots, \pi(|V| - 1)\). In fact, we need only know whether \(\pi(a)\) is even or odd.

Now suppose that we know a little bit less; we are given the parities

\[
p_v = (\pi(2v) + \pi(2v + 1)) \mod 2
\]

for \(v \in \{1, \ldots, |V|/2 - 1\}\) (since \(\pi\) is an odd permutation, \(p_0\) is uniquely determined by \(p_1, \ldots, p_{|V|/2 - 1}\)). In general, we are not able to determine the Euler trail from this information as there might be different Euler trails with identical values of \(p_v\) for all \(v\).

**Example 3** When \(n = 4\) (that is, \(|V| = 8\) and \(|A| = 16\), we have the following situation.

<table>
<thead>
<tr>
<th>Euler trail</th>
<th>((p_1, p_2, p_3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 1, 2, 5, 11, 6, 13, 10, 4, 9, 3, 7, 15, 14, 12, 8))</td>
<td>((0, 0, 0))</td>
</tr>
<tr>
<td>((0, 1, 2, 4, 9, 3, 6, 13, 10, 5, 11, 7, 15, 14, 12, 8))</td>
<td>((0, 0, 0))</td>
</tr>
<tr>
<td>((0, 1, 3, 7, 15, 14, 12, 9, 2, 5, 11, 6, 13, 10, 4, 8))</td>
<td>((0, 1, 0))</td>
</tr>
<tr>
<td>((0, 1, 3, 6, 12, 9, 2, 5, 11, 7, 15, 14, 13, 10, 4, 8))</td>
<td>((0, 1, 0))</td>
</tr>
<tr>
<td>((0, 1, 2, 4, 9, 3, 7, 15, 14, 10, 5, 11, 6, 12, 8))</td>
<td>((1, 0, 1))</td>
</tr>
<tr>
<td>((0, 1, 2, 5, 11, 7, 15, 14, 13, 10, 4, 9, 3, 6, 12, 8))</td>
<td>((1, 0, 1))</td>
</tr>
<tr>
<td>((0, 1, 3, 7, 15, 14, 10, 4, 9, 2, 5, 11, 6, 12, 8))</td>
<td>((1, 1, 1))</td>
</tr>
<tr>
<td>((0, 1, 2, 5, 11, 6, 12, 9, 3, 7, 15, 14, 13, 10, 4, 8))</td>
<td>((1, 1, 1))</td>
</tr>
</tbody>
</table>

Hence for each binary sequence of length three, there are exactly two Euler trails with \((p_1, p_2, p_3)\) equal to the sequence. A similar investigation for \(n = 5\) yields that the 2048 de Bruijn sequences of length \(2^n\) can be divided into 128 sets - one set for each of the 128 binary sequences \((p_1, \ldots, p_7)\) - and each such set contains exactly 16 sequences.

The general result is as follows: If we divide the Euler trails in \(G_n\) into sets in the manner just described, then each set contains just as many Euler trails as there are Euler trails in \(G_{n-1}\). The proof of a generalization of this result is given in Section 2.2.

We want to interpret the set of Euler trails in \(G_n\) as an abelian group. As we have mentioned several times, the number of Euler trails in \(G_n\) is a power of 2, which might suggest that the group should be a direct sum of copies of \(\mathbb{Z}_2\). However, our group is a little bit more complicated than that.

The construction is as follows. Introduce a free abelian group with generators \(g_0, \ldots, g_{|V|-1}\) and the relation \(g_0 = 0\); the rank of the group is \(|V| - 1\). Adding the relations
\[
2g_{2v+p} - g_v - g_{v+|V|/2} = 0
\]
for \(0 \leq v < |V|/2\) and \(p \in \{0, 1\}\), we obtain a quotient group \(\Phi(n)\), which is actually a finite group of order the number of Euler trails in \(G_n\) (see Section 3.1). The relations (2) may look a little bit strange, but they correspond in a very natural way to the columns in the Laplacian matrix defined in Subsection 1.3.1.

A bijection between the set \(E(n)\) of Euler trails in \(G_n\) and the group \(\Phi(n)\) is obtained as follows. Put

\[
\partial_h(2v + p) = g_{2v+p \mod |V|} - g_v
\]

\((0 \leq v < |V|, p \in \{0, 1\})\). This means that \(\partial_h(a)\) is equal to \(g_h - g_t\), where \(h\) is the head and \(t\) is the tail of the arc \(a\). In the Euler trail \(\pi\), consider 0 as the first element in the trail. The bijection \(E(n) \to \Phi(n)\) is given by

\[
\pi \mapsto \sum \partial_h(a),
\]

where we sum over all arcs \(a\) such that \(1 \leq a < |V|\) and such that \(a\) appears before \(a + |V|\) in \(\pi\) (with respect to the arc 0).

We should mention that (2) is equivalent to saying that

\[
\partial_h(2v + p) + \partial_h(2v + p + |V|) = 0.
\]

In a general digraph one defines \(\partial_h\) of an arc in the same manner, and each of the relations corresponding to (2) is of the form

\[
\sum \partial_h(a) = 0,
\]

where the summation ranges over all arcs with a given head. We refer to Section 3.1 for more details and the general result.

**Example 4** Let \(n = 4\). A bijection

\[
\Phi(4) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4
\]

is given by \(g_0 = g_4 = 0, g_1 \mapsto (1, 0, 0), g_5 \mapsto (1, 0, 2), g_2 \mapsto (0, 1, 3), g_6 \mapsto (0, 1, 1), \) and \(g_3 = g_7 \mapsto (0, 0, 1)\). We obtain the following table: arcs \(a\) such that \(1 \leq a < 8\) and such that \(a\) appears before \(a + 8\) are emphasized.
Euler trail
group element

\begin{align*}
(0, 1, 2, 5, 11, 6, 13, 10, 4, 9, 3, 7, 15, 14, 12, 8) & \quad (1, 0, 3) \\
(0, 1, 2, 4, 9, 3, 6, 13, 10, 5, 11, 7, 15, 14, 12, 8) & \quad (1, 1, 1) \\
(0, 1, 3, 7, 15, 14, 12, 9, 2, 5, 11, 6, 13, 10, 4, 8) & \quad (0, 0, 3) \\
(0, 1, 3, 6, 13, 10, 5, 11, 7, 15, 14, 12, 9, 2, 4, 8) & \quad (0, 1, 1) \\
(0, 1, 2, 4, 9, 3, 7, 15, 14, 13, 10, 5, 11, 6, 12, 8) & \quad (1, 0, 1) \\
(0, 1, 2, 5, 11, 7, 15, 14, 13, 10, 4, 9, 3, 6, 12, 8) & \quad (1, 1, 3) \\
(0, 1, 3, 7, 15, 14, 13, 10, 4, 9, 2, 5, 11, 6, 12, 8) & \quad (0, 1, 2) \\
(0, 1, 3, 7, 15, 14, 13, 11, 6, 12, 9, 2, 5, 10, 4, 8) & \quad (1, 1, 0) \\
(0, 1, 2, 5, 10, 4, 9, 3, 7, 15, 14, 13, 11, 6, 12, 8) & \quad (0, 1, 0) \\
(0, 1, 2, 5, 11, 6, 12, 9, 3, 7, 15, 14, 13, 10, 4, 8) & \quad (1, 1, 2) \\
(0, 1, 3, 6, 13, 10, 4, 9, 2, 5, 11, 7, 15, 14, 12, 8) & \quad (0, 0, 2) \\
(0, 1, 3, 6, 13, 11, 7, 15, 14, 12, 9, 2, 5, 10, 4, 8) & \quad (1, 0, 0) \\
(0, 1, 2, 5, 10, 4, 9, 3, 6, 13, 11, 7, 15, 14, 12, 8) & \quad (0, 0, 0) \\
(0, 1, 2, 5, 11, 7, 15, 14, 12, 9, 3, 6, 13, 10, 4, 8) & \quad (1, 0, 2)
\end{align*}

For example, in the third Euler trail, $a$ appears before $a+8$ if and only if $a \in \{1, 2, 3, 5, 7\}$; we obtain that

\begin{align*}
\sum_{a \in \{1, 2, 3, 5, 7\}} \partial_1(a) &= (g_1 - g_5) + (g_2 - g_1) + (g_6 - g_1) + (g_7 - g_2) + (g_7 - g_3) \\
&= -g_5 - g_1 + g_6 + g_7 \mapsto (0, 0, 3) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4.
\end{align*}

We conclude this introduction with an explicit formula for the group $\Phi(n)$;

\[ \Phi(n) \cong \bigoplus_{k=1}^{n-2} \mathbb{Z}_2^{2k-n-k} . \]

The proof of a more general result is found in Subsection 3.1.4.

Acknowledgments

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Chapter 1

Definitions and summaries

1.1 Basic concepts

A digraph (or, more precisely, a directed multigraph with loops permitted)

\[ G = (V, A, t, h) \]

consists of a set \( V = V_G \) of vertices, a set \( A = A_G \) of arcs, and two functions \( t, h : A_G \rightarrow V_G \); \( t(a) = t_G(a) \) is the tail of the arc \( a \) and \( h(a) = h_G(a) \) is the head of \( a \). Intuitively speaking, an arc is an arrow pointing from its tail to its head. Put \( H_v = h^{-1}(v) \) and \( T_v = t^{-1}(v) \) \((v \in V)\).

A path in \( G \) is a sequence \((a_1, \ldots, a_m)\) of arcs such that the head of \( a_i \) is equal to the tail of \( a_{i+1} \), that is, \( h(a_i) = t(a_{i+1}) \) for \( 1 \leq i < m \). If in addition \( h(a_m) = t(a_1) \), then \((a_1, \ldots, a_m)\) is a (directed) cycle. A (directed spanning) tree in \( G \) rooted in \( r \in V \) is a subdigraph \( G_0 \) of \( G \) such that there is a unique path from \( r \) to \( v \) in \( G_0 \) for every \( v \in V_r := V \setminus r \) and such that there is no arc with head \( r \). The vertex \( r \) is called the root of the tree. Since the vertex set is assumed to be fixed, one may identify a tree with its set of arcs. In fact, we will identify the tree \( G_0 \) with the function \( \tau : V_r \rightarrow A \) having the property that \( \tau(v) \) is the unique arc \( a \in G_0 \cap H_v \) for each \( v \in V_r \).

Let \( S_A \) be the symmetric group of permutations of \( A \) with multiplication defined by \( \pi_1(a) = \pi(\sigma(a)) \). A permutation \( \pi \in S_A \) is a \( G \)-permutation if \( h(a) = t(\pi(a)) \) for all \( a \in A \); this means that \( a \) ends where \( \pi(a) \) begins. The permutation \( \pi \) is an Euler trail in \( G \) if \( \pi \) is a cyclic \( G \)-permutation. A digraph containing Euler trails is an Eulerian digraph. For \( m > 0 \), an \( m \)-regular digraph is a digraph where, for each vertex \( v \), there are exactly \( m \) arcs with head \( v \) and \( m \) arcs with tail \( v \), that is, \(|H_v| = |T_v| = m \). Some 2-regular Eulerian digraphs are illustrated in Figure 1 in the Introduction and in Figure 2.6 in Section 2.3.

Given a permutation \( \pi \in S_A \) and a subset \( B \) of \( A \), let \( \pi^B \in S_B \) be the permutation obtained by removing all elements in \( A \setminus B \) from the cycle representation of \( \pi \). For example, if \( \pi = (1, 3, 5, 7)(2, 6)(4, 8) \) and \( B = \{1, 2, 5, 6\} \), then \( \pi^B = (1, 5)(2, 6) \).

The free abelian group on a given set \( X \) is denoted \( F(X) \). For a set \( \mathcal{P} \) of elements in a group \( \mathcal{P} \), let \( (\mathcal{P}) \) denote the subgroup of \( \mathcal{P} \) generated by \( \mathcal{P} \). Let \( \rho(\mathcal{P}) \) denote the rank of \( \mathcal{P} \).
Given two (totally ordered) sets $X_1$ and $X_2$ and a matrix $K$, we say that $K$ is an $X_1 \times X_2$-matrix if the rows and columns in $K$ are indexed by the elements in $X_1$ and $X_2$, respectively (in the given orders).

1.2 Summary of Chapter 2

We consider a 2-regular Eulerian digraph $G$ with vertex set $V := \{1, \ldots, n\}$ and with arc set $\{+1, \ldots, +n, -1, \ldots, -n\}$, where the head of $\pm k$ is $k$ (that is, $H_k = \{+k, -k\}$). Let $\pi$ be an Euler trail in $G$. Say that the vertices $j$ and $k$ intersect if $\pi \circ (+j, -j)(+k, -k)$ is an Euler trail and $j \neq k$. Cohn and Lempel [CL] defined a $V \times V$ intersection matrix

$$L(\pi) = (l_{jk})_{j,k \in V}$$

of $\pi$ by letting $l_{jk}$ be 1 if $j$ and $k$ intersect in $\pi$ and 0 otherwise. The interesting fact about $L(\pi)$ is the following result.

**Theorem 1.2.1 (Cohn & Lempel [CL])** Let $J$ be a subset of $V$ and let $\pi$ be an Euler trail. Then the nullity of the $J \times J$ submatrix of $L(\pi)$ is equal to the number of cycles minus one in the $G$-permutation

$$\sigma = \pi \circ \prod_{j \in J} (+j, -j).$$

\[\square\]

Note that $\sigma$ is equal to $\pi$ times a product of disjoint transpositions. Beck [Be] generalized Theorem 1.2.1 by considering products of not necessarily disjoint transpositions.

It is possible to give the elements in the matrix $L(\pi)$ signs such that Theorem 1.2.1 remains true over $\mathbb{Z}$; see Theorem 3.2.2 and Subsection 3.3.4 for details. The construction is as follows. Let the element indexed by $(u, v)$ in $L(\pi)$ be 1 if

$$\pi^{H_u \cup H_v} = (+u, +v, -u, -v)$$

and $-1$ if

$$\pi^{H_u \cup H_v} = (+u, -v, -u, +v).$$

Using methods from theoretical physics, Macris and Pulé [MP2] obtained the following result.

**Theorem 1.2.2 (Macris & Pulé [MP2])** The determinant of $L(\pi) + I$ is equal to the number of Euler trails in $G$, where $I$ is the identity matrix. \[\square\]

Their proof uses the Matrix-tree Theorem 1.3.1 as well as the BEST Theorem 1.3.2. A direct proof inspired by [CL] has been established by J. Lauri [La], who actually proved more:
Theorem 1.2.3 (Lauri [La]) Let $J$ be a subset of $V$ and let $\pi$ be an Euler trail. Then the determinant of the $J \times J$ submatrix of $L(\pi)$ is equal to 1 if the $G$-permutation

$$\sigma = \pi \circ \prod_{j \in J} (+j, -j).$$

is an Euler trail and 0 otherwise. \hfill \square

One of our own contributions is the following result (Theorem 2.1.1): If $\sigma = \pi \circ \prod_{j \in J} (+j, -j)$ is an Euler trail, then

$$L_J(\sigma) = L_J^{-1}(\pi).$$

In Chapter 3 we generalize the intersection matrix to arbitrary Eulerian digraphs (see Section 1.3).

In Section 2.2 we consider a (not necessarily 2-regular) Eulerian digraph $G$ in which the vertices can be divided into sets $U_1, \ldots, U_\varphi$ and into sets $W_1, \ldots, W_\varphi$ such that if $a$ is an arc with its tail in $U_k$, then the head of $a$ is in $W_k$. Another way of expressing this is that $t^{-1}(U_k) = h^{-1}(W_k)$ for every $k$.

Let $\pi$ be a fixed Euler trail in $G$ and let $\sigma$ be another Euler trail in $G$. Then $\sigma^{-1} \pi$ is a permutation of the arcs such that the set $A_k$ of arcs with heads in $W_k$ is mapped onto itself for all $k$. In particular, $\sigma^{-1} \pi$ can be restricted to $A_k$. Let $s_k$ be the sign of this restricted permutation. We obtain a sequence $(s_1, \ldots, s_{\varphi-1})$ of signs associated with the Euler trail $\sigma$; we omit the $\varphi$th sign, since it is uniquely determined by (and equal to) $\prod_{k=1}^{\varphi-1} s_k$.

We prove that the number of Euler trails with a given sequence of signs is independent of the sequence, that is, the number is the same for all $2^{\varphi-1}$ possible sequences (Theorem 2.2.2). Suppose in addition that $G$ is 2-regular, and let $v$ and $w$ be vertices such that there is an arc from $v$ to $w$. Then the number of Euler trails with a given sequence of signs and with $v$ and $w$ intersecting is again constant (Theorem 2.2.3).

In Section 2.3 the chapter is concluded with a discussion about arc digraphs. In an arc digraph there are sets $U_1, \ldots, U_\varphi, W_1, \ldots, W_\varphi$ with the property that for each $k \in [\varphi]$, $u \in U_k$, and $w \in W_k$, there is exactly one arc with tail $u$ and head $w$. In Subsection 2.3.2 we apply Theorem 2.2.2 to de Bruijn sequences.

