Morse theory of distance functions between algebraic hypersurfaces

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Summary

- We (re)develop **Morse theory for distance functions** between subsets of $\mathbf{R}^n$.
- We establish that the **nondegeneracy** of distance functions between algebraic hypersurfaces is **generic**.
- We also compute bounds for the number of **critical points** of such functions.

Differential theory for locally Lipschitz functions

Subdifferential

- Let $X \subseteq \mathbf{R}^n$ be a smooth submanifold, $f\colon X \to \mathbf{R}$ a locally Lipschitz function, and $x \in X$.
- Denote by $\Omega(f)$ the set of differentiable points of f, of full measure by Rademacher's theorem.
- The **subdifferential of f at x** is the convex body

$$\partial_x f := \operatorname{conv}\left\{ \lim_{\substack{x_k \to x \\ x_k \in \Omega(f)}} D_{x_k} f \;\middle|\; \text{the limit exists} \right\}.$$

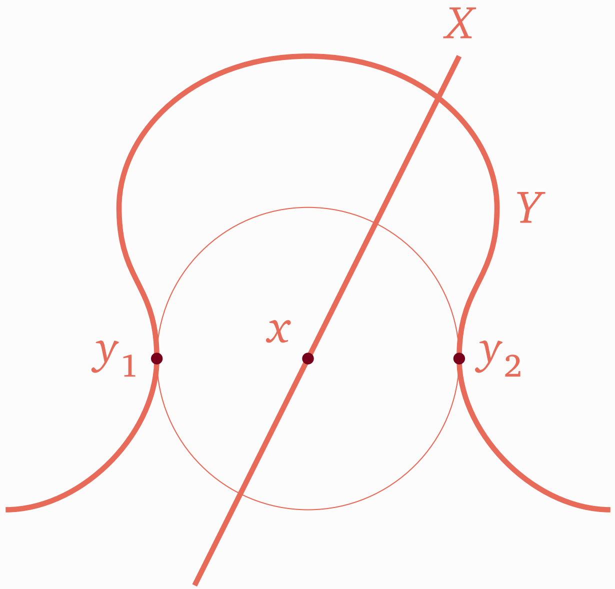
- The point x is **critical** if $0 \in \partial_x f$.

Proposition

- Let $X \subseteq \mathbf{R}^n$ be a submanifold and $Y \subseteq \mathbf{R}^n$ a closed semialgebraic set such that X is transverse to Y (and the closure of its medial axis).
- Then the subdifferential of $f = \mathrm{dist}_Y|_X$ at a point $x \in X$ is

$$\partial_x f = \mathrm{proj}_{T_x X} \operatorname{conv} \left\{ \frac{x - y}{\|x - y\|} \;\middle|\; y \in B(x, \mathrm{dist}_Y(x)) \cap Y \right\}.$$

Example:



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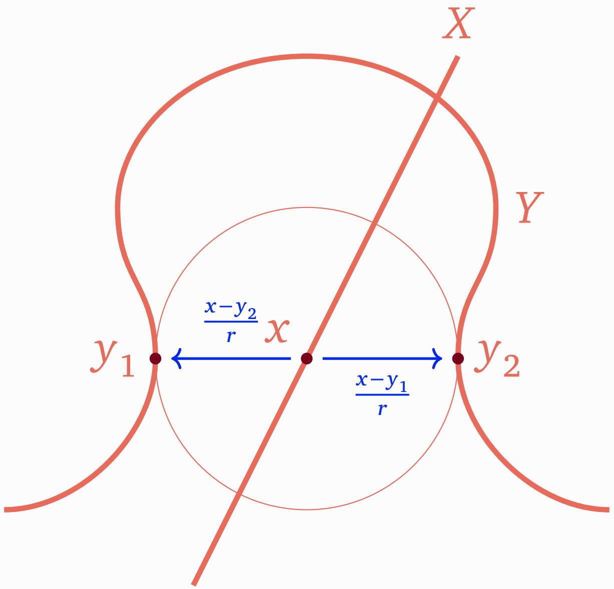

