Morse theory of distance functions between algebraic varieties

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Summary

- We (re)develop **Morse theory for distance functions** $\mathrm{dist}_Y|_X$ between subsets X and Y of $\mathbf{R}^n$.
- We establish that the **nondegeneracy** of distance functions between real complete intersections is **generic**.
- We also compute bounds for the number of **critical points** of such functions.

Differential theory for locally Lipschitz functions

Subdifferential

- Let $X \subseteq \mathbf{R}^n$ be a smooth submanifold, $f\colon X \to \mathbf{R}$ a locally Lipschitz function, and $x \in X$.
- Denote by $\Omega(f)$ the set of differentiable points of f, of full measure by Rademacher's theorem.
- The **subdifferential of f at x** is the convex body

$$\partial_x f := \operatorname{conv}\left\{ \lim_{\substack{x_k \to x \\ x_k \in \Omega(f)}} D_{x_k} f \;\middle|\; \text{the limit exists} \right\}.$$

- The point x is **critical** if $0 \in \partial_x f$.

Subdifferential

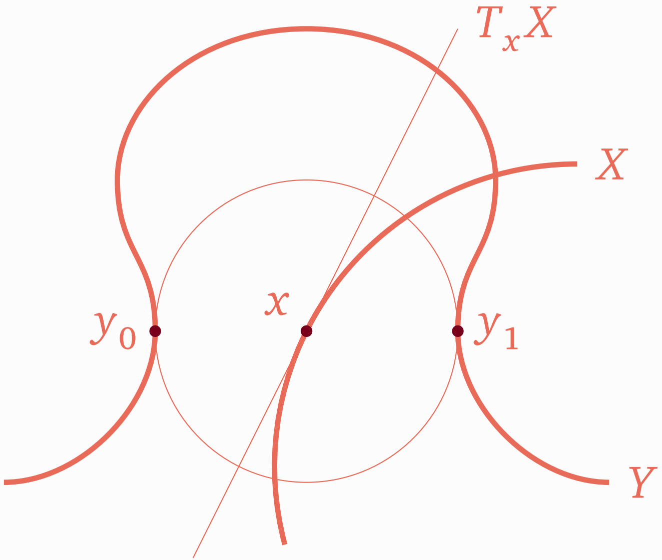
Proposition

- Let $X \subseteq \mathbf{R}^n$ be a submanifold and $Y \subseteq \mathbf{R}^n$ a closed semialgebraic set such that X is transverse to Y (and the closure of its medial axis).
- Then the subdifferential of $f = \operatorname{dist}_Y|_X$ at a point $x \in X$ is

$$\partial_x f = \operatorname{proj}_{T_x X} \operatorname{conv} \left\{ \frac{x - y}{\|x - y\|} \;\middle|\; y \in B(x, \operatorname{dist}_Y(x)) \cap Y \right\}.$$

Subdifferential

Example: in $\mathbf{R}^2$,



Subdifferential

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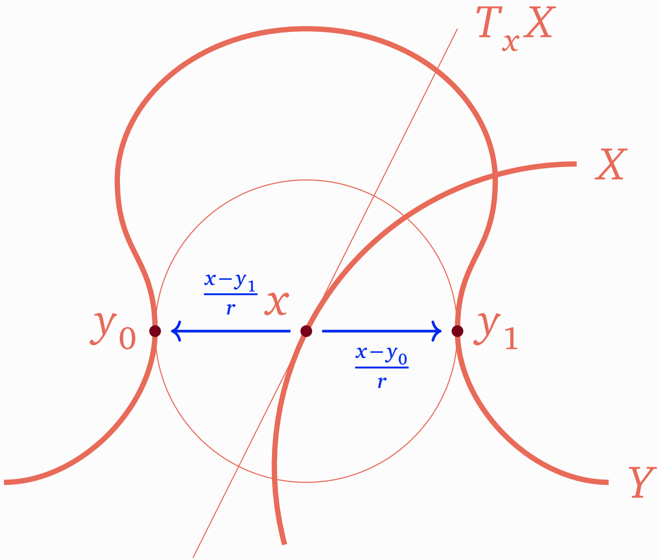