1.3 Summary of Chapter 3

There are two main themes in Chapter 3: The critical group of a digraph, which is discussed in Subsection 1.3.1, and a determinant formula for the number of directed spanning trees in Eulerian digraphs, which is discussed in Subsection 1.3.2. We will use standard results from the theory of linear algebra over the ring of integers; for more information, see [Mc].

1.3.1 The critical group

Given a digraph $G$, we want to establish a bijection between the set $X_r(G)$ of trees with a fixed root $r$ and a certain abelian group $\Phi_r(G)$. The group $\Phi_r(G)$
is constructed from the Laplacian of $G$, which is a matrix defined as follows. Let $p_{uv}$ be the number of arcs with tail $u$ and head $v$, that is,

$$ p_{uv} = |T_u \cap H_v|. $$

The Laplacian of $G$ is the $V \times V$ matrix $Q = (q_{uv})_{u,v \in V}$ defined by

$$ q_{uv} = \begin{cases} -p_{uv} & \text{if } u \neq v; \\ |H_v| - p_{uv} & \text{if } u = v. \end{cases} $$

For $r \in V$, let $Q_{(r)}$ be the $V_r \times V_r$ submatrix of $Q$ ($V_r = V \setminus \{r\}$). The following result has been discovered independently by several authors from different areas of mathematics; a discussion of the result can be found in [Tu].

**Theorem 1.3.1 (The Matrix-tree Theorem [BSST])** The determinant of the matrix $Q_{(r)}$ is equal to the number $|X_r(G)|$ of trees rooted in $r$. □

If $p_{ij} = p_{ji}$ for all $i,j$, then the digraph can be identified with the undirected graph with exactly $p_{ij}$ edges between $i$ and $j$. In this case parts of the theory developed here will coincide with results in [Bi].

Let $B_0(r) \subset F(V)$ be the group generated by $\{ \gamma_v : v \in V_r \}$, where $\gamma_v$ is the element represented by the column indexed by $v$ in $Q$;

$$ (1.1) \quad \gamma_v = - \sum_{u \in V} p_{uv} u + |H_v| v. $$

Let $\partial_0 : F(V) \to \mathbb{Z}$ be the homomorphism defined by $\partial_0(v) = 1$. Put $Z_0 = \ker \partial_0$ and

$$ \Phi_v(G) := Z_0/B_0(r); $$

this is well-defined since $\partial_0(\gamma_v) = 0$. $\Phi_v(G)$ is the critical group of $G$ with respect to the root $r$.

Unless $X_r(G)$ is empty, the number of elements in the set $X_r(G)$ of trees rooted in $r$ is equal to the number of elements in the group $\Phi_v(G)$; see Theorem 3.1.1. We have found an explicit bijection between the two sets; the bijection is easiest to describe in the case where all vertices in $V_r$ have indegree 2, that is, $|H_v| = 2$ for all $v \in V_r$. Fix a tree $\tau : V_r \to A$. For any tree $\hat{\tau}$, let $I_r$ be the set of vertices $v \in V_r$ with the property that $\tau(v) = \hat{\tau}(v)$. Then

$$ \hat{\tau} \mapsto \sum_{v \in I_r} (v - t \circ \tau(v)) = \sum_{v \in I_r} \partial_1(\tau(v)) $$

gives a bijection $X_r(G) \to \Phi_v(G)$, where $\partial_1(a) = h(a) - t(a)$. The general situation is treated in Section 3.1, where the bijection is constructed in a slightly different way.

### 1.3.2 A determinant formula for the number of trees in Eulerian digraphs

We want to generalize a determinant formula for the number of Euler trails in 2-regular Eulerian digraphs; the formula was discovered by Macris and Pulé [MP2] and discussed in the previous section. Recall that $G$ being 2-regular means that
\(|H_v| = |T_v| = 2\) for all \(v \in V\), where \(H_v\) and \(T_v\) are the sets defined in the beginning of Section 1.1. Our purpose is to construct a matrix associated with an Euler trail in an arbitrary Eulerian digraph such that the determinant of the matrix counts the number \(\chi(G) = \chi_r(G)\) of trees rooted in \(r\). The following result indicates how this will generalize Theorem 1.2.2.

**Theorem 1.3.2 (BEST Theorem, [AB])** Let \(G\) be an Eulerian digraph and let \(E_G\) be the number of Euler trails in \(G\). Then

\[
E_G = \chi(G) \cdot \prod_{v \in V(G)} (|H_v| - 1)!,
\]

The generalization is carried out in Sections 3.2 and 3.3; here we discuss the background and the most important ideas.

André Bouchet [Bo] has introduced a class of algebraic structures called isotropic systems. A special case is the class of isotropic systems associated with 2-regular Eulerian digraphs. Given such a digraph \(G\) with vertex set \(V\) and arc set \(A\), its isotropic system can be described as a pair \((M, C)\), where \(M\) is a vector space over \(GF(2)\) and \(C\) is a subspace. We define \(M\) to be a direct sum of 2-dimensional spaces \(M_v\), one for each \(v \in V\). \(M_v\) consists of the four elements \(0, ab = cd, ad = cb, ab + ad = cb + cd\), where \(\{a, c\} = H_v\) and \(\{b, d\} = T_v\). To define \(C\), let \(L_1\) be an \(|A|\)-dimensional vector space with basis \(A\). Let \(B_1\) be the subspace of \(L_1\) generated by the elements

\[
\sum_{a \in H_v} a + \sum_{b \in T_v} b,
\]

\(v \in V\). Define the linear map \(\delta : M \to L_1/B_1\) by \(\delta(ab) = a + b\) and let \(C\) be the kernel of \(\delta\). The most important facts from our point of view about \(C\) are summarized in the following theorem.

**Theorem 1.3.3 (Bouchet [Bo])** The dimension of \(C\) is \(|V|\). Moreover, let \(\sigma\) be a \(G\)-permutation and let \(M_\sigma^+\) be the \(|V|\)-dimensional subspace of \(M\) generated by \(\{a \sigma(a) : a \in A\}\). Then \(\dim(C \cap M_\sigma^+)\) is a vector space of dimension the number of cycles in \(\sigma\) minus one.

Though different in shape, Theorem 1.3.3 and Theorem 1.2.1 are just two ways of expressing the same result. For example, suppose that \(J = V\) in Theorem 1.2.1. Put \(M_\sigma^- = M_\sigma^+\) with notations as in Theorem 1.3.3; note that \(M\) is the direct sum of \(M_\sigma^+\) and \(M_\sigma^-\). For a certain basis for \(C\) (see Subsection 3.2.2) and the obvious basis for \(M, L(\pi)\) is the matrix for the projection of \(C\) on \(M_\sigma^-\) (consider \(M_\sigma^+\) and \(M_\sigma^-\) as orthogonal). In particular,

\[
\dim(C) = \dim(C \cap M_\sigma^+) + (\dim(C) - \text{null } L(\pi))
\]

by the dimension theorem.

Our purpose is to generalize these results to arbitrary Eulerian digraphs. We develop the theory over \(\mathbb{Z}\), that is, we consider free abelian groups instead of \(GF(2)\)-spaces. Namely, to be able to compute determinants over \(\mathbb{Z}\) we need matrices over \(\mathbb{Z}\).

In Theorem 3.2.2 and Theorem 3.3.1 we generalize Theorem 1.3.3 and Theorem 1.2.1, respectively. Theorem 1.2.2 is generalized in Theorem 3.3.2.
1.4 Some group theory

To obtain decent formulations and proofs of our results, some group theory will be useful. Namely, there is an obvious group-theoretic interpretation of $G$-permutations as follows.

Let $G$ be an Eulerian digraph. For a vertex $v \in V$, recall that $H_v = h^{-1}(v)$ is the set of arcs with head $v$. Let $Y_G$ be the Young subgroup of $S_A$ consisting of all permutations $\sigma$ such that $\sigma(H_v) = H_v$ for all $v \in V$. Clearly $Y_G$ is a group of the form $\prod_{v \in V} S_{H_v}$. Let $\pi$ be a fixed $G$-permutation; we claim that $\pi Y_G$ is the set of $G$-permutations. Namely, for any $\tau \in Y_G$, the permutation $\pi \tau$ is also a $G$-permutation since $t(\pi \tau(a)) = h(\pi(a)) = h(a)$. Moreover, if $\sigma$ is a $G$-permutation, then $\pi^{-1} \sigma \in Y_G$ since $h(\pi^{-1} \sigma(a)) = t(\sigma(a)) = h(a)$.

Conversely, every left coset $\pi Y$ of a permutation group of the form $Y = \prod_{v \in V} S_{A(v)}$, where the sets $A(v)$ are disjoint, can be interpreted as the set of $G$-permutations in a certain digraph $G$. Namely, put $V_G = V, A_G = \bigcup_{v \in V} A(v), h(a) = v$ for all $a \in A(v)$, and $t(a) = h(\pi^{-1}(a))$ for arbitrarily chosen $\pi \in \pi Y$. This is well defined, because with $\pi = \pi \tau$ we obtain that

$$h(\pi^{-1}(a)) = h(\pi(\pi^{-1} \pi^{-1}(a))) = h(\pi^{-1}(a)).$$

We say that $G$ is the digraph induced by $\pi Y$.

Let $B \subset A$, where $A$ is a finite set. We may consider $S_B$ as a subgroup of $S_A$ by defining $\pi(a) = a$ for all $\pi \in S_B$ and $a \in A \setminus B$. Recall the definition of $\pi^B \in S_B$, where $\pi$ is a permutation in $S_A$. Note that for each $b \in B$

$$\pi^B(b) = \pi^k(b),$$

where $k$ is the smallest positive number $j$ such that $\pi^j(b) \in B$. If $\pi(B) = B$, then $\pi^B$ is the restriction of $\pi$ to the set $B$. If $\pi, \sigma \in S_A$ and $\pi(a) = \sigma(a)$ for all $a \notin B$, then

$$\pi^B \sigma = \pi^{-1} \sigma \pi,$$

where the right-hand expression is restricted to $B$. To prove this, first assume that $B = A \setminus \{a\}$. Then $\pi(b) = \pi^B(b)$, unless $\pi(b) = a$ in which case $\pi^B(b) = \pi(a) = \sigma(a)$. The same is true for $\sigma$; hence

$$(\pi^B)^{-1} \sigma^B = (\pi^B)^{-1} \sigma^B \pi^B = \pi^{-1} \sigma \pi = \pi^{-1} \sigma \pi.$$ 

To prove (1.3) for general $B$, use induction over $|A \setminus B|$ (i.e., the number of elements in the set $J$).

For more information about permutation groups, see [DM].

1.5 Matrix notations

In Chapter 3 it will be of great importance for us to express elements in our free abelian groups in different bases. Therefore we introduce some matrix notations.

Let $\mathcal{X}$ and $\mathcal{Y}$ be totally ordered sets of elements in a free finitely generated abelian group such that $(\mathcal{Y}) \subseteq (\mathcal{X})$ and $\mathcal{X}$ is independent. Let $[\mathcal{X} \times \mathcal{Y}]$ be the $\mathcal{X} \times \mathcal{Y}$-matrix defined by letting the element with index $(r, s)$ be equal to $k_{xs}$, where
\[ y = \sum_{x \in \mathcal{X}} \kappa_{xy} x. \]

That is, the columns in \([\mathcal{X} | \mathcal{Y}]\) are the elements in \(\mathcal{Y}\) expressed in the basis \(\mathcal{X}\).

For \(\mathcal{X}_0 \subseteq \mathcal{X}\) and \(\mathcal{Y}_0 \subseteq \mathcal{Y}\), let \([\mathcal{X}_0 : \mathcal{X} | \mathcal{Y}_0]\) denote the \(\mathcal{X}_0 \times \mathcal{Y}_0\)-submatrix of \([\mathcal{X} | \mathcal{Y}]\); \([x : \mathcal{X} | \mathcal{Y}]\) denotes the row with index \(x\), while \([\mathcal{X}' | y]\) denotes the column with index \(y\) in \([\mathcal{X} | \mathcal{Y}]\).

Finally, suppose that \(\mathcal{Y}\) is an independent set and \((\mathcal{X})/(\mathcal{Y})\) is torsion-free. In the additive group of all \(\mathcal{Y} \times \mathcal{X}\) matrices, let \(\Gamma\) be the subgroup of all matrices \(\mathcal{N}\) such that \(\mathcal{N}[\mathcal{X} | \mathcal{Y}] = 0\). Let the matrix \(K\) satisfy

\[ K[\mathcal{X} | \mathcal{Y}] = [\mathcal{Y} | \mathcal{Y}] = \text{the identity matrix.} \]

We now define the “inverse” \([\mathcal{X} | \mathcal{X}]\) of \([\mathcal{X} | \mathcal{Y}]\) to be the coset \(K + \Gamma\).

We have the following simple product rule: If \((\mathcal{X}_1), (\mathcal{X}_2), (\mathcal{Y})\) are totally ordered with respect to inclusion and \((\mathcal{X}_i) \subseteq (\mathcal{Y})\) for at least one \(i\), then

\[ [\mathcal{X}_1 | \mathcal{Y}] [\mathcal{Y} | \mathcal{X}_2] = [\mathcal{X}_1 | \mathcal{X}_2]. \tag{1.4} \]

This is trivial unless \((\mathcal{X}_i) \subset (\mathcal{X}_{i-1}) \subset (\mathcal{Y})\) for \(i = 1\) or \(2\). But if for example \(i = 1\), then the left-hand side of (1.4) equals

\[ [\mathcal{X}_1 | \mathcal{X}_2][\mathcal{X}_2 | \mathcal{Y}] [\mathcal{Y} | \mathcal{X}_2] = [\mathcal{X}_1 | \mathcal{X}_2][\mathcal{X}_2 | \mathcal{X}_2] = [\mathcal{X}_1 | \mathcal{X}_2]. \]
Chapter 2

On the number of Euler trails

As the title indicates, the purpose of this chapter is to count Euler trails in directed graphs. The main tool is the intersection matrix of an Euler trail in a 2-regular Eulerian digraph; properties of the intersection matrix are discussed in Section 2.1. In Section 2.2 we discuss how to divide the set of Euler trails into smaller sets of the same size; these results are applied to arc digraphs in Section 2.3.

2.1 The intersection matrix

We are going to describe the intersection matrix of an Euler trail in a 2-regular digraph. Some small modifications compared to [MP1], [MP2], and [La] are made to make it possible to compare the intersection matrices for different Euler trails.

2.1.1 Definition

Let \( n > 0 \) be an integer; let \( G \) be a 2-regular Eulerian digraph with vertex set \( V = [n] \) and arc set \( A = \{+1, \ldots, +n, -1, \ldots, -n\} \), where the head of the arcs \(+k\) and \(-k\) is \( k \). Let \( \pi \) be an Euler trail in \( G \). Define the intersection matrix

\[
L(\pi) = (l_{jk})_{j,k \in [n]}
\]

of \( \pi \) as follows. For \( j, k \in [n] \), put \( B = \{+j, -j, +k, -k\} \) and

\[
l_{jk} = \begin{cases} 
1 & \text{if } \pi^B = (-j, +k, +j, -k); \\
-1 & \text{if } \pi^B = (-j, -k, +j, +k); \\
0 & \text{otherwise.}
\end{cases}
\]

In particular, \( l_{jj} = 0 \). Note that \( l_{jk} \) is nonzero if and only if \( j \) and \( k \) intersect in the sense described in the introduction. Also note that in this definition we obtain the transpose of the matrix defined in Section 1.2. We have made this choice for technical reasons; in Subsection 2.1.3 we find it handier to deal with row vectors than with column vectors.
One easily checks that our definition of $L(\pi)$ is the same as the corresponding definitions in [MP2] and [La]. See Subsection 2.1.3 for another equivalent definition.

2.1.2 The inverse of the intersection matrix

We will prove a result that requires our definition of the intersection matrix. Namely, to construct the intersection matrix, one must give the arcs signs. In this thesis these signs are fixed from the beginning, while in [La] the signs are not necessarily the same in different Euler trails.

For any set $J \subseteq [n]$, put $\tau_J = \prod_{j \in J} (+j, -j)$. Since $Y_G$ is the abelian group generated by the transpositions $(+1, -1), \ldots, (+n, -n)$, it is clear that $\pi \tau_J$ is a $G$-permutation whenever $\pi$ is a $G$-permutation. Moreover, the number of cycles in $\pi \tau_J$ is odd if and only if $|J|$ is even. In particular, if $\pi \tau_J$ is an Euler trail, then $|J|$ is even. Let $L_J(\pi)$ denote the $J \times J$ submatrix of $L(\pi)$.

**Theorem 2.1.1** Let $\pi$ be an Euler trail in a 2-regular digraph with vertex set $[n]$, and let $J \subseteq [n]$ be such that $\pi \tau_J$ is an Euler trail. Then

$$L_J(\pi \tau_J) = L_J^{-1}(\pi).$$

We will in fact prove more than Theorem 2.1.1:

**Lemma 2.1.2** With notations and assumptions as in Theorem 2.1.1, suppose that $J = [2m]$ ($2m \leq n$). Write

$$L(\pi) = \begin{pmatrix} L_J & -A^T \\ A & B \end{pmatrix},$$

where $L_J = L_J(\pi)$, and put

$$M(\pi, J) = \begin{pmatrix} L_J & 0 \\ A & I \end{pmatrix}.$$

Then

(2.1) \hspace{1cm} L(\pi) + I = M(\pi, J) \cdot (L(\pi \tau_J) + I)

or, equivalently,

(2.2) \hspace{1cm} L(\pi \tau_J) = \begin{pmatrix} L_J^{-1} & -L_J^{-1}A^T \\ -AL_J^{-1} & B + AL_J^{-1}A^T \end{pmatrix}.