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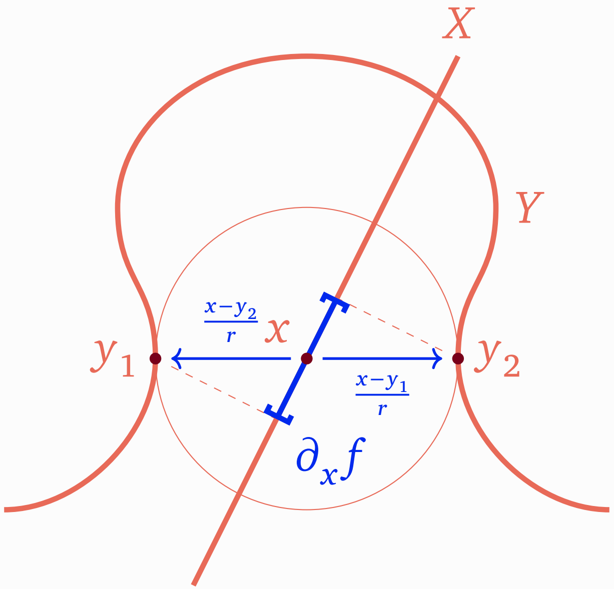

Continuous selections

- Let $f_0, \dots, f_m \colon X \to \mathbf{R}$ be continuous functions.
- A **continuous selection of** $f_0, \dots, f_m$ is a function $f \colon X \to \mathbf{R}$ if f is continuous and, for all $x \in X$, there exists $i \in \{0, \dots, m\}$ such that $f(x) = f_i(x)$.

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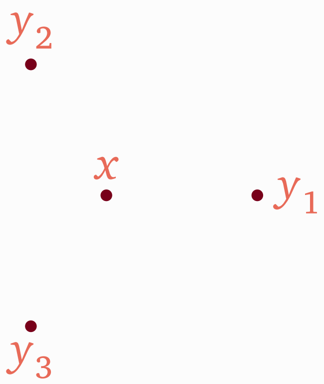
- For all $x \in X$, we define its **effective index set** as

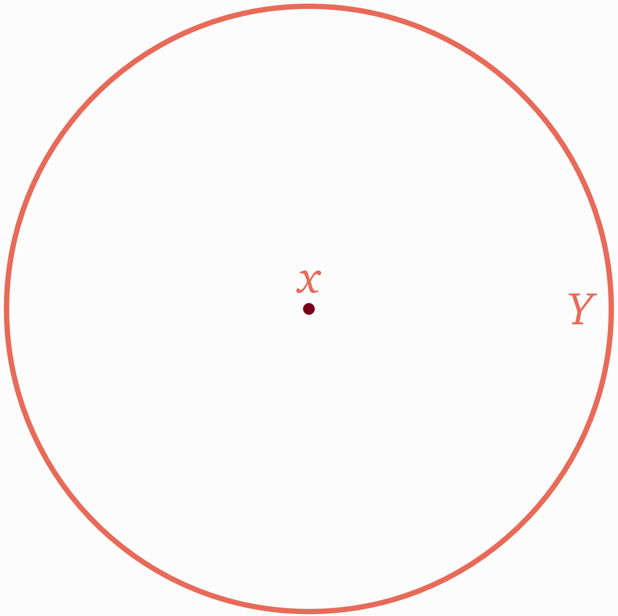
$$I(x) := \left\{ i \in \{0, \dots, m\} \mid x \in \operatorname{clos} \operatorname{int}\{y \in X \mid f(y) = f_i(y)\} \right\}.$$

- **Fact:** the subdifferential of f at x is $\operatorname{conv}\{D_x f_i \mid i \in I(x)\}$.