Subdifferential

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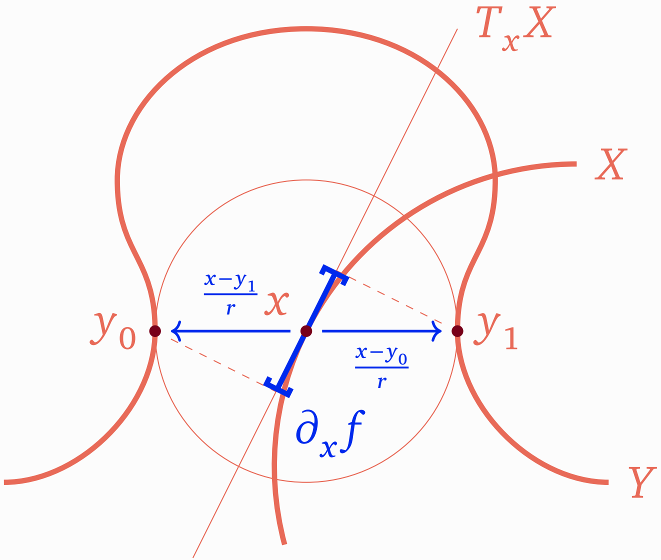

Continuous selections

- Let $f_0, \ldots, f_m : X \to \mathbf{R}$ be $\mathscr{C}^2$ functions.
- A **continuous selection of** $f_0, \ldots, f_m$ is a function $f : X \to \mathbf{R}$ if f is continuous and, for all $x \in X$, there exists $i \in \{0, \ldots, m\}$ such that $f(x) = f_i(x)$.

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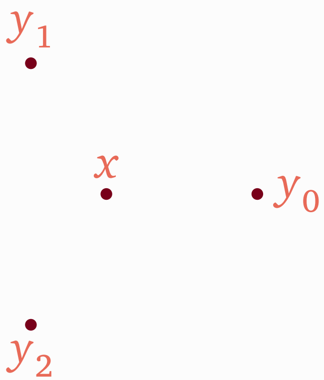
- For all $x \in X$, we define its **effective index set** as

$$I(x) := \left\{ i \in \{0, \dots, m\} \mid x \in \operatorname{clos} \operatorname{int}\{x' \in X \mid f(x') = f_i(x')\} \right\}.$$

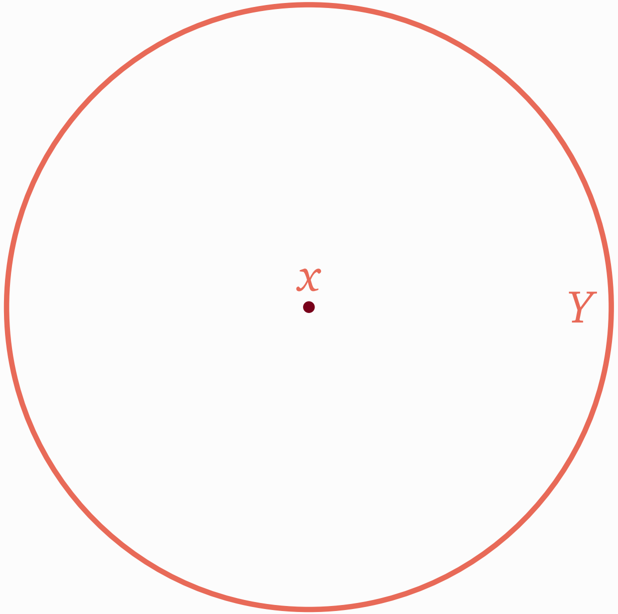
- **Fact:** the subdifferential of f at x is $\operatorname{conv}\{D_x f_i \mid i \in I(x)\}$.

Continuous selections

Example:



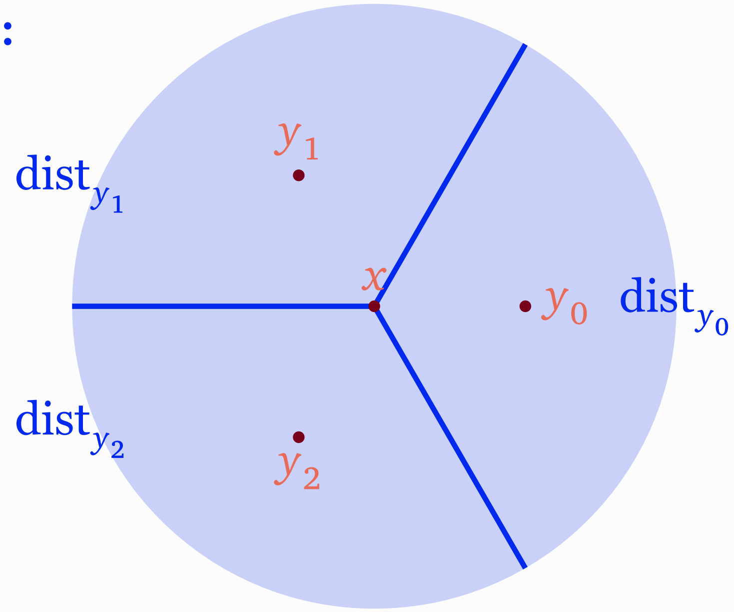
continuous selection



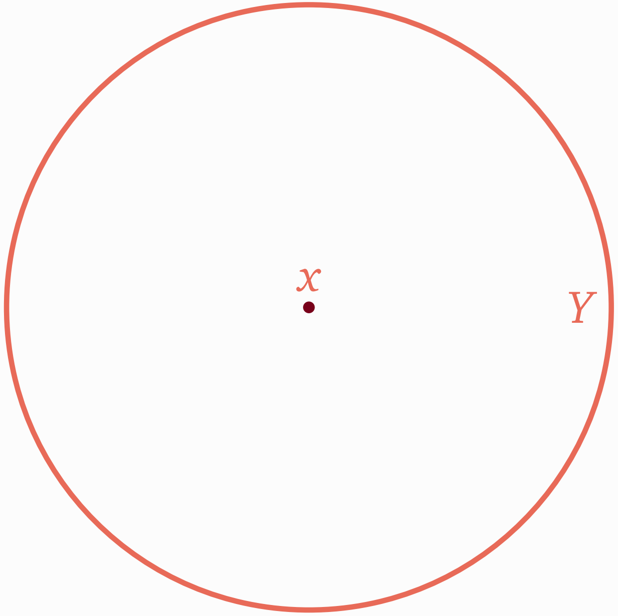
not a continuous selection

Continuous selections

Example:



continuous selection



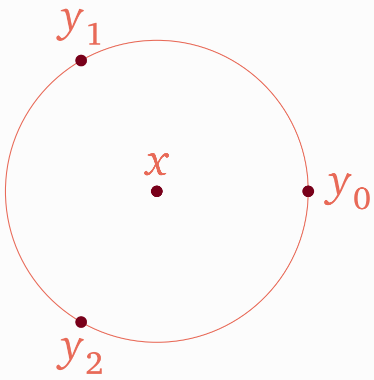
not a continuous selection

Nondegenerate critical points

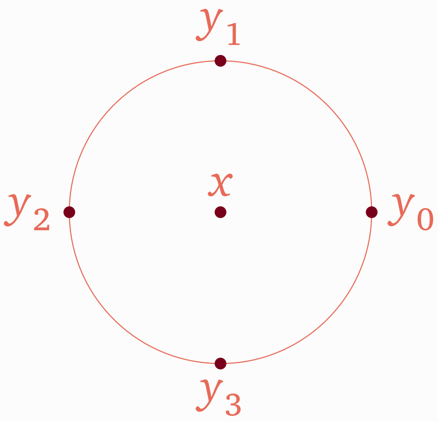
- A critical point x of a continuous selection f is **nondegenerate** if:
    1. for every $i \in I(x)$, the set of differentials $\{D_x f_j \mid j \in I(x) \setminus \{i\}\}$ is linearly independent; and
    2. writing $\sum_{i\in I(x)} \lambda_i D_x f_i = 0$ for the convex combination showing criticalness, the second differential of $\sum_{i\in I(x)} \lambda_i f_i$ is nondegenerate on $\bigcap_{i\in I(x)} \ker D_x f_i$.