**Proof.** First consider $J = \{1, 2\}$. We have to show that (2.2) holds. However, note that $l_{ij}(\pi \tau_J)$ only depends on $1, 2, i, j$. Thus it suffices to consider $n = 4$. In particular, there is only a finite number of cases, and these are easily checked; as in [La], we let the reader do this. Using the row vectors $R_i$ introduced later in Subsection 2.1.3, one may carry out a rigorous (but cumbersome) proof.

Induction over $m$ is used to prove (2.1); suppose that $m > 1$. There exist distinct numbers $i, j \leq 2m$ such that $\pi_1 = \pi \tau_J \setminus \{i, j\}$ is a cyclic permutation.
Namely, otherwise $L_{ij}(\pi \tau_J)$ would be the zero matrix. Assuming (without loss of generality) that $i = 1$ and $j = 2$, we may write

$$L(\pi_1) = \begin{pmatrix} D & -P^T & -Q^T \\ P & E & -R^T \\ Q & R & S \end{pmatrix},$$

where $D = L_{\{1, 2\}}(\pi_1), E = L_{\{3, \ldots, 2m\}}(\pi_1)$; the other matrices are defined in the obvious manner.

By induction,

$$L(\pi_1) + I = M(\pi_1, \{1, 2\}) \cdot (L(\pi \tau_J) + I)$$

and

$$L(\pi_1) + I = M(\pi_1, J \setminus \{1, 2\}) \cdot (L(\pi) + I),$$

where

$$M(\pi_1, J \setminus \{1, 2\}) = \begin{pmatrix} I & -P^T & 0 \\ 0 & E & 0 \\ 0 & R & I \end{pmatrix}.$$ 

This implies that

$$L(\pi) + I = M^{-1}(\pi_1, J \setminus \{1, 2\}) \cdot M(\pi_1, \{1, 2\}) \cdot (L(\pi \tau_J) + I)$$

Some easy computations yield that $M^{-1}(\pi_1, J \setminus \{1, 2\}) \cdot M(\pi_1, \{1, 2\})$ is equal to

$$(2.3) \begin{pmatrix} D + P^T E^{-1} P & P^T E^{-1} 0 \\ E^{-1} P & E^{-1} 0 \\ Q - RE^{-1} P & -RE^{-1} I \end{pmatrix}.$$ 

Computing $L(\pi) = M^{-1}(\pi_1, J \setminus \{1, 2\}) \cdot (L(\pi) + I) - I$, we immediately realize that (2.3) is equal to $M(\pi, J)$, which is exactly what we need to prove (2.1) and (2.2).

$\square$

**Remark 1** One may note that the lower right block in (2.2) does not have the same shape in [La]. This depends on the fact that Lauri defines the matrix corresponding to $\pi \tau_J$ in a slightly different way. However, the fact that Lauri considers $\tau_J \pi$ instead of $\tau_J \pi$ only affects the signs in the lower left and upper right blocks. By the way, (2.2) is the resulting matrix after the first step in a two-step method of computing the inverse of $L(\pi)$ (if the inverse exists); see for example [FF], pp. 161-163.

**Remark 2** In Chapter 3 we investigate the matrices $L(\pi)$ and $M(\pi, J)$ further. We will show that the rows in $L(\pi)$ can be interpreted as the vectors in a basis for a certain “cycle space”. The matrix $M(\pi, J)$ is the transformation matrix between two bases corresponding to the Euler trails $\pi$ and $\pi \tau_J$. 

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2.1.3 An application of the intersection matrix

We proceed by giving an alternative way of defining the intersection matrix of an Euler trail. Using this new definition, we prove a result that will be used and generalized in Section 2.2.

As usual we consider a 2-regular digraph $G$ with vertex set $[n]$ and with $H_i = \{+i, -i\}$ for $i \in [n]$. Let $\pi$ be an Euler trail in $G$. Put

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n;$$

the unit element in $e_i$ is on position $i$ ($1 \leq i \leq n$). Choose an arbitrary arc $a \in A$. Define the row vectors $R_b(a, \pi)$ and $L_j(\pi)$ ($b \in A, j \in [n]$) as follows.

$$R_a(a, \pi) = (0, \ldots, 0),$$
$$R_{\pi^{i+1}(a)}(a, \pi) = R_{\pi^i(a)}(a, \pi) + \text{sgn}(\pi^i(a)) \cdot e_{|\pi^i(a)|},$$

$(0 < i < 2n)$ and

$$L_j(\pi) = R_{+j}(a, \pi) - R_{-j}(a, \pi) + e_j.$$

The validity of the following statement is easily checked.

**Proposition 2.1.3** $L(\pi) = 
\begin{pmatrix}
L_1(\pi) \\
L_2(\pi) \\
L_3(\pi) \\
\vdots \\
L_n(\pi)
\end{pmatrix}
$ is the intersection matrix of $\pi$. \hfill \square

**Example 5** Consider the Euler trail

$$\pi = (-1, -2, -4, +2, -5, +4, -3, +1, +3, +5).$$

We obtain the following table.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\pi^i(-1)$</th>
<th>$R_{\pi^i(-1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>( 0, 0, 0, 0, 0)</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>(-1, 0, 0, 0, 0)</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
<td>(-1,-1, 0, 0, 0)</td>
</tr>
<tr>
<td>3</td>
<td>+2</td>
<td>(-1,-1, 0,-1, 0)</td>
</tr>
<tr>
<td>4</td>
<td>-5</td>
<td>(-1, 0, 0,-1, 0)</td>
</tr>
<tr>
<td>5</td>
<td>+4</td>
<td>(-1, 0, 0,-1,-1)</td>
</tr>
<tr>
<td>6</td>
<td>-3</td>
<td>(-1, 0, 0, 0,-1)</td>
</tr>
<tr>
<td>7</td>
<td>+1</td>
<td>(-1, 0,-1, 0,-1)</td>
</tr>
<tr>
<td>8</td>
<td>+3</td>
<td>( 0, 0,-1, 0,-1)</td>
</tr>
<tr>
<td>9</td>
<td>+5</td>
<td>( 0, 0, 0, 0,-1)</td>
</tr>
</tbody>
</table>

Thus

$$L(\pi) = 
\begin{pmatrix}
0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 \\
1 & 0 & 0 & 1 & 0
\end{pmatrix}.$$ \hfill \square

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Lemma 2.1.4 Let $G$ be a 2-regular digraph and suppose that there are two vertex sets $U = \{u_1, \ldots, u_k\}$ and $W = \{w_1, \ldots, w_k\}$ ($k > 0$) such that there is an arc with tail $u_i$ and head $w_i$ as well as an arc with tail $u_i$ and head $w_{i-1}$ for every $i \in [k]$; $u_0 = w_0$. Let $\pi$ be an Euler trail in $G$. Put $L^+ = L(\pi) + I, L^- = L(\pi) - I$. Then there exist numbers $\mu_i, \nu_i \in \{-1, +1\}, i = 1, \ldots, k$, such that

$$
\sum_{i=1}^k \mu_i L^+_{u_i} = \sum_{i=1}^k \nu_i L^-_{w_i}.
$$

Moreover, if $U$ and $W$ are disjoint, then the number of Euler trails $\sigma$ in $G$ such that $(\sigma^{-1} \pi)^{h^{-1}(W)}$ is odd is equal to the number of Euler trails $\sigma$ such that $(\sigma^{-1} \pi)^{h^{-1}(W)}$ is even.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.1.png}
\caption{The case $k = 3$.}
\end{figure}

**Proof.** With customary choices of numbers $\mu_i, \nu_i \in \{-1, +1\}$,

$$
\pi(\mu_i u_i) = \nu_i w_i, \pi(-\mu_i u_i) = -\nu_{i-1} w_{i-1}
$$

for all $i \in [k]$; $\nu_0 = \nu_k$ and $u_0 = w_k$. This implies that

$$
- (R_{\nu_i w_i} - R_{\mu_i u_i}) = - \mu_i e_{u_i},
$$

$$
R_{-\nu_{i-1} w_{i-1}} - R_{-\mu_i u_i} = - \mu_i e_{u_i}.
$$

Summing the left-hand sides, we obtain

$$
\sum_{i=1}^k \left( - (R_{\nu_i w_i} - R_{\mu_i u_i}) - R_{-\mu_i u_i} + R_{-\nu_{i-1} w_{i-1}} \right)
$$

$$
= \sum_{i=1}^k \left( - (R_{\nu_i w_i} - R_{\mu_i u_i}) + R_{-\nu_i w_i} - R_{-\mu_i u_i} \right)
$$

$$
= \sum_{i=1}^k (R_{\mu_i u_i} - R_{-\mu_i u_i}) - \sum_{i=1}^k (R_{\nu_i w_i} - R_{-\nu_i w_i})
$$

$$
= \sum_{i=1}^k \mu_i L^+_{u_i} - 2 \sum_{i=1}^k \mu_i e_{u_i} - \sum_{i=1}^k \nu_i L^-_{w_i}.
$$
Summing the right-hand sides, we obtain $-2 \sum_{i=1}^{k} \mu_i e_u$, which implies the first part of Lemma 2.1.4.

To prove the second part, let $K$ be the matrix with the property that

$$K_i = \begin{cases} L_i^- & \text{if } i \in W_i \\ L_i^+ & \text{otherwise.} \end{cases}$$

The first part of Lemma 2.1.4 implies that $\det K = 0$ (here we need the fact that $U$ and $W$ are disjoint). Expand $\det K$:

$$0 = \det K = \sum_{J \subseteq [n]} (-1)^{|W \setminus J|} \det L_J$$

$$= (-1)^{|W|} \sum_{J \subseteq [n]} (-1)^{|W \cap J|} \det L_J. \tag{2.4}$$

By Theorem 1.2.3, $\det L_J = 1$ if $\pi_{\tau J}$ is an Euler trail and 0 otherwise. Note that

$$\text{sgn} \left( ((\pi_{\tau J})^{-1} \pi)^{-1} h^{-1}(W) \right) = \text{sgn} \left( \prod_{j \in W \cap J} (+j, -j) \right) = (-1)^{|W \cap J|}.$$ 

Hence (2.4) implies the last part of Lemma 2.1.4. \qed

Lemma 2.1.4 is in fact a special case of Theorem 2.2.1, a result that will be stated and proved in the next section. It will appear that the last statement in Lemma 2.1.4 remains true with the weaker assumption that $U \neq W$.

### 2.2 Partitioning trails into sets of the same size

We prove some results about how to divide the set of Euler trails in a digraph into smaller sets of the same size.

#### 2.2.1 Main results

Let $G$ be an Eulerian digraph $G$ with vertex set $V$. Suppose that there exist nonempty vertex sets $U_1, \ldots, U_\varphi, W_1, \ldots, W_\varphi \subseteq V$ such that

$$V = \biguplus_{k=1}^{\varphi} U_k = \biguplus_{k=1}^{\varphi} W_k$$

($\biguplus$ denotes disjoint union) and

$$t^{-1}(U_k) = h^{-1}(W_k) \tag{2.5}$$

for $k \in [\varphi]$, that is, all arcs starting in $U_k$ have their heads in $W_k$ (we do not require that $U_k$ and $W_k$ are disjoint). Another way of describing the situation is as follows. Let $\pi$ be any $G$-permutation, and put $A_k = t^{-1}(U_k) = h^{-1}(W_k)$. Then for every $\tau \in Y_G$ and $k \in [\varphi]$,

$$\pi \tau \pi^{-1}(A_k) = \tau(A_k) = A_k. \tag{2.6}$$
Theorem 2.2.1 Let $G$ be an Eulerian digraph with vertex set $V$ and arc set $A$ such that $|H_v| \neq 1$ for each vertex $v \in V$. Let $E_G$ denote the set of Euler trails in $G$. Suppose that $A = \bigcap_{k=1}^\varphi A_k$ is a partition of $A$ satisfying (2.6) with $A_k$ nonempty and with $\varphi > 1$. Let $K$ be a nonempty proper subset of $[\varphi]$ and put $A^- = \bigcup_{k \in K} A_k$. Then, for any $G$-permutation $\pi$,

$$\sum_{\sigma \in E_G} \text{sgn}((\sigma^{-1} \pi)^{A^-}) = 0$$

That is, there are just as many Euler trails $\sigma$ in $G$ such that the restriction $(\sigma^{-1} \pi)^{A^-}$ to $A^-$ is odd as there are Euler trails $\sigma$ such that $(\sigma^{-1} \pi)^{A^-}$ is even.

Theorem 2.2.1 will be proved in Subsection 2.2.2. However, before we proceed, we will state another version of Theorem 2.2.1.

Theorem 2.2.2 Let the conditions in Theorem 2.2.1 be satisfied for the Eulerian digraph $G$. For any set $M \subseteq S_A$, let $E(M) = E_A(M)$ denote the set of cyclic permutations in $M$. Let $Y_0$ be the subgroup of $Y_G$ consisting of all permutations $\tau \in S_A$ such that the restriction of $\tau$ to $A_k$ is an even permutation for every $k \in [\varphi - 1]$. Then for any two $G$-permutations $\pi$ and $\sigma$,

$$|E(\pi Y_0)| = |E(\sigma Y_0)|.$$

In particular,

$$|E(\pi Y_0)| = 2^{-(\varphi - 1)} \cdot |E_G|$$

for every $G$-permutation $\pi$.

For any signs $s_1, \ldots, s_{\varphi - 1} \in \{-1, 1\}$, there is a permutation $\tau \in Y_G$ such that for all $i$ the restriction of $\tau$ to $A_i$ has sign $s_i$. Namely, $|H_v| \neq 1$ for every vertex $v \in V$. In particular, $Y_0$ divides $Y_G$ into $2^{\varphi - 1}$ cosets. Hence the second statement in Theorem 2.2.2 is a consequence of the first statement. In Subsection 2.2.3 we prove Theorem 2.2.2 using Theorem 2.2.1.

In the case of $G$ being a 2-regular digraph, Theorem 2.2.2 can be refined as follows. Recall that the vertices $u$ and $w$ intersect in an Euler trail $\pi$ if $\pi \circ (+u,-u)(+w,-w)$ is an Euler trail; $H_u = \{+u,-u\}$ and $H_w = \{+w,-w\}$.

Theorem 2.2.3 Let the conditions in Theorem 2.2.1 be satisfied for the 2-regular Eulerian digraph $G$ and let $Y_0$ be as in Theorem 2.2.2. Let $u$ and $w$ be vertices such that there is an arc with tail $u$ and head $w$. Then for any two $G$-permutations $\pi$ and $\sigma$, the number of Euler trails in $\pi Y_0$ such that $u$ and $w$ intersect is equal to the number of Euler trails in $\sigma Y_0$ such that $u$ and $w$ intersect.

With the convention that $u$ always intersects itself, the result remains valid if $u = w$. By Theorem 2.2.2, Theorem 2.2.3 is equivalent to the following statement: For any two $G$-permutations $\pi$ and $\sigma$, the number of Euler trails in $\pi Y_0$ such that $u$ and $w$ do not intersect is equal to the number of Euler trails in $\sigma Y_0$ such that $u$ and $w$ do not intersect. We will prove Theorem 2.2.3 by verifying this statement in Subsection 2.2.4.
2.2.2 Proof of Theorem 2.2.1

We recall that Theorem 2.2.1 considers a partition \( A = \bigsqcup_{j=1}^{m} A_j \) satisfying (2.6) and a set \( A^- = \bigcup_{k \in K} A_k \), where \( K \not\subseteq [\varphi] \) is nonempty. Put \( A^+ = A \setminus A^- \) and let \( \pi \) be a fixed \( G \)-permutation; we may without loss of generality assume that \( \pi \) is an Euler trail. Our first goal is to show how the problem can be reduced to the case \( \pi(A^-) = A^+ \).

Put \( B^+ = \pi^{-1}(A^-) \cap A^+, B^- = \pi^{-1}(A^+) \cap A^- \), and \( B = B^+ \cup B^- \). By (2.6), this definition of \( B^+ \) and \( B^- \) does not depend on the choice of \( \pi \); \( B^+ \) is the set of arcs \( b \) with the property that the arcs with tail \( h(b) \) belong to \( A^- \). This implies for every \( v \in V \) that if \( H_v \cap B \neq \phi \), then \( H_v \subset B \). Hence \( |H_v \cap B| \neq 1 \). Note that \( \pi^{-1}(A^-) \neq A^- \). Namely, since \( \pi \) is a cyclic permutation, equality would imply that \( A^- = A \). In particular, \( B^+ \) and \( B^- \) are nonempty.