Example:



continuous selection



not a continuous selection

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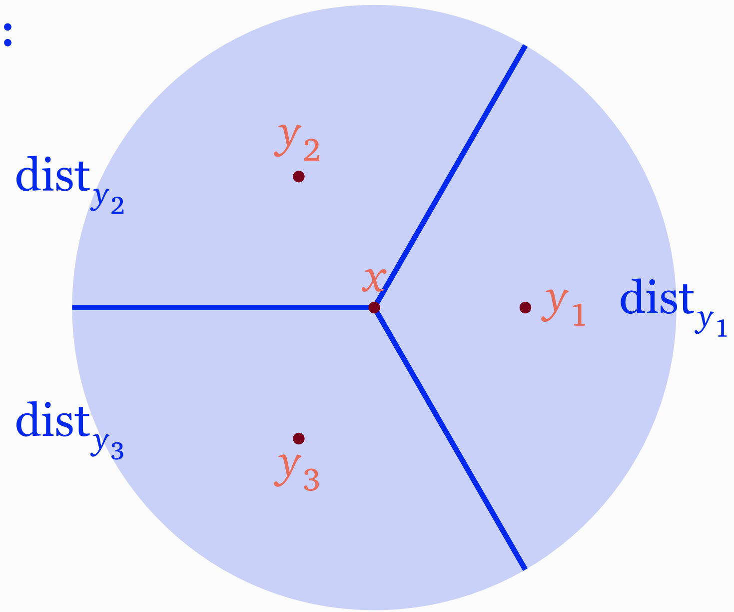

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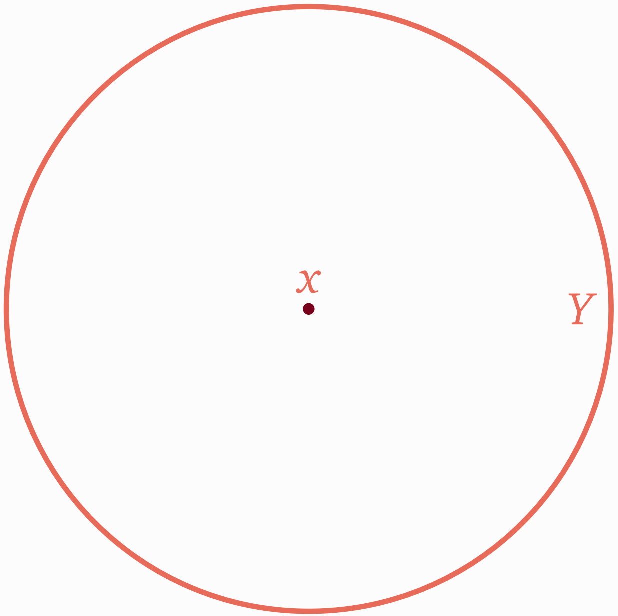

not a continuous selection

- A critical point x of f is **nondegenerate** if:
  1. for every $i \in I(x)$, the set of differentials $\{D_x f_j \mid j \in I(x) \setminus \{i\}\}$ is linearly independent; and
  2. writing $\sum_{i \in I(x)} \lambda_i D_x f_i = 0$ for the convex combination showing criticalness, the second differential of $\sum_{i \in I(x)} \lambda_i f_i$ is nondegenerate on $\bigcap_{i \in I(x)} \ker D_x f_i$.

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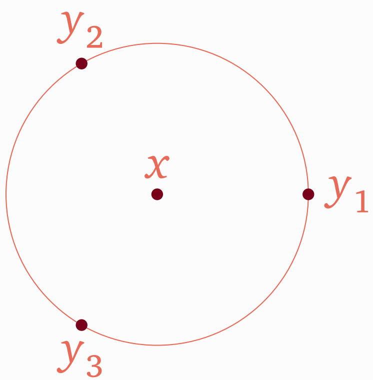
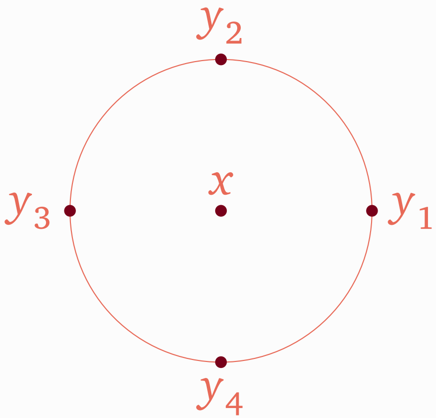
- We denote by $k(x) := \#I(x) - 1$ the **piecewise linear index of** x and by $\iota(x)$ the negative inertia index of the above restricted second differential, which we call the **quadratic index of** x.
  - We denote by $C_{k,\iota}(X, Y)$ the set of nondegenerate critical points with piecewise linear index k and quadratic index ι.