Nondegenerate critical points

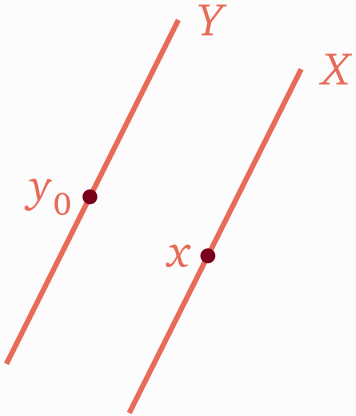
Example: in $\mathbf{R}^3$,



nondegenerate critical point



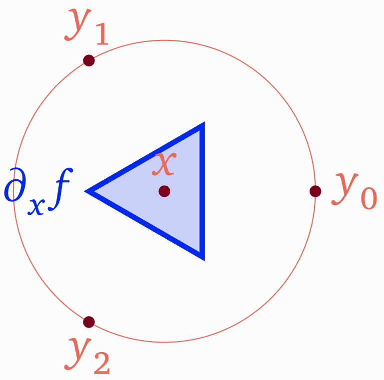
degenerate critical point for the first condition



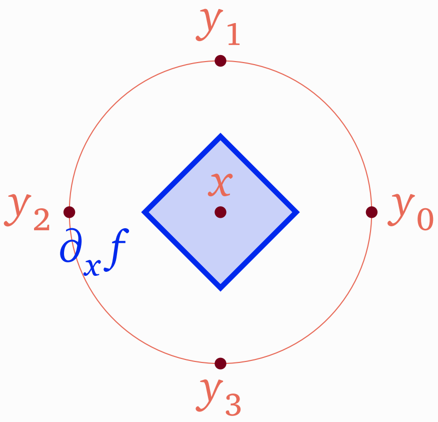
degenerate critical point for the second condition

Nondegenerate critical points

Example: in $\mathbf{R}^3$,



nondegenerate critical point



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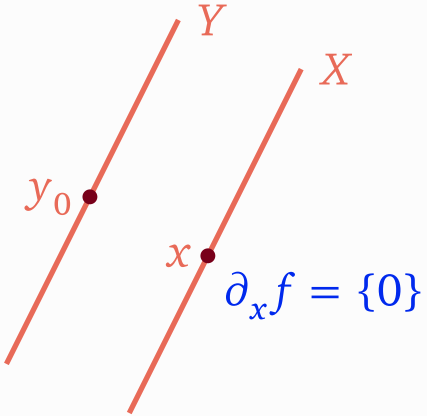


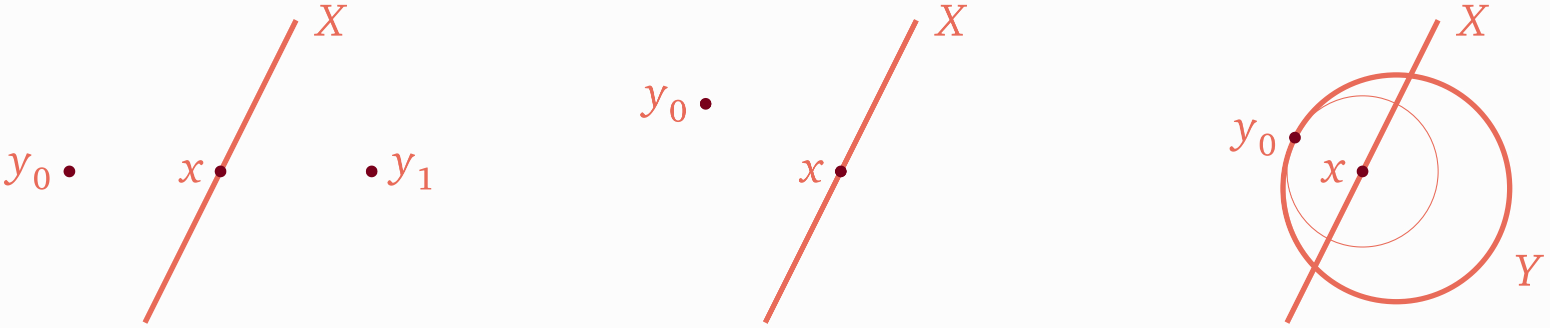
degenerate critical point for the second condition

Critical indices

- We denote by $k(x) := \#I(x) - 1$ the **piecewise linear index of x** and by $\iota(x)$ the negative inertia index of the above restricted second differential, which we call the **quadratic index of x**.
- We denote by $C_{k,\iota}(X, Y)$ the set of nondegenerate critical points with piecewise linear index k and quadratic index ι.