Consider a \( G \)-permutation \( \sigma \) and recall the construction of \( \sigma^B \) from (1.2). Put \( Y_B = Y_G \cap S_B \). Since \( \rho Y_B = \sigma Y_B \) if \( \rho \in \sigma Y_B \),

\[
E_G = E_A(\pi Y_G) = \prod_{i=1}^{m} E_A(\sigma_i Y_B)
\]

for some \( \sigma_1, \ldots, \sigma_m \in E_G \); \( E_A(M) \) is the set of cyclic permutations in \( M \subseteq S_A \). Thus it suffices to show that

\[
\sum_{\rho \in E_A(\sigma_i Y_B)} \text{sgn}((\rho^{-1}(\pi)A^-) = 0
\]

for every \( i \in [m] \). Let \( G_i \) be the digraph induced by \( \sigma_i^B Y_B \subset S_B \) (see Section 1.4). This means that \( Y_{G_i} = Y_B \) (restricted to \( S_B \)) and that \( \sigma_i^B \) is an Euler trail in \( G_i \). An important observation is that \( \rho \in \sigma_i Y_B \) is an Euler trail in \( G \) if and only if \( \rho^B \) is an Euler trail in \( G_i \). Namely, if \( \rho^B \) happens to be an Euler trail in \( G_i \) without \( \rho \) being an Euler trail in \( G \), then \( \rho \) contains some cycle with arcs exclusively from \( A \setminus B \). However, since \( \rho^{-1}\sigma_i \) leaves all elements in \( A \setminus B \) fixed, the very same cycle will occur in \( \sigma_i \), which is a contradiction to the fact that \( \sigma_i \) is an Euler trail. Thus

\[
\sum_{\rho \in E_A(\sigma_i Y_B)} \text{sgn}((\rho^{-1}(\pi)A^-) = \\
= \sum_{\rho \in E_A(\sigma_i Y_B)} \text{sgn}((\rho^{-1}\sigma_i)A^-) \text{sgn}((\sigma_i^{-1}\pi)A^-) \\
= \sum_{\rho^B \in E_{G_i}} \text{sgn}((\rho^{-1}\sigma_i)B^-) \text{sgn}((\sigma_i^{-1}\pi)A^-).
\]

Here, the first identity is justified by the fact that all permutations are restrictions to \( A^- \), while the second identity follows from the fact that the restriction of \( \rho^{-1}\sigma_i \) to \( A^- \setminus B^- \) is the identity permutation. The conclusion is that (2.7) is equivalent to

\[
\sum_{\rho^B \in E_{G_i}} \text{sgn}((\rho^{-1}\sigma_i)B^-) = 0.
\]
Note that the conditions in Theorem 2.2.1 are satisfied if $G$, $A$, $A^-$, and $\pi$ are replaced by $G_i$, $B$, $B^-$, and $\sigma_i^B$, respectively. Namely, one easily convinces oneself that for every $\rho \in \sigma_i Y_B$, $(\rho^{-1} B^+)$ is $B^-$. Moreover, as we have already mentioned, $B^+$ and $B^-$ are nonempty and $|H_0 \cap B| \neq 1$ for all $v \in V$. Thus we may assume that $A = B$; that is, $\pi(A^-) = A^+$.

Our aim is to reduce the problem to the situation in Lemma 2.1.4. Let $a \mapsto (u, w)$ mean that the tail of $a$ is $u$ and the head of $a$ is $w$. Choose an arbitrary arc $d_0 \in A^- \setminus \{ a \}$; we have $d_0 \mapsto (u_1, w_0)$ for some $u_1 \in h(A^+)$ and $w_0 \in h(A^-)$. The set $t^{-1}(u_1)$ contains at least one arc $d_{-1}$ different from $d_0$; assume that $d_{-1} \mapsto (u_1, w_1)$. The set $H_{u_1}$ contains at least one member $d_1$ different from $d_{-1}$; assume that $d_1 \mapsto (u_2, w_1)$. Find an arc $d_{-2} \in t^{-1}(u_2)$ in the same manner as we found $d_{-1}$ (that is, $d_{-2} \neq d_1$ and $d_{-2} \mapsto (u_2, w_2)$ for some $w_2$). Continue in this zig-zag manner until for the first time $u_{k+1} = u_{j+1}$ or $w_k = w_j$ for some $j, k, k > j$. There is no loss of generality assuming that $j = 0$. In Figure 2.2 the case $w_3 = w_0$ is illustrated. The reader may note the similarities between this figure and Figure 2.1 in Subsection 2.1.3.

![Figure 2.2: The case $w_3 = w_0$](image)

Put $U = \{u_1, \ldots, u_k\}$, $W = \{w_1, \ldots, w_k\}$, and

$$D^- = \{d_1, \ldots, d_k, d_{-1}, \ldots, d_{-k}\} \subseteq A^-.$$ 

For the Euler trail $\sigma$, put

$$D^+ \sigma = \rho^{-1}(D^-) \subseteq A^+$$

and $D_\sigma = D^+ \sigma \cup D^-$. Suppose that $\rho \in \sigma Y_{D_\sigma}$, where $Y_{D_\sigma} = Y_G \cap S_{D_\sigma}$. Then $\rho(a) = \sigma(a)$ if $a \in A^+ \setminus D^+ \sigma$, which implies that $\rho(D^+ \sigma) = \sigma(D^+ \sigma) = D^-$ since $\rho(A^+) = \sigma(A^+)$. Hence $D_\rho = D_\sigma$ and $\rho Y_{D_\rho} = \sigma Y_{D_\sigma}$. In particular, there are Euler trails $\sigma_1, \ldots, \sigma_4$ such that $E_G = \bigcup_{j=1}^4 E_A(\sigma_j Y_{D_{\sigma_j}})$. Therefore it suffices to prove

$$\sum_{\rho^D_{\sigma} \in E_{D_{\sigma}}(\sigma^D_{D_{\sigma}})} \text{sgn}((\rho^{-1} \sigma)^{D^-}) = 0$$

for any $\sigma \in E_G$ by computations similar to those implying (2.8). However, since $\sigma^D_{D_{\sigma}} Y_{D_{\sigma}}$ induces a 2-regular Eulerian digraph satisfying the conditions in Lemma 2.1.4, (2.9) follows immediately. $\square$
2.2.3 Proof of Theorem 2.2.2

To prove Theorem 2.2.2, we need some names on the $2^{v-1}$ different cosets in $Y_G$ given by $Y_0$. First we fix a $G$-permutation $\pi$. For a vector $y = (y_1, \ldots, y_{v-1})$ of elements from \{0,1\}, let $E(y)$ be the set of Euler trails $\rho$ such that the sign of the permutation $(\sigma^{-1} \pi)A_k$ is equal to $(-1)^{y_k}$ for each $k \in [\varphi - 1]$. We want to prove that $|E(y)|$ is the same for all vectors $y$. Therefore, let $y = (y_1, \ldots, y_{\varphi-1})$ and $y' = (y'_1, \ldots, y'_{\varphi-1})$ be different vectors in \{0,1\}^{\varphi-1}. Consider the sum

\[
\sum_{x \cdot (y' - y) \equiv 1} \left( \sum_{(z-y) \cdot x \equiv 0} |E(z)| - \sum_{(z-y) \cdot x \equiv 1} |E(z)| \right),
\]

where the outer and inner summations range over all \{0,1\}^{\varphi-1}-vectors $x$ and $z$, respectively, satisfying the indicated relations; the dot products are computed modulo 2. Let $x \neq 0$ be a fixed vector; put $A^- = \bigcup_{x_i=1} A_i$ and $A^+ = A \setminus A^+$. Note that $A^+$ is nonempty since $A_{\varphi} \subseteq A^+$. Theorem 2.2.1 implies that

\[
\sum_{(z-y) \cdot x \equiv 0} |E(z)| - \sum_{(z-y) \cdot x \equiv 1} |E(z)| = 0.
\]

Namely, let $\sigma$ be such that $\text{sgn}((\sigma^{-1} \pi)A_i) = (-1)^{y_i}$ for $1 \leq i \leq \varphi - 1$. The first sum counts the number of Euler trails $\rho$ such that $(\rho^{-1} \sigma)A_i$ is an even permutation, while the second sum counts the other kind of permutations. In particular, (2.10) vanishes. We want to prove that (2.10) is equal to

\[
2^{\varphi-2} \cdot (|E(y)| - |E(y')|);\]

$2^{\varphi-2}$ is the number of vectors $x$ such that $x \cdot (y' - y) \equiv 1$. Namely, this will imply that $|E(y)| = |E(y')|$, which is exactly what we want to prove.

Obviously, the coefficient in front of $|E(y')|$ in (2.10) is equal to $2^{\varphi-2}$, while the coefficient of $|E(y')|$ is equal to $-2^{\varphi-2}$. Now, let $z$ be a vector such that $z \neq y, y'$. The coefficient of $|E(z)|$ is computed as follows. Since $z - y$ and $y' - y$ are linearly independent over $GF(2)$, there is a vector $x'$ such that $(z - y) \cdot x' \equiv 1$ (mod 2) and $(y' - y) \cdot x' \equiv 0$ (mod 2). Note that $x \mapsto (x + x')$ mod 2 is a permutation of the set consisting of vectors $x$ such that $x \cdot (y' - y) \equiv 1$ (mod 2). Thus since

\[
\sum_{x \cdot (y' - y) \equiv 1} (-1)^{(z-y) \cdot x} = (-1)^{(z-y) \cdot x'} \sum_{x \cdot (y' - y) \equiv 1} (-1)^{(z-y) \cdot x} = \sum_{x \cdot (y' - y) \equiv 1} (-1)^{(z-y) \cdot (x+x')} = \sum_{(x+x') \cdot (y' - y) \equiv 1} (-1)^{(z-y) \cdot (x+x')} = \sum_{x \cdot (y' - y) \equiv 1} (-1)^{(z-y) \cdot x},
\]

the coefficient in front of $|E(z)|$ is 0. The theorem is proved. \qed
2.2.4 Proof of Theorem 2.2.3

By assumption there is an arc, say \( +w \), with tail \( u \) and head \( w \). Say that the tail of the second arc \(-w\) with head \( w \) is \( w' \) and let \( a \) be the second arc with tail \( w \); say that the head of \( a \) is \( w' \). If \( u = w \), then there is nothing to prove, since in this case \( \pi \circ (+u, -u)(+w, -w) = \pi \) for all Euler trails \( \pi \). If \( u = w' \) then there are two arcs from \( u \) to \( w \); it is obvious that \( u \) and \( w \) intersect in all Euler trails. If instead \( u = w' \) or \( u' = w \), then there will be a loop at \( u \) or \( w \), which means that \( u \) and \( w \) never intersect in any Euler trail. Thus assume that \( u, w, u', w' \) are all different.

Let \( H \) be the 2-regular Eulerian digraph obtained from \( G \) as follows: We remove the vertex \( w \) and the arcs \(+w, -w\) from \( G \). Moreover, the arcs with tail \( w \) in \( G \) will have \( u \) as their tail in \( H \). Finally, the arc \( a \), whose tail is \( u \) in \( G \), will have \( u' \) as its tail in \( H \). The situation is illustrated in Figure 2.3.

![Figure 2.3: The construction of \( H \) from \( G \).](image)

Consider an Euler trail \( \pi \) in \( G \) where \( u \) and \( w \) do not intersect. This means that \( \pi \) is of the form

\[
(a, S_1, \alpha u, +w, S_2, -w, S_3, -\alpha u),
\]

where \( S_i \) are some sequences of arcs and \( \alpha \) is \(+1\) or \(-1\). \( \pi \) can be divided into three blocks: \([a, S_1, \alpha u], [+w, S_2, -w], \) and \([S_3, -\alpha u]\). Removing \(+w\) and \(-w\) and swapping block 2 and block 3, we obtain the permutation

\[
\hat{\pi} = (a, S_1, \alpha u, S_3, -\alpha u, S_2),
\]

which is an Euler trail in \( H \). Since the blocks can be recovered from \( \hat{\pi} \) and since any Euler trail in \( H \) is of the form (2.11), we have obtained a one-to-one correspondence between the set of Euler trails in \( G \) where \( u \) and \( w \) do not intersect and the set of Euler trails in \( H \).

We want to find sets \( \hat{U}_1, \ldots, \hat{U}_\varphi \) and \( \hat{W}_1, \ldots, \hat{W}_\varphi \) in \( H \) satisfying (2.5); recall that \( U_k = t_G^{-1}(A_k) \) and \( W_k = h^{-1}_G(A_k) \). Put \( \hat{W}_k = W_k \setminus \{w\} \). Construct the sets \( \hat{U}_j \) by first removing \( u \) and \( w \) and then adding \( u \) to \( \hat{U}_j \), where \( U_j \) is the set containing \( w \). One readily verifies from this construction that (2.5) is satisfied for these sets in \( H \). Put \( \hat{A} = h^{-1}(\hat{W}_k) \) and \( \hat{A} = A \setminus \{+w, -w\} \).

Let \( Y_0 \) be the subgroup \( Y_H \) consisting of all permutations \( \tau \in S_A \) such that the restriction of \( \tau \) to \( \hat{A} \) is an even permutation for every \( k \in [\varphi - 1] \). By Theorem 2.2.2 we have for any \( H \)-permutations \( \hat{\pi} \) and \( \hat{\sigma} \) that
Fix an Euler trail $\pi$ in $G$ where $u$ and $w$ do not intersect; $\pi$ corresponds to the permutation $\hat{\pi}$ in the manner described above. For a vector $y = (y_1, \ldots, y_{\varphi - 1})$ of elements from $\{0, 1\}$, let $\hat{E}(y)$ be the set of Euler trails $\hat{\sigma}$ in $H$ such that the sign of the permutation $(\hat{\sigma}^{-1} \pi)^{A^h}$ is $(-1)^y_k$ for each $k \in [\varphi - 1]$. Let $E^*(y)$ be the set of Euler trails $\sigma$ in $G$ where $u$ and $w$ do not intersect such that the sign of the permutation $(\sigma^{-1} \pi)^{A^h}$ is $(-1)^y_k$ for each $k \in [\varphi - 1]$. Finally, let $E(y)$ be as in the proof of Theorem 2.2.2.

Suppose that $u \in W_i$ and $w \in W_j$; $i$ and $j$ might be equal. Let $z = (z_1, \ldots, z_{\varphi - 1})$ be defined by $z_k = 0$ if $k \neq i, j$, $z_i = z_j = 1$ if $i \neq j$, and $z_i = z_j = 0$ if $i = j$. To prove Theorem 2.2.3, it suffices to prove that

$$|E^*(y)| + |E^*(y + z)| = |E^*(y')| + |E^*(y' + z)|$$

for every $y, y' \in \{0, 1\}^{\varphi - 1}$, where the vector sums are computed modulo 2. Namely, there is an obvious bijection between $E(y) \setminus E^*(y)$ and $E(y + z) \setminus E^*(y + z)$ given by $E(y) \leftrightarrow E^*(y)$. Hence (2.13) and Theorem 2.2.2 imply Theorem 2.2.3.

Consider an Euler trail $\sigma$ in $G$ where $u$ and $w$ do not intersect. The sign of $(\sigma^{-1} \pi)^{A^h}$ is equal to the sign of $(\hat{\sigma}^{-1} \hat{\pi})^{A^h}$ if $k \neq i, j$. Namely, the procedure $\sigma \leftrightarrow \hat{\sigma}$ only modifies the successors of $u, u'$, and $w$, and they are all in $A_i \cup A_j$. This means exactly that

$$|E^*(y)| + |E^*(y + z)| = |\hat{E}(y)| + |\hat{E}(y + z)|$$

$$|E^*(y')| + |E^*(y' + z)| = |E^*(y')| + |E^*(y' + z)|,$$

where the second equality follows from (2.12). Theorem 2.2.3 is proved. □

2.3 2-regular arc digraphs

We consider arc digraphs and give interpretations of Theorems 2.2.2 and 2.2.3; we will concentrate on 2-regular digraphs. The section is concluded with an application of the results to de Bruijn sequences.

2.3.1 Interpretations of Theorem 2.2.2

The arc digraph $K(G)$ of a digraph $G$ is defined by $V_{K(G)} = A_G$,

$$A_{K(G)} = \{(a, b) : a, b \in A_G, h_G(a) = t_G(b)\},$$

t_{K(G)}((a, b)) = a$, and $h_{K(G)}((a, b)) = b$. That is, the arcs in $G$ are the vertices in $K(G)$ and there is an arc from $a$ to $b$ in $K(G)$ if and only if the head of $a$ is equal to the tail of $b$ in $G$.

We obtain in a natural way sets $A_1, \ldots, A_\varphi$ in $K(G)$ satisfying (2.6), where $\varphi$ is the number of vertices in $G$. Namely, for each $k \in V_G$, we put

$$A_k = t_{K(G)}^{-1}(h_{K(G)}^{-1}(k)) = h_{K(G)}^{-1}(t_G^{-1}(k)),$$
that is, $A_k$ consists of all arcs $(a, b)$ in $K(G)$ such that the head of $a$ and the tail of $b$ in $G$ is $k$. Putting $U_k = h^{-1}_G(k)$ and $W_k = t^{-1}_G(k)$, we may notice the similarity between (2.14) and (2.5). The local shape of a 2-regular arc digraph is illustrated in Figure 2.4.

![Diagram](attachment:image.png)

Figure 2.4: A vertex $k$ in $G$ and the corresponding arc set $A_k$ in $K(G)$

From our point of view, the most interesting result about 2-regular arc digraphs and Euler trails is the following striking correspondence between the numbers of Euler trails in a digraph and its arc digraph.