Example:

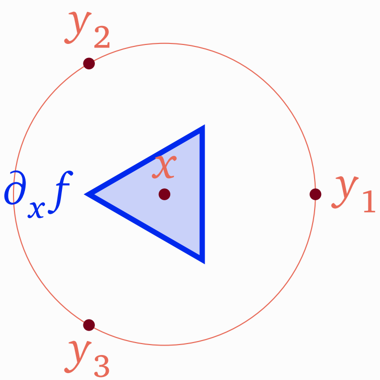


nondegenerate critical point

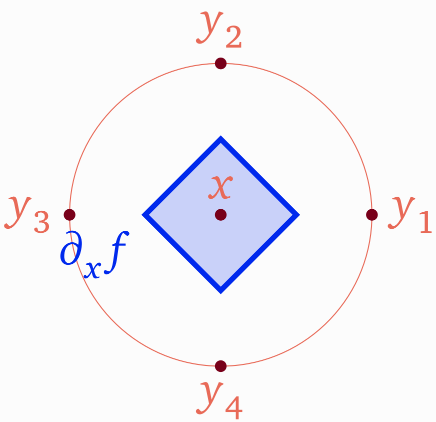


degenerate critical point

Example:



nondegenerate critical point



degenerate critical point

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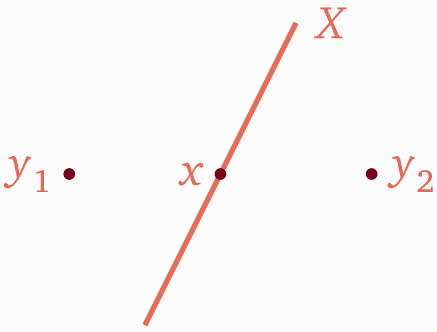
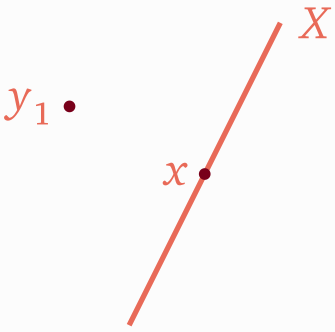


index $(1, 0)$



index $(0, 0)$

Normal forms

Proposition [Jongen-Pallaschke 1988]

- For a continuous selection $f\colon X \to \mathbf{R}$ and a nondegenerate critical point $x \in X$ with piecewise linear index k and quadratic index ι, there exists a neighborhood U of x and a locally Lipschitz homeomorphism $\psi\colon \mathbf{R}^k \times \mathbf{R}^{n-k} \to U$ such that

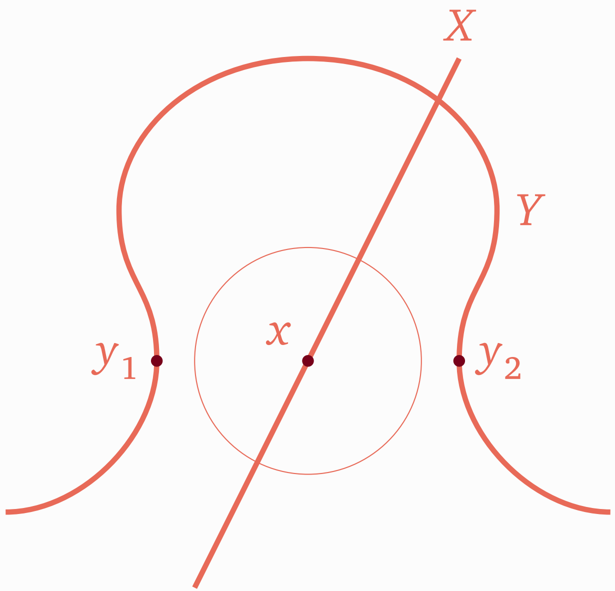
$$f(\psi(t_1, \dots, t_n)) = f(x) + \ell(t_1, \dots, t_k) - \sum_{j=k+1}^{k+\iota} t_j^2 + \sum_{j=k+\iota+1}^{n} t_j^2,$$