Critical indices

Example: in $\mathbb{R}^2$,



index $(1, 0)$ index $(0, 0)$ index $(0, 1)$

Normal forms

Proposition [Jongen-Pallaschke 1988]

- For a continuous selection $f\colon X \to \mathbf{R}$ and a nondegenerate critical point $x \in X$ with piecewise linear index k and quadratic index ι, there exists a neighborhood U of x and a locally Lipschitz homeomorphism $\psi\colon \mathbf{R}^k \times \mathbf{R}^{n-k} \to U$ such that

$$f(\psi(t_1, \dots, t_n)) = f(x) + \ell(t_1, \dots, t_k) - \sum_{j=k+1}^{k+\iota} t_j^2 + \sum_{j=k+\iota+1}^{n} t_j^2,$$

where ℓ is a continuous selection of $t_1, \dots, t_k, -(t_1 + \cdots + t_k)$.

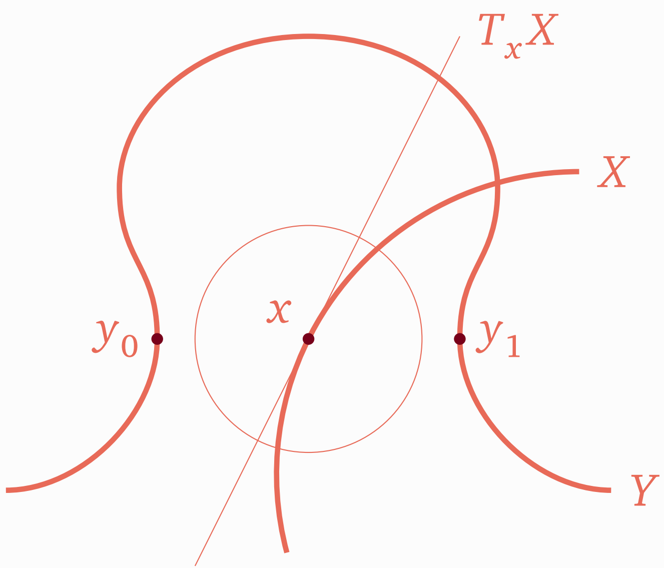
Sufficient condition for continuous selection

Proposition

- Let $Y \subseteq \mathbf{R}^n$ be a smooth submanifold, $\varphi\colon NY \to \mathbf{R}^n$ the exponential map, sending (y, v) to $y + v$, and $x \in \mathbf{R}^n$ a regular value of φ.
- Then:
  1. $B(x, \mathrm{dist}_Y(x)) \cap Y$ is a finite set $\{y_0, \ldots, y_k\}$; and
  2. $\mathrm{dist}_Y|_{B(x,\delta)}$ is a continuous selection of the functions $\mathrm{dist}_{B(y_i,\varepsilon)\cap Y}|_{B(x,\delta)}$.

Sufficient condition for continuous selection

Example:



Sufficient condition for continuous selection

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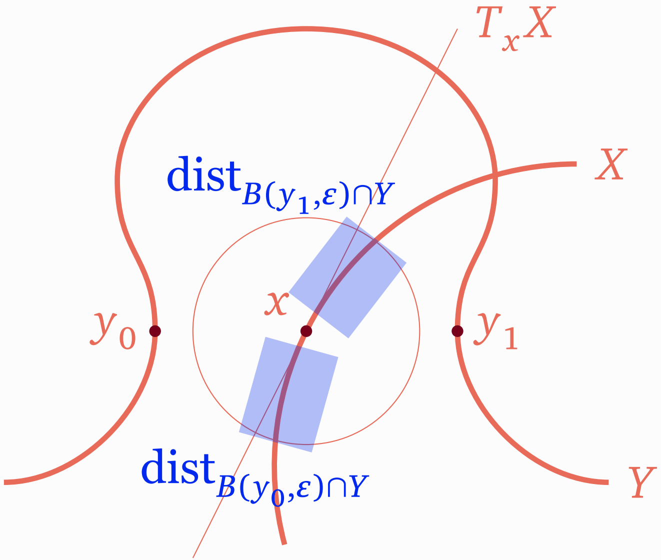

Morse theory for distance functions

Between critical values

Proposition [Clarke 1976, Agrachev-Pallaschke-Scholtes 1997]

- Let $f = \mathrm{dist}_Y|_X$ and $[a, b] \subseteq \mathbf{R}$ an interval containing no critical values.
- Then the space $\{f \leq b\}$ deformation retracts to the space $\{f \leq a\}$.