**Theorem 2.3.1 (de Bruijn [Br]; see [MHa])** If $G$ is a 2-regular Eulerian digraph, then

$$|E_{K(G)}| = 2^\varphi - 1 |E_G|,$$

where $\varphi$ is the number of vertices in $G$.

Using a result by D. E. Knuth [Kn], one may prove Theorem 2.3.1 from the BEST Theorem 1.3.2.

Theorems 2.2.2 and 2.3.1 imply the following result.

**Theorem 2.3.2** Let $G$ be a 2-regular Eulerian digraph with vertex set $[\varphi]$, and let $A_k$ be defined as in (2.14) for $k \in [\varphi]$. Let $Y_0$ be the group of permutations $\tau \in Y_{K(G)}$ such that $\text{sgn}(\tau^a_k) = 1$ for $k \in [\varphi - 1]$. Then

$$|E_{A_{K(G)}}(\pi Y_0)| = |E_G|$$

for any $K(G)$-permutation $\pi$.

**Proof.** By Theorem 2.2.2,

$$|E_{A_{K(G)}}(\pi Y_0)| = 2^{1-\varphi} |E_{K(G)}|;$$

hence Theorem 2.3.2 is a consequence of Theorem 2.3.1.

The interpretation of Theorem 2.3.3 is somewhat more delicate.

**Theorem 2.3.3** Use the same notations as in Theorem 2.3.2 and let $a$ and $b$ be arcs in $G$ such that the head of $a$ equals the tail of $b$. Then for any $G$-permutation $\pi$, the number of Euler trails in $E_{A_{K(G)}}(\pi Y_0)$ such that the vertices $a$ and $b$ in $K(G)$ intersect is equal to the number of Euler trails $\sigma$ in $E_G$ such that $\sigma(a) \neq b$. 

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PROOF. By Theorem 2.3.2, Theorem 2.3.3 is equivalent to the following statement: The number of Euler trails in $E_{A_k(G)}(\pi Y_b)$ such that $a$ and $b$ do not intersect in $K(G)$ is equal to the number of Euler trails $\sigma$ in $E_G$ such that $\sigma(a) = b$.

Let $a'$ be the arc with the same head as $a$ in $G$; hence $a'$ is a vertex in $K(G)$ such that there is an arc from $a'$ to $b$. Let $b'$ be the arc with the same tail as $b$ in $G$, that is, there is an arc from $a$ to $b'$ and from $a'$ to $b'$ in $K(G)$, $K(G)$ is illustrated in the left part of Figure 2.5. Construct a digraph $Q$ from $K(G)$ in the same manner as we constructed $H$ in the proof of Theorem 2.2.3: Remove the vertex $b$ and the arcs with head $b$. Moreover, the arcs with tail $b$ in $K(G)$ will have $a$ as their tail in $Q$. Finally, the arc with tail $a$ and head $b'$ in $K(G)$ will have $a'$ as its tail in $Q$ (and still $b'$ as its head).

Following the proof of Theorem 2.2.3, we realize that the number of Euler trails in $K(G)$ such that $a$ and $b$ do not intersect is equal to $|E_Q|$. In $Q$ there are two arcs from $a'$ to $b'$. Construct the digraph $\hat{Q}$ from $Q$ by removing $b'$ together with the two arcs from $a'$ to $b'$ and by letting $a'$ be the new tail of the arcs with old tail $b'$ in $Q$. We have that $|E_\hat{Q}| = 2|E_Q|$. Namely, given an Euler trail $(+a', S_1, -a', S_2)$ in $\hat{Q}$, we obtain two Euler trails $(+d', +b', S_1, -a', -b', S_2)$ and $(+a', -b', S_1, -a', +b', S_2)$ in $Q$; $S_1$ and $S_2$ are sequences of arcs.

The graph $\hat{Q}$ can be obtained directly from $K(G)$ as follows. First remove $b$ and $b'$ together with the arcs with head $b$ or $b'$. Then let $a$ and $a'$ be the new tails of the arcs with old tails $b$ and $b'$, respectively (see Figure 2.5).

![Diagram](image)

Figure 2.5: The construction of $\hat{Q}$ from $K(G)$.

Now, construct a digraph $H$ from $G$ by removing $b$, $b'$, and the head of $a$ and $a'$; let the head of $a$ in $H$ be the head of $b$ in $G$; let the head of $a'$ in $H$ be the head of $b'$ in $G$. By inspection, one realizes that $K(H) = \hat{Q}$. Theorem 2.3.1 implies that $|E_\hat{Q}| = 2^{e-2}|E_H|$. Thus the number of Euler trails in $E_{K(G)}$ such that $a$ and $b$ do not intersect is equal to

$$|E_Q| = 2|E_\hat{Q}| = 2^{e-1}|E_H|,$$

which is equal to the number of Euler trails $\sigma$ in $G$ such that $\sigma(a) = b$. Hence Theorem 2.3.3 follows from Theorem 2.2.3.

**Example 6** Let $G$ and $K$ be the digraphs in Figure 2.6; $K$ is isomorphic to the arc digraph $K(G)$ of $G$. The vertex set of $K$ is $V = [8]$ and the arc set is $A = \pm[8]$. Note that $Y_K$ is the subgroup of $S_A$ generated by $(+k, -k)$, $1 \leq k \leq 8$ and that $h_K(\pm k) = k$.  

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Put $U_1 = \{4, 6\}, U_2 = \{1, 7\}, U_3 = \{2, 8\}, U_4 = \{3, 5\}$ and $W_1 = \{1, 5\}, W_2 = \{2, 6\}, W_3 = \{3, 7\}, W_4 = \{4, 8\}$. The names of the vertices in $G$ are chosen in correspondence to the sets $W_1, \ldots, W_4$; $\tau_G^{-1}(15) = \{1, 5\}$ and so on.

Consider the $K$-permutation
\[
\pi = (-1, +6, -5, +4)(-4, +5, -8, +3)(-3, +8, -7, +2)(-2, +7, -6, +1).
\]

For a vector $y = (y_1, y_2, y_3) \in \{0, 1\}^3$, let $E(y)$ be the set of Euler trails $\pi \tau_J = \pi \circ \prod_{j \in J}(+j, -j)$ such that $|J \cap W_k| \equiv y_k \pmod{2}$. We obtain the following table, showing for each $y = (y_1, y_2, y_3)$ all sets $J$ such that $\pi \tau_J \in E(y)$.

<table>
<thead>
<tr>
<th>$(y_1, y_2, y_3)$</th>
<th>$J$</th>
<th>$\tau_J$</th>
<th>$E(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0, 0)$</td>
<td>${6, 2, 8}$</td>
<td>${6, 2, 4}$</td>
<td>${1, 8, 5}$</td>
</tr>
<tr>
<td>$(1, 1, 0)$</td>
<td>${1, 2, 8}$</td>
<td>${1, 2, 4}$</td>
<td>${6, 8, 5}$</td>
</tr>
<tr>
<td>$(0, 1, 1)$</td>
<td>${6, 7, 8}$</td>
<td>${1, 7, 4}$</td>
<td>${1, 3, 4}$</td>
</tr>
<tr>
<td>$(1, 0, 0)$</td>
<td>${6, 2, 3}$</td>
<td>${1, 7, 3}$</td>
<td>${6, 2, 5}$</td>
</tr>
<tr>
<td>$(0, 1, 0)$</td>
<td>${6, 7, 3}$</td>
<td>${1, 2, 5}$</td>
<td>${6, 8, 4}$</td>
</tr>
<tr>
<td>$(0, 0, 1)$</td>
<td>${6, 2, 3}$</td>
<td>${1, 7, 5}$</td>
<td>${1, 3, 5}$</td>
</tr>
<tr>
<td>$(1, 1, 1)$</td>
<td>${6, 2, 3}$</td>
<td>${1, 2, 3}$</td>
<td>${6, 3, 5}$</td>
</tr>
</tbody>
</table>

As Theorem 2.3.2 states, the number of sets in each row is equal to $|E_G| = 4$. The last column of sets corresponds to the set of Euler trails where 1 and 6 do not intersect, while the emphasized sets correspond to Euler trails where 1 and 2 intersect. Note that the underlying digraph $G$ contains four Euler trails, namely

$$(1, 6, 5, 8, 7, 3, 4), (1, 2, 3, 4, 5, 8, 7, 6),$$

$$(1, 2, 7, 6, 5, 8, 3, 4), (1, 2, 3, 8, 7, 6, 5, 4).$$

1 is followed by 6 in one trail and by 2 in the other trails. Thus we have verified Theorem 2.3.3 for $K$ when $u = 1$ and $w = 2, 6$. \qed

### 2.3.2 de Bruijn digraphs

We conclude this chapter with an application of Theorem 2.3.2. More precisely, we consider the de Bruijn sequences discussed in the Introduction of this thesis.

Let $n > 1$, $b > 1$. Define the digraph $G_{b, n}$ as follows. Let
\[ V = \{(c_1, \ldots, c_{n-1}) : c_1, \ldots, c_{n-1} \in \{0, \ldots, b - 1\} \} \]

and
\[ A = \{(c_1, \ldots, c_n) : c_1, \ldots, c_n \in \{0, \ldots, b - 1\} \} \]

Moreover, let the tail of \((c_1, \ldots, c_n)\) be \((c_1, \ldots, c_{n-1})\) and let the head be \((c_2, \ldots, c_n)\). The digraph \(G_{b,n}\) is called a de Bruijn digraph. The vertices and arcs in a de Bruijn digraph can be interpreted as the \(b\)-ary representations of integers.

The equivalence class
\[ S = (s_0, \ldots, s_{k-1}) \]

of rotations of a sequence \((s_0, \ldots, s_{k-1})\) is called a cyclic sequence. A word \((c_1, \ldots, c_n)\) is contained in \(S\) if there is an \(i\) \((0 \leq i < k)\) such that \(s_{i+j} = c_j\) for \(1 \leq j \leq n\) (the indices are taken modulo \(k\)). The Euler trails in \(G_{b,n}\) correspond to cyclic sequences \(S = (s_0, \ldots, s_{k-1})\) with the property that each \(b\)-ary word of length \(n\) is contained exactly once in \(S\). Such cyclic sequences are called de Bruijn sequences. The Euler trail corresponding to the cyclic sequence \((s_0, \ldots, s_{k-1})\) is the Euler trail \(\pi \in E_{G_{b,n}}\) with the property that
\[ \pi(s_k, s_{k+1}, \ldots, s_{k+n-1}) = (s_{k+1}, s_{k+2}, \ldots, s_{k+n}) \]

for all \(k\); the indices are computed modulo \(b^n\).

In the following we will only deal with the 2-regular case; therefore put \(G_n = G_{2,n}\). For any binary sequence \(S = (s_1, \ldots, s_k)\) \((k \geq 1)\) and binary numbers \(x_1, \ldots, x_a, y_1, \ldots, y_b\) \((a, b \geq 0)\), put
\[ (x_1, \ldots, x_a, S, y_1, \ldots, y_b) = (x_1, \ldots, x_a, s_1, \ldots, s_k, y_1, \ldots, y_b). \]

In \(G_n\) the vertex sets
\[ U_S = \{(0,S), (1,S)\} \]

and
\[ W_S = \{(S,0), (S,1)\} \]

\((S \in \{0,1\}^{n-2})\) satisfy (2.5). Moreover, it is not difficult to check that \(G_n\) is isomorphic to the arc digraph of \(G_{n-1}\).

Let \(n > 2\). An \(n\)-de Bruijn sequence is a binary de Bruijn sequence of length \(2^n\). Theorem 2.3.1 and induction imply that the number \(|E_{G_n}|\) of \(n\)-de Bruijn sequences is equal to \(2^{2^{n-1}-n}\).

**Theorem 2.3.4** Let
\[ \{p_S : S = (s_1, \ldots, s_{n-2}) \in \{0,1\}^{n-2} \} \]

be a set of binary numbers. Then the number of \(n\)-de Bruijn sequences containing exactly one of the sequences \((0,S,0,0)\) and \((0,S,1,p_S)\) for each \(S \in \{0,1\}^{n-2} \setminus \{(0,0,\ldots,0)\}\) is equal to the number \(2^{2^{n-2}-(n-1)}\) of \((n-1)\)-de Bruijn sequences.
Proof. Let $\pi$ be the $G_n$-permutation given by $\pi(0, S, 0) = (S, 0, 1)$ and $\pi(0, S, 1) = (S, 1, p_S)$ for all $S \in \{0, 1\}^{n-2}$. Let $Y_0$ be the group of permutations $\tau \in Y_{G_n}$ such that the first digit in $\tau(0, S, 0)$ is equal to the first digit in $\tau(0, S, 1)$ for all $S \in \{0, 1\}^{n-2} \setminus \{(0, 0, \ldots, 0)\}$, that is, $Y_0$ is defined as in Theorem 2.3.2. Then $\pi Y_0$ contains all Euler trails corresponding to $n$-de Bruijn sequences with the property in Theorem 2.3.4. Theorem 2.3.2 implies that

$$|E_{A_{G_n}}(\pi Y_0)| = |E_{G_{n-1}}|,$$

which is equal to the number of $(n - 1)$-de Bruijn sequences. \qed
Chapter 3

Groups associated with spanning trees and Euler trails

While the previous chapter was quite combinatorial in nature, we will use a lot of linear algebra and group theory in this chapter. In Section 3.1 we discuss the critical group of a digraph and prove some bijective results relating this group with sets of spanning trees and Euler trails. In Sections 3.2 and 3.3 we concentrate on Eulerian digraphs and generalize Theorem 1.2.2.

3.1 The critical group

This section is devoted to the critical group $\Phi_r(G)$, which was introduced in Subsection 1.3. In Subsection 3.1.1 we review some elementary properties of $\Phi_r(G)$. Recall that $X_r(G)$ is the set of trees rooted in the vertex $r$; we will always assume that $X_r(G)$ is nonempty. A bijection from $X_r(G)$ to $\Phi_r(G)$ is constructed in Subsection 3.1.2; this bijection is interpreted in the Eulerian case in Subsection 3.1.3.

3.1.1 Basic properties of the critical group

Let $G$ be a digraph with vertex set $V$ and arc set $A$. Let $r$ be a fixed vertex; put $\chi_r(G) = |X_r(G)|$ and $V_r = V \setminus r$. Everything in this Subsection is known when the Laplacian matrix is symmetric; see Norman Biggs’ article [Bi]. Sections 2, 7, and 26.

Theorem 3.1.1 If $X_r(G)$ is nonempty, then the critical group $\Phi_r(G)$ is a finite group of size $\chi_r(G)$. Moreover, if $G$ is Eulerian, then $\Phi_r(G)$ is the same group for all $r \in V$; in particular, $\chi_r(G)$ is the same for all $r \in V$.

Proof. Note that $E = \{v - r : v \in V_r\}$ is a basis for $Z_0$. Let $\gamma_v$ be the element defined in (1.1). One readily verifies that the column vector $[E|\gamma_v]$ (see Subsection 1.5 for notation) is equal to the column indexed by $v$ in $Q(v)$. Theorem 1.3.1 hence implies that $|\Phi_r(G)| = \chi_r(G)$. If $G$ is Eulerian, then the sum of the columns in $Q$ is zero, that is,
\[ \gamma_r = -\sum_{v \neq r} \gamma_v, \]

which immediately implies that \( B_0(r) \) is independent of the choice of \( r \). \( \square \)

Let \( Z_1 \) be the kernel of the homomorphism \( \partial_1 : F(A) \to F(V) \) defined by

\[ \partial_1(a) = h(a) - t(a) \]

for \( a \in A \). We say that \( Z_1 \) is the arc cycle group of \( G \). Let \( B_1(r) \) be the subgroup of \( F(A) \) generated by

\[ \{ \beta_v = \sum_{a \in H_v} a : v \neq r \}. \]

**Theorem 3.1.2** With notations as above,

\[ F(A)/(Z_1 + B_1(r)) \cong \Phi_*(G). \tag{3.1} \]

Moreover, \( \rho(Z_1) = |A| - |V| + 1 \) and \( Z_1 + B_1(r) = Z_1 \oplus B_1(r) \).

**Proof.** Note that \( \partial_1(\beta_v) = \gamma_v \), which implies that \( \partial_1(B_1(r)) = B_0(r) \).

Since \( Z_1 = \ker \partial_1 \), it is clear that

\[ F(A)/(Z_1 + B_1(r)) \cong \partial_1(F(A))/\partial_1(B_1(r)) = \partial_1(F(A))/B_0(r). \]

We claim that \( \partial_1(F(A)) = Z_0 \). Namely, \( E \subset \partial_1(F(A)) \), where \( E \) is the set in the proof of Theorem 3.1.1. This follows from the fact that for each \( v \in V_r \), there is a path \( (a_1, \ldots, a_m) \) of arcs in \( G \) with \( t(a_1) = r \) and \( h(a_m) = v \); clearly \( \partial_1(\sum_{j=1}^m a_j) = v - r \), and (3.1) follows. Since

\[ 0 \longrightarrow Z_1 \longrightarrow F(A) \xrightarrow{\partial_1} F(V) \xrightarrow{\partial_1} Z \longrightarrow 0 \]

is exact, we have \( \rho(Z_1) = |A| - |V| + 1 \). Thus since \( \rho(B_1(r)) = |V| - 1 \) and \( \rho(Z_1 + B_1(r)) = |A| \) by (3.1), we obtain that \( Z_1 + B_1(r) = Z_1 \oplus B_1(r) \). \( \square \)

**Corollary 3.1.3** Let \( Z \) be a basis for \( Z_1 \) and let \( B \) be a basis for \( B_1(r) \). Then

\[ [A][Z \cup B] \]

is a matrix with determinant \( \pm \chi_r(G) \).