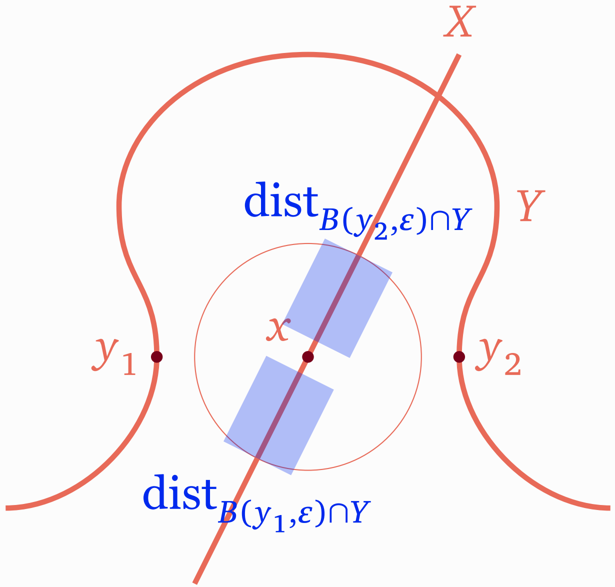
where ℓ is a continuous selection of $t_1, \dots, t_k, -(t_1 + \cdots + t_k)$.

Proposition

- Let $Y \subseteq \mathbf{R}^n$ be a smooth submanifold, $\varphi\colon NY \to \mathbf{R}^n$ the exponential map, sending (y, v) to $y + v$, and $x \in \mathbf{R}^n$ a regular value of φ.
- Then:
  1. $B(x, \mathrm{dist}_Y(x)) \cap Y$ is a finite set $\{y_0, \dots, y_k\}$; and
  2. $\mathrm{dist}_Y|_{B(x,\delta)}$ is a continuous selection of the functions $\mathrm{dist}_{B(y_i,\varepsilon)\cap Y}|_{B(x,\delta)}$.

Differential theory for locally Lipschitz functions





X
Y
$\mathrm{dist}_{B(y_2,\varepsilon)\cap Y}$
x
y_1
y_2
$\mathrm{dist}_{B(y_1,\varepsilon)\cap Y}$

Morse theory for distance functions

Between critical values

Proposition [Agrachev-Pallaschke-Scholtes 1997] actually it's Clarke

- Let $f = \mathrm{dist}_Y|_X$ and $[a, b] \subseteq \mathbf{R}$ an interval containing no critical values.
- Then the space $\{f \leq b\}$ deformation retracts to the space $\{f \leq a\}$.

Passing a critical value

Proposition, follows from [Agrachev-Pallaschke-Scholtes 1997]

- Let $X \subseteq \mathbf{R}^n$ be a smooth manifold and $Y \subseteq \mathbf{R}^n$ a closed semialgebraic set.
- Let $c > 0$ be a critical value of $f = \operatorname{dist}_Y|_X$ such that the associated critical points $x_1, \ldots, x_m$ are all nondegenerate. Then

$$H^*\left(\{f \leq c + \varepsilon\}, \{f \leq c - \varepsilon\}\right) \cong \bigoplus_{i=1}^{m} \tilde{H}^*\left(S^{k(x_i)+\iota(x_i)}\right).$$