Passing a critical value

Proposition, follows from [Agrachev-Pallaschke-Scholtes 1997]

- Let $X \subseteq \mathbf{R}^n$ be a smooth manifold and $Y \subseteq \mathbf{R}^n$ a closed semialgebraic set.
- Let $c > 0$ be a critical value of $f = \mathrm{dist}_Y|_X$ such that the associated critical points $x_1, \ldots, x_m$ are all nondegenerate. Then

$$H^*(\{f \leq c + \varepsilon\}, \{f \leq c - \varepsilon\}) \cong \bigoplus_{i=1}^{m} \tilde{H}^*\left(S^{k(x_i)+\iota(x_i)}\right).$$

Morse inequalities

Proposition

- Let $X \subseteq \mathbf{R}^n$ be a smooth, compact, semialgebraic manifold and $Y \subseteq \mathbf{R}^n$ a closed semialgebraic set such that all critical points of $\mathrm{dist}_Y|_X$ are nondegenerate.
- Then, for every integer $\lambda \geq 0$,

$$\sum_{i=0}^{\lambda}(-1)^{i+\lambda} b_i(X) \leq \sum_{i=0}^{\lambda}(-1)^{i+\lambda}\left(b_i(X \cap Y) + \sum_{k+\iota=i} \#C_{k,\iota}(X, Y)\right),$$

where the b_i are cohomology dimensions.

Morse inequalities

- Consequently,

$$\chi(X \cap Y) + \sum_{k,\iota \geq 0} (-1)^{k+\iota} \# C_{k,\iota}(X, Y) = \chi(X).$$

- If Y is also smooth and compact, and $\mathrm{dist}_X|_Y$ has only nondegenerate critical points, then

$$\chi(Y) + \sum_{k,\iota \geq 0} (-1)^{k+\iota} \# C_{k,\iota}(X, Y) = \chi(X) + \sum_{k,\iota \geq 0} (-1)^{k+\iota} \# C_{k,\iota}(Y, X).$$

Genericity for complete intersections

Complete intersections

- Consider the set of **complete intersections** in $\mathbf{R}^n$ of codimension m whose defining polynomials all have degree at most d.
- Denote by $\mathscr{C}_d^m$ the open subset of $(\mathbf{R}[x_1, \dots, x_n]_{\leq d})^m$ whose elements generate such complete intersections.

Complete intersections

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- Denote by $\mathscr{C}_d^m$ the open subset of $(\mathbf{R}[x_1, \dots, x_n]_{\leq d})^m$ whose elements generate such complete intersections.
- Let $\vec{p} \in \mathscr{C}_{d_1}^{\ell}$ and $\vec{q} \in \mathscr{C}_{d_2}^{m}$ be tuples of n-variable polynomials, $X := Z(\vec{p})$ and $Y := Z(\vec{q})$, and consider $\mathrm{dist}_Y|_X$.
    - We will show that, generically, the function $\mathrm{dist}_Y|_X$ is “Morse”, i.e. all of its critical points are nondegenerate.

Related settings

- We recover real versions of previously studied notions:
  - When $\vec{p} = \{0\}$ (and so $X = \mathbf{R}^n$), a critical point of piecewise linear index k is a **real geometric $(k+1)$-bottleneck** [Di Rocco et al. 2023].
  - The **real bottleneck degree** is the number of such geometric 2-bottlenecks.
  - When $Y = \{y\}$ is a generic point, the number of critical points is related to the **Euclidean distance degree**.
- Our bounds on the number of critical points complement and generalize the known bounds on these values.

Proposition

- For $d \geq 2$ and generic $\vec{q} \in \mathscr{C}_d^m$, for all $x \in \mathbf{R}^n$, the set $B(x, \mathrm{dist}_Y(x)) \cap Y$ is a nondegenerate simplex.

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Idea of proof:

- We follow the strategy of [Yomdin 1981].
- In particular, we use the parametric transversality theorem of [Hirsch 1976].

Theorem

- For $d_1, d_2 \geq 3$ and generic $\vec{p} \in \mathscr{C}^{\ell}_{d_1}$ and $\vec{q} \in \mathscr{C}^{m}_{d_2}$, there are a finite number of critical points.
- The number of critical points with piecewise linear index k is bounded above by $c(k, \ell, m, n) d_1^n d_2^{n(k+1)}$.

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Idea of proof:

- We follow a similar approach to [Di Rocco et al. 2023], defining necessary algebraic equations for critical points of $\mathrm{dist}_Y|_X$.
- We use a parametric transversality result to show that this set is finite.
- The upper bound follows from a bound on the Betti numbers of an algebraic set [Basu-Rizzie 2018].