A simple way of constructing a basis for \( Z_1 \) is as follows. Let \( r \) be a tree rooted in \( r \). We want to find \( |A| - |V| + 1 \) linearly independent vectors spanning \( Z_1 \). For each arc \( a \in A \setminus \tau(V_r) \), we have a digraph consisting of \( a \) and the arcs in \( \tau(V_r) \). The arc cycle group \( Z_1 \) of this digraph is a group of rank 1. We obtain a uniquely determined generator \( g_a \) of the group if we require that the coefficient of \( a \) is +1, that is,

\[ g_a = a + \sum_{v \in V_r} \lambda_{av} \tau(v), \]

for some integers \( \lambda_{av} \).

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Proposition 3.1.4 The set $\mathcal{Z}_r = \{g_a : a \in A \setminus \tau(V_r)\}$ is a basis for $Z_1$.

Proof. By construction, each $g_a$ is contained in $Z_1$. Clearly the matrix

$$[A \setminus \tau(V_r) : A|\mathcal{Z}_r]$$

is the identity matrix and $\rho(Z_1) = |A \setminus \tau(V_r)|$, which immediately implies that $\mathcal{Z}_r$ is a basis for $Z_1$. \hfill \Box

If $G$ is an Eulerian digraph, then $\Phi_r(G)$ is the same group $\Phi(G)$ for all $r$. Let $B_1$ be the group generated by all elements $\beta_v = \sum_{a \in H_v} a$. $B_1$ is a group of rank $|V|$, and since $G$ is Eulerian, $Z_1 + B_1 = Z_1 + B_1(r)$. Namely, $\sum_{v \in V} \beta_v = \sum_{a \in A} a$ is an element in $Z_1$. This implies the following result.

Corollary 3.1.5 If $G$ is an Eulerian digraph, then

$$F(A)/(Z_1 + B_1) \cong \Phi(G).$$

Moreover, $Z_1 \cap B_1$ is the free group generated by the single element $\sum_{a \in A} a$. \hfill \Box

3.1.2 The critical group and trees

Consider a digraph $G$ and let $r$ be a fixed vertex. We will show how to construct a bijective function from $X_r(G)$ to $\Phi_r(G)$. We are able to prove that the function is injective without using the Matrix-tree Theorem 1.3.1, but we need this result to prove that the function is surjective. Hence we give a solution to a problem stated in [Bi], Section 24, about finding a bijective correspondence between $\Phi_r(G)$ and $X_r(G)$, but our solution is not complete in the sense that we have to use the Matrix-tree Theorem in the proof.

Fix a total order $\leq_v$ of the arcs in $H_v$ for each $v \in V = V \setminus \{r\}$. We may extend the total orders $\leq_v$ to a partial order $\leq$ on $A$ by letting $a$ and $b$ be incomparable if $h(a) \neq h(b)$.

Theorem 3.1.6 For a tree $\tau \in X_r(G)$, put

$$\alpha(\tau) = \sum_{v \in V_r} \left( \sum_{a \geq \tau(v)} \partial_1(a) \right) \in Z_0.$$  

Then the induced function $\hat{\alpha} : X_r(G) \to \Phi_r(G)$ is a bijection.

Proof. As we mentioned above, we will show that $\hat{\alpha}$ is injective. This and the Matrix-tree Theorem 1.3.1 will imply that $\hat{\alpha}$ is bijective. Let $\tau_1, \tau_2 \in X_r(G)$. We have that

$$\alpha(\tau_1) - \alpha(\tau_2) = \sum_{v \in V_r} \left( \sum_{a \geq \tau_1(v)} \partial_1(a) - \sum_{a \geq \tau_2(v)} \partial_1(a) \right).$$

If $\tau_1(v) \leq \tau_2(v)$, then

$$\sum_{a \geq \tau_1(v)} \partial_1(a) - \sum_{a \geq \tau_2(v)} \partial_1(a) = \sum_{\tau_1(v) \leq a < \tau_2(v)} \partial_1(a),$$

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while if $\tau_2(v) < \tau_1(v)$, then
\[
\sum_{a \geq \tau_1(v)} \partial_1(a) - \sum_{a \geq \tau_2(v)} \partial_1(a) = - \sum_{\tau_2(v) \leq a < \tau_1(v)} \partial_1(a) \equiv \sum_{a < \tau_2(v)} \partial_1(a) + \sum_{a \geq \tau_1(v)} \partial_1(a)
\]
modulo $B_0(r)$. This implies that
\[
\alpha(\tau_1) - \alpha(\tau_2) \equiv \sum_{u \in U} \sum_{a \in J_u} \partial_1(a) \pmod{B_0(r)},
\]
where $U$ is the set of vertices $u$ in $V_r$ such that $\tau_1(u) \neq \tau_2(u)$ and $J_u \subset H_u$ is a set containing $\tau_1(u)$ but not $\tau_2(u)$ for all $u \in U$. Put
\[
\psi_u = \sum_{a \in J_u} \partial_1(a) \in \mathbb{Z}_0.
\]
Let $\gamma_v$ be defined as in (1.1). Note that $\bar{\alpha}(\tau_1) - \bar{\alpha}(\tau_2) = 0$ if and only if there are integers $\lambda_v$ such that
\[
(3.2) \quad \sum_{v \in V_r} \lambda_v \gamma_v = \sum_{u \in U} \psi_u.
\]
We want to show that this implies that $\tau_1 = \tau_2$. Let $V^+$ be the family of vertices $v$ such that $\lambda_v \geq 1$ and let $V^-$ be the family of vertices $v$ such that either $\lambda_v \leq -1$ or $\lambda_v = 0$ and $v \in U$. To prove that $\tau_1 = \tau_2$, it suffices to show that $V^+ = V^- = \emptyset$. Namely, then $U = \emptyset$, which implies that $\tau_1(v) = \tau_2(v)$ for all $v$.

We can rewrite (3.2) as
\[
\sum_{u \in V^+} (\lambda_u \gamma_u - \psi_u) = \sum_{u \in V^-} (-\lambda_u \gamma_u + \psi_u).
\]
Let $x = \sum_{v \in V^+} x_v v$ denote the element in the left-hand side of this equality. Note that $\lambda_u \gamma_u - \psi_u$ is equal to $\partial_1(y)$, where $y \in F(A)$ is a nonnegative sum of arcs with head $u$. This follows from the facts that $\lambda_u \geq 1$, $\gamma_u$ is the sum of all arcs with head $u$, and $\psi_u$ is the sum of some arcs with head $u$. In particular, $x_u \leq 0$ if $v \notin V^+$. Since $x$ is also equal to the right-hand side, $x_v \leq 0$ if $v \notin V^-$. Now $V^+$ and $V^-$ are disjoint, which implies that all coefficients are at most zero. Since the coordinate sum of $x$ is zero, $x$ itself must hence be zero.

Suppose that $V^+ \neq \emptyset$. For each $v \in V^+$, the element $\lambda_v \gamma_v - \psi_v$ is an element with a strictly positive coefficient of $v$ and with all other coefficients less than or equal to zero. Namely, $\gamma_v - \psi_v$ is the image under $\partial_1$ of the sum of all arcs in $H_v \setminus J_v$, which is nonempty since it contains $\tau_2(v)$. In particular, if $u \notin V^+$, then for each $v \in V^+$ the coefficient of $u$ in $\lambda_v \gamma_v - \psi_v$ is zero; otherwise we would have $x_u < 0$, which is a contradiction. However, there must be some $v \in V^+$ such that the tail $u$ of $\tau_2(v)$ is not in $V^+$, because otherwise $\tau_2(V^+)$ would contain a directed cycle. This means that the coefficient of $u$ in $\lambda_v \gamma_v - \psi_v$ must be strictly negative, and we have obtained a contradiction; hence $V^+ = \emptyset$.

Next suppose that $V^- \neq \emptyset$. Again, for each $v \in V^-$ the element $-\lambda_v \gamma_v + \psi_v$ is a column vector with a strictly positive coefficient of $v$, because either $-\lambda_v \geq 1$ or $\psi_v \neq 0$. Since $\tau_1(V^-)$ does not contain any directed cycle, there must be a $v \notin V^-$ such that the tail $u$ of $\tau_1(v)$ is not in $V^-$. However, this means that the coefficient of $u$ in $-\lambda_v \gamma_v + \psi_v$ is strictly negative, and again a contradiction is obtained. \qed

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Remark 3 Theorem 3.1.6 implies that we may define a group operation on \(X_r(G)\) by putting

\[
\tau_1 + \tau_2 = \tilde{\alpha}^{-1}(\tilde{\alpha}(\tau_1) + \tilde{\alpha}(\tau_2)).
\]

An obvious problem is to find a way of adding trees without using \(\alpha\).

3.1.3 The critical group and Euler trails

Let \(G\) be an Eulerian digraph. Theorem 3.1.6 gives a bijection between the set \(X_r(G)\) of trees rooted in \(r\) and the critical group \(\Phi(G) = \Phi_r(G)\), but it can also be used to construct a bijection between a partition of the set of Euler trails and \(\Phi(G)\). The procedure is as follows. Let \(r\) be a fixed root vertex and fix an arc \(a\) ending in \(r\). For an Euler trail \(\pi\) and a vertex \(v \in V_r = V \setminus \{r\}\), put

\[
\tau_\pi(v) = \pi^{-1} \cup [a](a),
\]

that is, \(\tau_\pi(v)\) is the first arc with head \(v\) after \(a\) in \(\pi\). \(\tau_\pi\) is easily seen to define a tree with root \(r\). Following the proof in [AB] (see [HP]) of Theorem 1.3.2, one readily verifies the following result.

**Theorem 3.1.7** For each tree \(\tau\) rooted in \(r\) and each set \(\{\sigma_v : v \in V\}\) of cyclic permutations \(\sigma_v : H_v \to H_v\), there is exactly one Euler trail \(\pi\) such that \(\tau = \tau_\pi\) and \(\pi^{-1} \cup [a](a)\) for all \(v \in V\).

Let \(B = \{b_v : v \in V_v\}\) be a subset of \(h(V_r)\) such that \(b_v \in H_v\) for each \(v \in V_r\). For \(a \in H_r\) and \(b \in H_r\setminus\{b_v\}\), put

\[
z_\pi(a, b) = \begin{cases} 1 & \text{if } \pi^{-1}(a, b) = (a, b, b_v); \\ 0 & \text{if } \pi^{-1}(a, b) = (a, b_v, b). \end{cases}
\]

Theorem 3.1.6 and Theorem 3.1.7 imply the following result.

**Theorem 3.1.8** With notations as above and with \(a \in H_r\), for each element \(x \in \Phi(G)\) and each family \(\{\sigma_v : v \in V\}\) of cyclic permutations \(\sigma_v : H_v \to H_v\), there is exactly one Euler trail \(\pi\) such that \(\pi^{-1} \cup [a](a)\) for all \(v\) and

\[
\sum_{b \in h(V_r) \setminus B} z_\pi(a, b)\delta_1(b) = x,
\]

where \(\delta_1 : F(A) \to \Phi(G)\) is the homomorphism induced by \(\partial_1\).

**Proof.** Consider a fixed set \(P = \{\sigma_v : v \in V\}\) of cyclic permutations \(\sigma_v : H_v \to H_v\). Note that \(\sigma_v\) induces an order of the elements in \(H_v\):

\[
\sigma_v(b_v) < \sigma_v^2(b_v) < \ldots < \sigma_v^{H_v^{-1}-1}(b_v) < b_v,
\]

where \(b_v\) is the element in \(B \cap H_v\). Let \(E_P\) be the set of Euler trails \(\pi\) with \(\pi^{-1} \cup [a](a)\) for all \(v \in V\). With the given orders on the sets \(H_v\) and with \(\alpha\) as in Theorem 3.1.6, one readily verifies that

\[
\alpha(\pi) = \sum_{b \in h(V_r) \setminus B} z_\pi(a, b)\partial_1(b) + \sum_{v \in V_r} \partial_1(b_v).
\]
Since the second sum is constant for all \( \pi \in E_P \),
\[
\pi \mapsto \sum_{b \in A^+} z_\pi(a, b) \delta_1(b)
\]
is a bijection \( E_P \to \Phi(G) \) by Theorems 3.1.6 and 3.1.7. \( \square \)

**Remark 4** Theorem 3.1.8 is more natural than Theorem 3.1.6 in the sense that things only depend on the choice of the set \( B \) (and, of course, the choice of the root \( r \) and the arc \( a \in H_r \)); in the construction of \( \alpha \) in Theorem 3.1.6, we were forced to order all elements in each \( H_v \).

### 3.1.4 The critical group and arc digraphs

Recall the definition of the arc digraph \( K(G) \) of a digraph \( G \) from Section 2.3.

We will compute the critical group of the arc digraph of a \( d \)-regular Eulerian digraph. In particular, we will be able to determine the critical groups of the de Bruijn digraphs considered in Subsection 2.3.2.

**Theorem 3.1.9** Let \( d \geq 2 \) and let \( G \) be a \( d \)-regular Eulerian digraph with \( n \) vertices. Suppose that
\[
\Phi(G) \cong \mathbb{Z}_{r_1} \oplus \mathbb{Z}_{r_2} \oplus \cdots \oplus \mathbb{Z}_{r_{n-1}}
\]
(some \( r_i \) might be equal to 1). Then
\[
\Phi(K(G)) \cong \mathbb{Z}_d^{n(d-2)} \oplus \mathbb{Z}_{d r_1} \oplus \mathbb{Z}_{d r_2} \oplus \cdots \oplus \mathbb{Z}_{d r_{n-1}}.
\]
In particular, \( \Phi(G) \cong \partial \Phi(K(G)) = \{ dx : x \in \Phi(K(G)) \} \).

**Remark 5** The Smith normal form of a matrix \( R \) with integer coefficients is the diagonal matrix \( \text{diag}(g_1, g_2, g_1, \ldots, g_n/g_{n-1}) \), where \( g_i \) is the greatest common divisor of the determinants of the \( i \times i \) submatrices of \( R \) \((0/0 = 0)\). Theorem 3.1.9 states that if the Smith normal form of the Laplacian of \( G \) is
\[
\text{diag}(r_1, r_2, \ldots, r_{n-1}, 0),
\]
then the Smith normal form of the Laplacian of \( K(G) \) is
\[
\text{diag}\left((1)^n, (d)^{(d-2)n}, dr_1, dr_2, \ldots, dr_{n-1}, 0\right),
\]
where \((k)^m\) is the sequence \((k, \ldots, k)\) with \( m \) identical elements. See [Mc] for more information.

**Remark 6** It would be nice with a bijective proof of the last statement in Theorem 3.1.9.

**Proof.** First of all, note that Hall’s Theorem [PHa] implies that there is a set \( A_d \) of arcs in \( G \) such that \( |A_d| = n \) and such that each vertex in \( G \) is the head of exactly one arc and the tail of exactly one arc in \( A_d \). Namely, given any set \( U \) of vertices, the set \( W = h(t^{-1}(U)) \) of vertices that can be reached from
$U$ via an arc satisfies $|W| \geq |U|$; the number of arcs in $t^{-1}(U)$ is $d|U|$, which implies that the number of vertices in $W$ is at least $d|U|/d = |U|$. Let $A_d$ be as above and let $A_1, \ldots, A_{d-1}$ be subsets of $A$ such that $A$ is the disjoint union of $A_1, \ldots, A_d$ and such that each $A_i$ contains exactly one edge from each head set $H_i$. The Laplacian of $K(G)$ is an $A \times A$ matrix of the form

$$
\begin{pmatrix}
    -P_1 & d \cdot I & \cdots & d \cdot I - P_{d-1} & -P_d \\
    d \cdot I - P_2 & -P_1 & \cdots & -P_{d-1} & -P_d \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    d \cdot I - P_d & d \cdot I - P_{d-1} & \cdots & -P_{d-1} & -P_d \\
    -P_d & -P_{d-1} & \cdots & d \cdot I & 0
\end{pmatrix},
$$

where the blocks are $A_i \times A_j$ matrices; the $j$th blocks from the left contain the columns indexed by $A_j$. With $A_d$ as above, we obtain that $P_d$ is a permutation matrix and hence invertible over $\mathbb{Z}$.

Let us say that two matrices are *Smith equivalent* if they have the same Smith normal form. One readily verifies that two matrices $K_1$ and $K_2$ are Smith equivalent if $K_1$ can be obtained from $K_2$ via elementary row and column operations over $\mathbb{Z}$.