Morse inequalities

Proposition

- Let $X \subseteq \mathbf{R}^n$ be a smooth, compact, semialgebraic manifold and $Y \subseteq \mathbf{R}^n$ a closed semialgebraic set such that all critical points of $\mathrm{dist}_Y|_X$ are nondegenerate.
- Then, for every integer $\lambda \geq 0$,

$$\sum_{i=0}^{\lambda}(-1)^{i+\lambda} b_i(X) \leq \sum_{i=0}^{\lambda}(-1)^{i+\lambda}\left(b_i(X \cap Y) + \sum_{k+\iota=i} \#C_{k,\iota}(X,Y)\right),$$

where the b_i are cohomology dimensions.

- Moreover,

$$\chi(X \cap Y) + \sum_{k,\iota \geq 0} (-1)^{k+\iota} \# C_{k,\iota}(X, Y) = \chi(X).$$

- If Y is also smooth and compact, and $\operatorname{dist}_X|_Y$ has only nondegenerate critical points, then

$$\chi(Y) + \sum_{k,\iota \geq 0} (-1)^{k+\iota} \# C_{k,\iota}(X, Y) = \chi(X) + \sum_{k,\iota \geq 0} (-1)^{k+\iota} \# C_{k,\iota}(Y, X).$$

Genericity for algebraic hypersurfaces

Setup

- Let p and q be two real n-variable polynomials, $X := Z(p)$, and $Y := Z(q)$, and consider $\mathrm{dist}_Y|_X$.

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- Let p and q be two real n-variable polynomials, $X := Z(p)$, and $Y := Z(q)$, and consider $\mathrm{dist}_Y|_X$.
- We recover previously studied notions:
  - When $p = 0$ (and so $X = \mathbf{R}^n$), a critical point of piecewise linear index k is a **real geometric $(k+1)$-bottleneck** [Di Rocco et al. 2023].
  - The **real bottleneck degree** is the number of such geometric 2-bottlenecks.
  - When $Y = \{y\}$ is a generic point, the number of critical points is (related to) the **Euclidean distance degree**.
- Our bounds on the number of critical points complement and generalize the known bounds on these values.

Theorem (Parametric transversality, Hirsch 1976)

- Let M, N, P be smooth manifolds, $W \subseteq N$ a submanifold of N, and $F\colon P \times M \to N$ a smooth map.
- For all p in P, we denote by F_p the map $F(p, -)\colon M \to N$.
- If F is transverse to W, then the set $\{p \in P \mid F(p, -) \pitchfork W\}$ is residual in P.
  - In other words, for generic $p \in P$, the map F_p is transverse to W.
  - If M and W have small enough dimensions, then F_p misses W entirely.

Theorem (Multijet parametric transversality)

- Now consider $P = \mathscr{P}_d$ the space of polynomials of degree at most d, $M = ((\mathbf{R}^n)^k \setminus \Delta)$ the space of k distinct points in $\mathbf{R}^n$, $N = {}_kJ^r(\mathbf{R}^n, \mathbf{R})$ the space of k-multijets of order r, and

$$F\colon \begin{cases} \mathscr{P}_d \times ((\mathbf{R}^n)^k \setminus \Delta) & \to & {}_kJ^r(\mathbf{R}^n,\mathbf{R}) \\ (p, y_1, \dots, y_k) & \mapsto & \left(\dfrac{\partial^{|\alpha|} p}{\partial x^\alpha}(y_i)\right)_{\substack{\alpha\in\mathbf{N}^n, |\alpha|\le r, \\ i\in\{1,\dots,k\}}} \end{cases}.$$

- Then there exists a function $d(k,r)$ such that, for all $d \geq d(k,r)$, the map F is a submersion (and so is transverse to every submanifold of N).

Proposition

- For q generic of degree ≥ 2, for all $x \in \mathbf{R}^n$, the set $B(x, \mathrm{dist}_Y(x)) \cap Y$ is a nondegenerate simplex.

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Idea of proof:

- We follow the strategy of [Yomdin 1981].
- In particular, we use the original parametric transversality theorem.