- Specifically, we define

$$F\colon \begin{cases} \mathscr{C}^{\ell}_{d_1} \times \mathscr{C}^{m}_{d_2} \times \mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \setminus \Delta) \times \mathbf{R}^L & \to J^1(\mathbf{R}^n, \mathbf{R}^\ell) \times {}_{k+1}J^1(\mathbf{R}^n, \mathbf{R}^m) \times \mathbf{R}^L \\ (\vec{p}, \vec{q}, x, \vec{y}, \vec{\lambda}, \vec{\mu}, \Xi, r) & \mapsto \begin{array}{c}(x, \vec{p}(x), \nabla\vec{p}(x), \vec{y}, \vec{q}(\vec{y}), \nabla\vec{q}(\vec{y}), \\ \vec{\lambda}, \vec{\mu}, \Xi, r)\end{array} \end{cases},$$

$$W := \left\{ \begin{array}{rcl|l} (x, \vec{s}, \vec{u}) & \in & J^1(\mathbf{R}^n, \mathbf{R}^\ell), & x = \sum_{j=1}^{\ell} \mu_j u_j + \sum_{i=0}^{k} \lambda_i y_i, \\ (\vec{y}, T, V) & \in & {}_{k+1}J^1(\mathbf{R}^n, \mathbf{R}^m), & \sum_{i=0}^{k} \lambda_i = 1, \\ \vec{\lambda} & \in & \mathbf{R}^{k+1}, & \forall j \in \{1, \dots, \ell\},\ s_j = 0, \\ \vec{\mu} & \in & \mathbf{R}^{\ell}, & \forall i \in \{0, \dots, k\},\ \forall j \in \{1, \dots, m\},\ t_{ij} = 0, \\ \Xi & \in & \mathbf{R}^{(k+1)\times m}, & \forall i \in \{0, \dots, k\},\ \|x - y_i\|^2 = r^2, \\ r & \in & \mathbf{R} & \forall i \in \{0, \dots, k\},\ \sum_{j=1}^{m} \xi_{ij} v_{ij} = x - y_i \end{array} \right\}.$$

- The intersection $\operatorname{im} F(\vec{p}, \vec{q}, -) \cap W$ defines **algebraic k-critical points** w.r.t. $\vec{p}$, $\vec{q}$.

Proposition

- For $d_1 \geq 3$ and $d_2 \geq 4$, and generic $\vec{p} \in \mathscr{C}_{d_1}^{\ell}$ and $\vec{q} \in \mathscr{C}_{d_2}^{m}$, the distance function $\mathrm{dist}_Y|_X$ is a continuous selection around each of its critical points.

Proposition

- For $d_1 \geq 3$ and $d_2 \geq 4$, and generic $\vec{p} \in \mathscr{C}^{\ell}_{d_1}$ and $\vec{q} \in \mathscr{C}^{m}_{d_2}$, the distance function $\mathrm{dist}_Y|_X$ is a continuous selection around each of its critical points.

Idea of proof:

- We show that the critical points of $\mathrm{dist}_Y|_X$ are all regular values of the exponential map of Y.

Theorem

- For $d_1, d_2 \geq 4$ and generic $\vec{p} \in \mathscr{C}^{\ell}_{d_1}$ and $\vec{q} \in \mathscr{C}^{m}_{d_2}$, the critical points of $\mathrm{dist}_Y|_X$ are all nondegenerate.

Theorem

- For $d_1, d_2 \geq 4$ and generic $\vec{p} \in \mathscr{C}^{\ell}_{d_1}$ and $\vec{q} \in \mathscr{C}^{m}_{d_2}$, the critical points of $\operatorname{dist}_Y|_X$ are all nondegenerate.

Idea of proof:

- We define necessary algebraic equations for degenerateness, and then generically avoid this set.

Summary

- We (re)develop Morse theory for distance functions between subsets of $\mathbf{R}^n$ using the notion of **continuous selections**.
- We establish that the **nondegeneracy** of distance functions between algebraic hypersurfaces **is generic**.
- We also compute bounds for the number of critical points of such functions, which **generalize bounds** on the bottleneck degree and the Euclidean distance degree.
  - Our results should hold in the complex case as well.

Thank you for your attention :)

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