Subtracting the first row block from the other row blocks and adding all column blocks to the first column block, we obtain the matrix

$$
\begin{pmatrix}
    Q & -P_2 & \cdots & -P_{d-1} & -P_d \\
    0 & d \cdot I & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & d \cdot I & 0 \\
    0 & 0 & \cdots & 0 & d \cdot I
\end{pmatrix},
$$

where $Q$ is the Laplacian of $G$; this matrix is obviously Smith equivalent to the Laplacian of $K(G)$. Since $P_d$ is invertible, the matrix is Smith equivalent also to

$$
\begin{pmatrix}
    Q & -P_2 & \cdots & -P_{d-1} & -P_d \\
    0 & d \cdot I & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & d \cdot I & 0 \\
    d \cdot P_d^{-1}Q & -d \cdot P_d^{-1}P_2 & \cdots & -d \cdot P_d^{-1}P_{d-1} & 0
\end{pmatrix},
$$

which in turn is Smith equivalent to

$$
\begin{pmatrix}
    0 & 0 & \cdots & 0 & -P_d \\
    0 & d \cdot I & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & d \cdot I & 0 \\
    d \cdot P_d^{-1}Q & 0 & \cdots & 0 & 0
\end{pmatrix},
$$

Since $d \cdot P_d^{-1}Q$ and $P_d$ are Smith equivalent to $d \cdot Q$ and $I$, respectively, the proof is finished.

**Corollary 3.1.10** With notations as in Subsection 2.3.2, the critical group of the de Bruijn digraph $G_{d,n}$ is

$$
(3.3) \quad \left( \bigoplus_{k=1}^{n-2} \mathbb{Z}^{(d-1)^3d^{n-k-1}} \right) \oplus \mathbb{Z}^{d^{n-2}}.
$$
**Proof.** Recall that the number of vertices in $G_{d,n}$ is $d^{n-1}$. The corollary is certainly true for $n = 1$. Assume by induction that (3.3) holds for some $n$, that is,

$$
\Phi(G_{d,n}) \cong \mathbb{Z}^{d^n}_{d^{-1}} \oplus \left( \bigoplus_{k=1}^{n-2} \mathbb{Z}^{(d-1)^2d^{n-k-2}}_{d^{k+1}} \right) \oplus \mathbb{Z}^{d^{-2}}_{d^{k-1}}
$$

(now the sum of the exponents is equal to the number of vertices minus 1). Since $G_{d,n+1}$ is the arc digraph of $G_{d,n}$, Theorem 3.1.9 implies that

$$\Phi(G_{d,n+1}) \cong \mathbb{Z}^{d^n}_{d^{-1}(d-2)} \oplus \mathbb{Z}^{d^n}_{d^{-1}} \oplus \left( \bigoplus_{k=1}^{n-2} \mathbb{Z}^{(d-1)^2d^{n-k-2}}_{d^{k+1}} \right) \oplus \mathbb{Z}^{d^{-2}}_{d^{k-1}},$$

and we are done.

\[ \square \]

### 3.2 Isotropic systems associated with Eulerian digraphs

In Subsection 1.3.2 we described the isotropic system $\langle M, C \rangle$ associated with a 2-regular Eulerian digraph. We want to consider the general case where the Eulerian digraph is not necessarily 2-regular. In Subsection 3.2.1 we show how to define the free group $M$, while Subsection 3.2.2 is devoted to the definition of the subgroup $C$. The main result is Theorem 3.2.2, which generalizes the $\mathbb{Z}$-version of Theorem 1.3.3.

#### 3.2.1 The free group $M$

Let $G$ be an Eulerian digraph with vertex set $V$ and arc set $A$. We will assume that $|H_v| \geq 2$ for all $v \in V$. For each $v \in V$, let $N_v \subseteq F(H_v) \oplus F(T_v)$ be the group generated by the elements $(a, b)$, $a \in H_v, b \in T_v$. Note that $N_v$ is the kernel of the linear functional $F(H_v) \oplus F(T_v) \rightarrow \mathbb{Z}$ given by $(a, 0) \rightarrow 1$, $(0, b) \rightarrow -1$; hence $\rho(N_v) = 2|H_v| - 1$. Since $(a, b)$ also denotes a permutation in $S_A$, we will write $ab$ instead of $(a, b)$ to avoid confusion. The element $ab$ is a pair of half-arcs (compare to [Bo], where the term half-edge is used).

Consider the subgroup $D_v = F(\Delta_v)$ of $N_v$, where

$$\Delta_v = \left( \sum_{a \in H_v} a, \sum_{b \in T_v} b \right) = \sum_{a \in H_v} a \sigma(a);$$

$\sigma : H_v \rightarrow T_v$ is any bijective function. For each $v \in V$, let $\langle \cdot, \cdot \rangle_v : N_v \times N_v \rightarrow \mathbb{Z}$ be the symmetric bilinear form defined by

$$\langle ab, cd \rangle_v = \begin{cases} 
1 & \text{if } a = c \text{ and } b \neq d; \\
-1 & \text{if } a \neq c \text{ and } b = d; \\
0 & \text{otherwise}
\end{cases}$$

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for each \(a, b \in H_v\) and \(c, d \in T_v\). Consider the element
\[
x = \left( \sum_a \lambda_a a, \sum_b \mu_b b \right).
\]
Note that \(x \in D_v\) if and only if there is an \(s \in \mathbb{Z}\) such that \(s = \lambda_a = \mu_b\) for all \(a \in H_v, b \in T_v\). Since
\[
\langle ab, x \rangle_v = \lambda_a - \mu_b,
\]
we obtain that \(x \in D_v\) if and only if \(\langle ab, x \rangle_v = 0\) for all \(a \in H_v, b \in T_v\). Put
\[
N = \bigoplus_{v \in V} N_v,
\]
\[
D = \bigoplus_{v \in V} D_v,
\]
and
\[
M = N/D \cong \bigoplus_{v \in V} N_v/D_v.
\]
We have that
\[
\rho(M) = \rho(N) - \rho(D) = \sum_v (2|H_v| - 1) - |V| = 2(|A| - |V|).
\]
Note that \(\langle \cdot, \cdot \rangle_v\) induces a map \(\langle \cdot, \cdot \rangle : N_v/D_v \times N_v/D_v \to \mathbb{Z}\). Let \(\langle \cdot, \cdot \rangle : M \times M \to \mathbb{Z}\) be defined as
\[
\langle \sum_{v \in V} x_v, \sum_{v \in V} y_v \rangle = \sum_{v \in V} \langle x_v, y_v \rangle_v
\]
for \(x_v, y_v \in N_v/D_v\).

For each \(v \in V\), choose an arbitrary arc \(\bar{a}_v \in H_v\) and form the set
\[
A^- = \{\bar{a}_v : v \in V\}.
\]
This set will be fixed for the rest of this chapter. For \(a \in H_v\), put \(\bar{a} = \bar{a}_v; a \mapsto \bar{a}\) is a function from \(A\) to \(A^-\). Furthermore, put
\[
A^+ = A \setminus A^-
\]
and
\[
H^+_v = H_v \setminus \{\bar{a}_v\}.
\]
For each \(G\)-permutation \(\pi\), put
\[
M^+_v = \{a \pi(a) : a \in A^+\},
\]
\[
M^-_v = \{a \pi(\bar{a}) : a \in A^+\}.
\]
and

\[ M_\pi = M_\pi^+ \cup M_\pi^- . \]

Put \( M_\pi^+ = (M_\pi^+) \) and \( M_\pi^- = (M_\pi^-) \).

**Lemma 3.2.1** For any \( G \)-permutation \( \sigma \) and \( x \in M \),

\[ x = \sum_{\alpha \in A^+} (x, a\sigma(a))a\sigma(\hat{a}) - \sum_{\alpha \in A^+} (x, \hat{a}\sigma(a))a\sigma(a). \]

In particular, \( M_\sigma \) is a basis for \( M \).

**Remark 7** Note that the coefficients of \( a\sigma(a) \) and \( a\sigma(\hat{a}) \) in (3.5) only depend on \( a, \hat{a}, \) and \( \sigma(a) \) and not on \( \sigma(\hat{a}) \).

**Proof.** Since \( |M_\sigma| = \rho(M) \), we need only prove that \( M_\sigma \) spans \( M \). Consider \( \sigma(d) \), where \( h(c) = h(d) = v \). Straightforward computations yield that the sum in (3.5) equals

\[
\begin{cases}
\sigma(\hat{c}) & \text{if } c \neq \hat{a}_v, d = \hat{a}_v, \\
\sigma(\hat{a}_v) + d\sigma(d) - d\sigma(\hat{a}_v) & \text{if } c, d \neq \hat{a}_v, \\
- \sum_{\alpha \in H_+^+ \setminus \{d\}} a\sigma(a) - d\sigma(\hat{a}_v) & \text{if } c = \hat{a}_v, d \neq \hat{a}_v, \\
- \sum_{\alpha \in H_+^+} a\sigma(a) & \text{if } c = d = \hat{a}_v.
\end{cases}
\]

One readily verifies that (3.5) holds in all four cases. \( \square \)

### 3.2.2 The half-arc cycle group \( C \)

Next we want to define a group consisting of cycles in \( G \). Let \( \delta : N \to F(A) \) be the homomorphism defined by \( \delta(ab) = b - a \) for \( a \in H_v, b \in T_v \). With an arbitrary \( G \)-permutation \( \sigma \) and with \( \Delta_v \) defined as in (3.4), note that

\[
\sum_v \lambda_v \delta(\Delta_v) = \sum_v \sum_{\alpha \in H_v} \lambda_v \delta(a\sigma(a)) = \sum_{\alpha \in A} (\lambda_{\alpha(a)} - \lambda_{\hat{a}(a)})a.
\]

This sum is 0 if and only if \( \lambda_{\alpha(a)} = \lambda_{\hat{a}(a)} \) for all arcs \( a \). Since \( G \) is an Eulerian digraph, this implies that \( \lambda_u = \lambda_v \) for all \( u, v \in V \). In particular, \( \rho(\delta(D)) = |V| - 1 \).

\( \delta \) induces a map \( \tilde{\delta} : M \to F(A)/\delta(D) \). Put \( Z = \ker \delta \) and

\[
C = \ker \tilde{\delta} = (Z + D)/D \subseteq M.
\]

We will say that \( C \) is the half-arc cycle group of \( G \), \( C \) is in the upper right corner in Figure 3.1, which is a commutative diagram where all rows and columns are exact.
Figure 3.1: A diagram illustrating the construction of $C$

We want to determine the rank of $C$ and certain subgroups of $C$. Let $\sigma \in S_A$. Define $\sigma^* : A \rightarrow M$ as follows. For $a \in A$, let $(c_1, \ldots, c_m)$ be the cycle in $\sigma$ containing $a$. Put

$$\sigma^*(a) = \sum_{j=0}^{m-1} c_j c_{j+1};$$

c_0 = c_m$. Note that $\hat{\delta}(\sigma^*(a)) = 0$; thus $\sigma^* : A \rightarrow C$. For any $G$-permutation $\pi$ and $a \in A$, put $\pi_a = \pi \circ (a, \hat{a})$; this means that $\pi^*_a(a) = (\pi \circ (a, \hat{a}))^*(a)$. The following result is a generalized $\mathbb{Z}$-version of Theorem 1.3.3.

**Theorem 3.2.2** If $\sigma$ is a $G$-permutation with $p+1$ cycles, then the rank of $C \cap M^+_\sigma$ is $p$. Moreover, if $\pi$ is an Euler trail in $G$, then

$$C_\pi = \{\pi^*_a(a) : a \in A^+\}.$$

is a basis for $C$. In particular, $\rho(C) = |A| - |V| = \frac{1}{2}\rho(M)$.

**Proof.** Let $\tau_1, \ldots, \tau_{p+1}$ be the $p+1$ different cycles in $\sigma$ and let $c_j \in \tau_j$, $1 \leq j \leq p+1$, be arbitrarily chosen arcs. We claim that $C \cap M^+_\sigma$ is generated by

$$\{\sigma^*(c_j) : j = 1, \ldots, p+1\}.$$

The claim is proved as follows. Consider an element $x \in C \cap M^+_\sigma$; let $y \in Z \subset N$ be such that $x = y + D$. Since $x \in M^+_\sigma$ and since any element in $D$ can be written as a linear combination of the elements in $\{a\sigma(a) : a \in A\}$ (consider (3.4)), we can write

$$y = \sum_{a \in A} \lambda_a a\sigma(a)$$

for some $\lambda_a \in \mathbb{Z}$. Since

$$0 = \delta(y) = \sum_{a \in A} \lambda_a (\sigma(a) - a) = \sum_{a \in A} (\lambda_{\sigma^{-1}(a)} - \lambda_a)a,$$

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\( \lambda_a = \lambda_{\sigma^{-1}(a)} \) for all \( a \in A \). Hence
\[
x = \sum_{j=1}^{p+1} \lambda_{c_j} \sigma^*(c_j),
\]
and the claim is proved.

Since \( \sigma^*(c_{p+1}) = -\sum_{j=1}^{p} \sigma^*(c_j) \), it follows that
\[
C \cap M^+_\sigma = \{ \sigma^*(c_j) : j = 1, \ldots, p \}.
\]
To prove the first claim in the Theorem, it therefore suffices to prove that
\( \{ \sigma^*(c_j) : j = 1, \ldots, p \} \) is an independent set in \( (C \cap M^+_\sigma) \). For this purpose, consider a linear combination
\[
x = \sum_{j=1}^{p+1} \lambda_j \sigma^*(c_j) = 0.
\]
We have to show that \( \lambda_1 = \ldots = \lambda_{p+1} \); there is nothing to prove if \( p = 0 \). Use induction over \( p \). Since there are Euler trails in \( G \), there are arcs \( a, c \) with the same head such that \( a \) and \( c \) belong to different cycles; say that \( a \in \tau_p \) and \( c \in \tau_p \). Note that \( 0 = \langle x, a \sigma(c) \rangle = \lambda_p - \lambda_{p+1} \). Simple calculations show that
\[
\sigma^*(a) + \sigma^*(c) = (\sigma \circ (a, c))^*(a) - a \sigma(c) - a \sigma(a) + a \sigma(c) = (\sigma \circ (a, c))^*(a)
\]
in \( M \). Thus since \( \lambda_p = \lambda_{p+1} \), we have that
\[
x = \sum_{j=1}^{p} \lambda_j (\sigma \circ (a, c))^*(c_j).
\]
By induction, it follows that \( \lambda_1 = \ldots = \lambda_p \), and we are finished.

Now consider an Euler trail \( \pi \). We want to show that the elements in \( C_\pi \) are independent. Note that
\[
(3.8) \quad \pi^*_\pi(a) = a \pi(\hat{a}) + \sum_{k=1}^{m-1} \pi^k(\hat{a}) \pi^{k+1}(\hat{a}) = a \pi(\hat{a}) + y,
\]
where \( m \) is the length of the cycle in \( \pi \circ (a, \hat{a}) \) containing \( a \) and \( y \in M^+_\pi \) (when \( \pi^k(\hat{a}) \in A^- \), apply (3.5) to show that \( \pi^k(\hat{a}) \pi^{k+1}(\hat{a}) \) is an element in \( M^+_\pi \)). This implies that \( \pi^*_\pi(a) \equiv a \pi(\hat{a}) \) (mod \( M^+_\pi \)). In particular, \( M = (C_\pi) \oplus (M^+_\pi) \), and since \( M^-_\pi \) is an independent set in \( M \), \( C_\pi \) is an independent set. Hence to prove that \( C = (C_\pi) \) we only have to show that \( M^+_\pi \cap C = 0 \). Since this is a consequence of the first statement in the Theorem, we are finished. \( \square \)

**Example 7** Let \( G \) be a digraph with vertex set \( \{1, 2\} \), three arcs \( a, b, c \) from \( 2 \) to \( 1 \), and three arcs \( x, y, z \) from \( 1 \) to \( 2 \), that is, \( H_1 = \{a, b, c\} \) and \( H_2 = \{x, y, z\} \). Put \( \hat{a} = c \) and \( \hat{a} = z \). Let \( \pi = (a, x, b, y, c, z) \) and \( \sigma = (a, y)(b, z, c, x) \). Then
\[
M_\pi = M^+_\pi \cup M^-_\pi = \{ax, by, xb, yc\} \cup \{az, bz, xa, ya\}.
\]
\[
C_\pi = \{az + za, bz + z + xa + ax + xb, xa + ax, ya + ax + xb + by\}.
\]
\[
M_\sigma = \{ay, bx, zb, ya\} \cup \{ax, bx, xc, yc\}.
\]
However, \( C_\sigma \) cannot be defined since \( \sigma \) is not an Euler trail.
3.3 Matrix representations of the half-arc cycle group

Our next purpose is to examine the set $C_\pi$ for an Euler trail $\pi$. In particular, we are going to study the matrices $[M_\sigma : C_\pi]$ for different $G$-permutations $\sigma$ (see Section 1.5 for matrix notations).

The order of the rows and columns in a matrix will be of great importance, especially since we are going to compute determinants; the signs of the determinants will be of considerable interest. Choose any order $<$ of the vertices in $V$ and let the order $<$ on the arc set $A$ be such that $a < c$ if $a \in H^+_c$, $c \in H^+_a$, and $v < w$ and such that $a < c$ if $a \in A^+$ and $c \in A^-$. Moreover, if $v < w$, then let $\hat a_v < \hat a_w$. For any $G$-permutation $\pi$, we want to define an order, also denoted by $<$, of the elements in $M_\pi$. For $a, c \in A^+$ such that $a < c$, let

$$a\pi(a) < c\pi(c) < a\pi(\hat a) < c\pi(\hat c).$$

Moreover, if $\pi$ is an Euler trail, then define the order of the elements in $C_\pi$ by

$$\pi^+_a(a) < \pi^-_c(c) \iff a\pi(\hat a) < c\pi(\hat c) \iff a < c$$

In Example 7 in Subsection 3.2.2 the order of the arcs is

$$a < b < x < y < c < z;$$

c and $z$ belong to $A^-.$

3.3.1 Main results

The following result is a mere reformulation of Theorem 3.2.2.

**Theorem 3.3.1** Let $\pi$ be an Euler trail and let $\sigma$ be a $G$-permutation. Then the nullity of $[M^-_\sigma : M_\sigma[C_\pi]]$ is equal to the number of cycles minus one in $\sigma$.

**Proof.** Let $p + 1$ be the number of cycles in $\sigma$ and define $\omega : C \to M^{-}_\sigma$ as $\omega(x) = x^-$, where $x = x^+ + x^-$; $x^+ \in M^+_\sigma$ and $x^- \in M^-_\sigma$. We want to show that $\rho(\omega(C)) = \rho(C) - p$; namely,

$$\text{null } [M^-_\sigma : M_\sigma[C_\pi]] = \rho(C) - \rho(\omega(C)).$$

Since $\rho(\omega(C)) = \rho(C) - \rho(\ker \omega)$ and $\ker \omega = C \cap M^+_\sigma$, it suffices to prove that $\rho(C \cap M^+_\sigma) = p$. However, this is the first statement in Theorem 3.2.2. \qed

Our next goal is to obtain a new determinant formula for the number $\chi_r(G)$ of trees with a fixed root $r$ in $G$. We have already noticed that $\chi_r(G)$ is independent of the root $r$ when $G$ is Eulerian. Therefore we will simply write $\chi(G)$ instead of $\chi_r(G)$.

Let $\epsilon : N \to F(A)$ be the homomorphism given by

$$\epsilon(ab) = a$$
for each half-arc $ab$. Note that $\epsilon$ induces a homomorphism $\hat{\epsilon} : M \to F(A)/\epsilon(D)$ and that $F(A)/\epsilon(D) \cong F(A^+)$. Identifying these groups in the obvious manner, we obtain that

$$\hat{\epsilon}(M^-_\pi) = \hat{\epsilon}(M^+_\pi) = A^+.$$ 

It is clear that

(3.9) \hspace{1cm} [a : A^+ \mid \hat{\epsilon}(C_\pi)] = [a\pi(a) : M_\pi | C_\pi] + [a\pi(\hat{a}) : M_\pi | C_\pi].

The matrix $[A^+ \mid \hat{\epsilon}(C_\pi)]$ can be defined directly in $F(A)/\epsilon(D)$ without involving $M$. Namely, $\hat{\epsilon}(\pi_\alpha(a))$ is just the sum of the arcs in the $\pi_\alpha$-cycle containing $a$.

**Theorem 3.3.2** Let $\pi$ be an Euler trail in $G$. Then

$$\det[A^+ \mid \hat{\epsilon}(C_\pi)] = \det Q_\pi = \chi(G)$$

where $Q$ is the Laplacian of $G$ and $r \in V$.

The proof of Theorem 3.3.2 is given in Subsection 3.3.2.

**Theorem 3.3.3** Let $\pi, \sigma$ be two Euler trails in $G$. Then

$$[M^-_\pi : M_\sigma | C_\pi] = [C_\pi | C_\sigma]$$

and

$$\det[M^-_\pi : M_\sigma | C_\pi] = \det[C_\pi | C_\sigma] = 1.$$ 

In particular,

$$[M^-_\sigma : M_\pi | C_\sigma] = [M^-_\pi : M_\pi | C_\sigma]^{-1}$$

Theorem 3.3.3 is proved in Subsection 3.3.3. As an immediate consequence of Theorems 3.3.1 and 3.3.3 we obtain the following result, which is a generalization of the main result in [La] (see Subsection 3.3.4 for details).

**Corollary 3.3.4** Let $\pi$ be an Euler trail and let $\sigma$ be a $G$-permutation. Then

$$\det[M^-_\pi : M_\sigma | C_\pi] = \begin{cases} 1 & \text{if } \sigma \text{ is an Euler trail} \\ 0 & \text{otherwise.} \end{cases}$$

\hfill \Box

### 3.3.2 Proof of Theorem 3.3.2

First we want to prove that

$$(F(A)/\epsilon(D))/\hat{\epsilon}(C) \cong \Phi(G).$$

Namely, then Corollary 3.1.5 will imply that $\det [A^+ \mid \hat{\epsilon}(C_\pi)] = \pm \det Q_\pi$. Since $\epsilon(D) = B_1$, it suffices to prove that $\epsilon(Z) = Z_1$ ($B_1$ and $Z_1$ are defined in Subsection 3.1.1). Clearly $\epsilon(Z) \subseteq Z_1$; it remains to show that $Z_1 \subseteq \epsilon(Z)$. The
claim will follow from Proposition 3.1.4 if for a given tree $t$ and for each arc $a$ not in the tree we can find an element $x \in Z$ such that $\epsilon(x) = g_a$. The element $g_a$ is either of the form $\sum_{i=1}^{s} a_i$, where $a_s = a$ and $(a_1, \ldots, a_s)$ is a directed cycle or of the form $\sum_{i=1}^{s} a_i - \sum_{i=1}^{t} b_i$, where $a_s = a$ and $(a_1, \ldots, a_s)$ and $(b_1, \ldots, b_t)$ are directed paths with $t(a_1) = t(b_1)$ and $h(a_s) = h(b_t)$. In the first case
\[
\epsilon(a_1 a_2 + \cdots + a_{s-1} a_s + a_s a_1) = g_a.
\]
In the second case
\[
\epsilon(c a_1 + a_1 a_2 + \cdots + a_{s-1} a_s + a_s d - c b_1 - b_1 b_2 - \cdots - b_1 b_t - b_t d) = g_a,
\]
where $c$ is an arc with $h(c) = t(a_1) = t(b_1)$ and $d$ is an arc with $t(d) = h(a_s) = h(b_t)$. Thus $\epsilon(Z) = Z_1$.

It remains to prove that the sign of the determinant of
\[
K = [A^+ | \delta(C_n)]
\]
is positive, which will follow if we prove that all real eigenvalues of $K$ are positive. We claim that $K$ is equal to the sum of two matrices $P$ and $R$ with the following properties: The matrix $P$ is skew-symmetric with all $H^+_v \times H^+_v$ submatrices equal to zero, while $R$ is a matrix with all $H^+_v \times H^+_w$ submatrices $(v \neq w)$ equal to zero and with $H^+_v \times H^+_v$ submatrices of the form
\[
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]
Here we assume (without loss of generality) that our order of the elements in $H^+_v$ coincides with the order in which the elements occur in $\pi$, that is,
\[
\pi^{H^+_v} = (c_1, c_2, \ldots, c_s),
\]
with $c_1 < c_2 < \cdots < c_s = \hat{a}_v$. To prove the claim, consider two arcs $a, b \in A^+$ with different heads. One easily checks that $[a : A^+ | \pi^*_a(b)] = -[b : A^+ | \pi^*_a(a)]$ for all possible cases of $\pi^{(a,b,a,b)}$. For example, if
\[
\pi^{(a,b,a,b)} = (a, b, \hat{a}, \hat{b}),
\]
then (3.9) and Lemma 3.2.1 imply that
\[
[a : A^+ | \delta(\pi^*_a(b))] = -[b : A^+ | \delta(\pi^*_a(a))] = 1.
\]
Namely, $\pi^*_a(b)$ is a sum containing the half-arc $a \pi(a)$ but not the half-arc $\hat{a} \pi(\hat{a})$, while $\pi^*_a(a)$ is a sum containing $b \pi(\hat{b})$ but not $b \pi(b)$.

Next consider two arcs $a, b \in A^+$ with the same head; assume that $a < b$. We have already made the assumption that $\pi^{(a,b,a,b)} = (a, b, \hat{a}, \hat{b})$, which implies that $\pi^*_a(a)$ does not contain $b \pi(b)$, while $\pi^*_a(b)$ contains $a \pi(a)$; hence $[a : A^+ | \pi^*_a(b)] = 1$ and $[b : A^+ | \pi^*_a(a)] = 0$. One readily verifies that $[a : A^+ | \pi^*_a(a)] = 1$ for all $a \in A^+$; hence the claim is verified.
Let $u$ be a real eigenvector of $K$. Note that

$$u^t Ku = u^t Ru = u^t R^* u,$$

where $R^*$ is the matrix obtained from $R$ by replacing the above $H^+_v \times H^+_v$ matrix with the corresponding symmetric matrix

$$
\begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 1 & \cdots & \frac{1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 1
\end{pmatrix}.
$$

This is a positive definite matrix, which means that the eigenvalue of $u$ must be positive; thus we are done. 

\[ \square \]

### 3.3.3 Proof of Theorem 3.3.3

Let $\pi$ and $\sigma$ be Euler trails. Then

$$[C_\sigma|C_\pi] = [C_\sigma|M_\sigma][M_\sigma|M_\pi][M_\pi|C_\pi].$$

By (3.8),

$$[\sigma(\tilde{e}) : M_\sigma]|\sigma_\pi^*(a)] = \begin{cases} 
1 & \text{if } a = c; \\
0 & \text{otherwise}
\end{cases}$$

for any $a,c \in A^+$, that is, $[M^-_\sigma : M_\sigma|C_\sigma]$ is the identity matrix. Therefore, the coset $[C_\sigma|M_\sigma]$ contains the matrix with ones on positions indexed by $(\sigma_\pi^*(a),\sigma(\tilde{e}))$ and with zeroes on all other positions. This means that $[C_\sigma|M_\sigma]$ picks out the rows indexed by $M^-_\sigma$ in $[M_\sigma|C_\sigma]$. Thus the first part of the Theorem is proved.

For the second part, note that $[C_\sigma|C_\pi]$ is equal to $[\tilde{e}(C_\sigma)|\tilde{e}(C_\pi)]$ and that

$$[A^+|\tilde{e}(C_\sigma)] = [A^+|\tilde{e}(C_\sigma)][\tilde{e}(C_\sigma)|\tilde{e}(C_\sigma)].$$

Thus by Theorem 3.3.2,

$$\chi(G) = \det[A^+|\tilde{e}(C_\sigma)] = \det[A^+|\tilde{e}(C_\sigma)]\det[\tilde{e}(C_\sigma)|\tilde{e}(C_\sigma)] = \chi(G)\cdot\det[\tilde{e}(C_\sigma)|\tilde{e}(C_\sigma)].$$

Since $\chi(G) \neq 0$, our proof is finished. 

\[ \square \]

### 3.3.4 The 2-regular case

We want to show that Theorem 3.3.2 is a generalization of Theorem 1.2.2 and that Theorem 3.3.1 is a generalization of (a Z-version of) Theorem 1.2.1. Let $G$ be 2-regular and put $\{v^+,v^-\} = H^+_v$; let $A^+ = \{v^+ : v \in V\}$ and $A^- = \{v^- : v \in V\}$. Let $\pi$ be an Euler trail. First of all, note that $[v^+\pi|v^+]$:

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\( \mathcal{M}_\pi | \pi_{w^+} (w^+) \) is nonzero if and only if \( v \) and \( w \) intersect in the sense described before Theorem 1.2.1; more precisely,

\[
[v^+ \pi(v^+) : \mathcal{M}_\pi | \pi_{w^+} (w^+)] = \begin{cases} 
1 & \text{if } \pi_{H_v \cup H_w} = (v^+, w^+, v^-, w^-); \\
-1 & \text{if } \pi_{H_v \cup H_w} = (v^+, w^-, v^-, w^+); \\
0 & \text{otherwise.}
\end{cases}
\]

This immediately implies that \( [M^+_\pi : \mathcal{M}_\pi | \mathcal{C}_\pi] \) is equal to (the transpose of) the matrix \( L(\pi) \) in Theorem 1.2.2. In particular, since the matrix \( [M^-_\pi : \mathcal{M}_\pi | \mathcal{C}_\pi] \) is easily seen to be the identity matrix, we obtain that

\[
[A^+ | \mathcal{C}_\pi] = L(\pi) + I;
\]

hence Theorem 3.3.2 is a generalization of Theorem 1.2.2.

Now, let \( \sigma \) be a \( G \)-permutation. By Lemma 3.2.1,

\[
[v^+ \sigma(v^-) : \mathcal{M}_\sigma | \mathcal{C}_\pi] = \begin{cases} 
[v^+ \pi(v^-) : \mathcal{M}_\pi | \mathcal{C}_\pi] & \text{if } \sigma(a) = \pi(a); \\
[v^+ \pi(v^+) : \mathcal{M}_\pi | \mathcal{C}_\pi] & \text{if } \sigma(a) \neq \pi(a).
\end{cases}
\]

As we indicated above, the row vector \( [v^+ \pi(v^-) : \mathcal{M}_\pi | \mathcal{C}_\pi] \) is the unit vector indexed by \( v \), which means that the nullity of \( [M^-_\pi : \mathcal{M}_\sigma | \mathcal{C}_\pi] \) is equal to the nullity of the matrix obtained from \( [M^+_\pi : \mathcal{M}_\pi | \mathcal{C}_\pi] \) by removing rows and columns corresponding to vertices \( v \) such that \( \sigma(v^+) = \pi(v^+) \). However, this matrix is equal to the \( J \times J \) submatrix of \( L(\pi) \), where \( J = \{v : \sigma(v^+) \neq \pi(v^+)\} \); thus Theorem 3.3.1 is a generalization of Theorem 1.2.1. The same argument yields that the last statement in Theorem 3.3.3 is a generalization of Theorem 2.1.1.

### 3.3.5 Some simple examples

We illustrate the results with two very simple examples. For any \( G \)-permutations \( \sigma \) and \( \pi \), put

\[
K^\pi_\sigma = [\mathcal{M}_\sigma : \mathcal{M}_\pi].
\]

One easily realizes that \( K^\sigma_\tau = K^\pi_\sigma \) for any \( \tau \in Y_G \) and any \( G \)-permutations \( \sigma, \pi \), where \( Y_G \) is the subgroup of \( S_A \) consisting of all permutations \( \tau \) such that \( \tau(H_v) = H_v \) for all \( v \in V \). In particular, \( \tau \mapsto K^\pi_\tau \) is a matrix representation of \( Y_G \):

\[
K^\pi_{\tau_1} K^\pi_{\tau_2} = K^\pi_{\tau_1 \tau_2}, \quad K^\pi_{\tau_1} = K^\pi_{\tau_1 \tau_1}.
\]

Now fix an Euler trail \( \pi \). Put \( K^\pi = K^\pi_\pi \) and

\[
L_\sigma = [\mathcal{M}_\sigma^- : \mathcal{M}_\sigma | \mathcal{C}_\pi] = [\mathcal{M}_\sigma^- : \mathcal{M}_\pi | \mathcal{M}_\pi | \mathcal{C}_\pi].
\]

Our first example is the trivial example with a digraph consisting of one single vertex \( 1 \) and the arc set \( A = \{a, b\} \). Put \( A^- = \{b\} \) and \( \pi = \{a, b\} \); \( \pi \) is the only Euler trail in the digraph. Note that \( \mathcal{M}_\pi = \{ab, aa\}, \mathcal{C}_\pi = \{aa\}, [\mathcal{M}_\pi | \mathcal{C}_\pi] = (0) \), and
\[ K_{(a,b)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; K_{(a)(b)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Obviously, \( L_{(a,b)} = 1 \) and \( L_{(a)(b)} = 0. \)

A less trivial example is the digraph consisting of the same vertex 1 and the arc set \( A = \{a, b, c\} \); put \( A^- = \{c\} \) and \( \pi = (a, b, c); \pi \) is one of two Euler trails in the digraph. Easy computations yield that \( \mathcal{M}_{\pi} = \{ab, bc, aa, ba\}, \)
\( \mathcal{C}_{\pi} = \{aa, ba + ab\}, \)
\[ [\mathcal{M}_{\pi}|\mathcal{C}_{\pi}] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \]
and
\[ K_{(a,b,c)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad K_{(a,c,b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}; \]
\[ K_{(a)(b)(c)} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 \end{pmatrix}; \quad K_{(a,c)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}; \]
\[ K_{(a)(b,c)} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad K_{(a,b)(c)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}. \]

Here, the columns in \( K_{\sigma} \) correspond (in order from the left to the right) to \( a\sigma(a), b\sigma(b), a\sigma(c), \) and \( b\sigma(c), \) respectively. We obtain that
\[ L_{(a,b,c)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad L_{(a,c,b)} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \]
\[ L_{(a)(b)(c)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad L_{(a,c)(b)} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}; \]
\[ L_{(a)(b,c)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad L_{(a,b)(c)} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}. \]

In all cases null \( L_{\sigma} \) is equal to the number of cycles minus one in \( \sigma \) and the determinant of \( L_{\sigma} \) is either 0 or 1.
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