- For all $k \in \{0, \dots, n+1\}$, define

$$F_k \colon \begin{cases} \mathscr{P}_d \times \mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \setminus \Delta) & \to & {}_{k+1}J^0(\mathbf{R}^n, \mathbf{R}^2) \\ (q, x, \overline{y}) & \mapsto & (\overline{y}, \dots, q(y_i), \dots, \|x - y_i\|^2, \dots), \end{cases}$$

$$W_k := \left\{ (\overline{z}, \overline{s}, \overline{t}) \in {}_{k+1}J^0(\mathbf{R}^n, \mathbf{R}^2) \;\middle|\; \begin{array}{l} \forall i \in \{0, \dots, k\},\ s_i = 0, \\ \forall i \in \{1, \dots, k\},\ t_i = t_0 \end{array} \right\}.$$

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- We show $F_k \pitchfork W_k$, so $F_k(q, -) \pitchfork W_k$ for generic q by parametric transversality.

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- This implies that $y_0, \dots, y_k \in B(x, \mathrm{dist}_Y(x)) \cap Y$ the $\{y_i - y_0\}_{i=1}^k$ are linearly independent.

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- This implies that $y_0, \dots, y_k \in B(x, \mathrm{dist}_Y(x)) \cap Y$ the $\{y_i - y_0\}_{i=1}^k$ are linearly independent.
- The case $k = n+1$ implies that $B(x, \mathrm{dist}_Y(x)) \cap Y$ has at most $n+1$ elements, and these elements thus form a nondegenerate simplex.

Theorem

- For p and q generic of degree ≥ 3, there are a finite number of critical points.
- The number of critical points with piecewise linear index k is bounded above by $c(k,n)\deg(p)^n\deg(q)^{n(k+1)}$.

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Idea of proof:

- We follow a similar approach to [Di Rocco et al. 2023], defining necessary algebraic equations for critical points.
- The set of polynomials and points satisfying these equations is our W, and we apply parametric transversality.
- The upper bound follows from a bound on the Betti numbers of an algebraic set [Basu-Rizzie 2018].

- Specifically, we define

$$F^1 : \begin{cases} \mathscr{P}_{d_1} \times \mathscr{P}_{d_2} \times \mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \setminus \Delta) \times \mathbf{R}^{k+3} & \to J^1(\mathbf{R}^n, \mathbf{R}) \times {}_{k+1}J^1(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}^{k+3} \\ (p, q, x, \overline{y}, \overline{\lambda}, \mu, r) & \mapsto \begin{matrix} (x, p(x), \nabla p(x), \overline{y}, q(\overline{y}), \\ \nabla q(\overline{y}), \overline{\lambda}, \mu, r) \end{matrix} \end{cases},$$

$$W^1 := \left\{ \begin{array}{rcl|l} (x, s, u) & \in & J^1(\mathbf{R}^n, \mathbf{R}), & x = \mu u + \sum_{i=0}^k \lambda_i y_i, \\ (\overline{y}, \overline{t}, \overline{v}) & \in & {}_{k+1}J^1(\mathbf{R}^n, \mathbf{R}), & \sum_{i=0}^k \lambda_i = 1, \\ \overline{\lambda} & \in & \mathbf{R}^{k+1}, & s = 0, \\ \mu, r & \in & \mathbf{R} & \forall i \in \{0, \dots, k\},\ t_i = 0, \\ & & & \forall i \in \{0, \dots, k\},\ \|x - y_i\|^2 = r^2, \\ & & & \forall i \in \{0, \dots, k\},\ \operatorname{rk}(x - y_i, v_i) \leq 1 \end{array} \right\}.$$

- The intersection $\operatorname{im} F^1(p, q, -) \cap W^1$ defines **algebraic k-critical points**.

Proposition

- For p and q generic of degrees ≥ 3 and ≥ 4, respectively, the distance function $\mathrm{dist}_Y|_X$ is a continuous selection around each of its critical points.

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Idea of proof:

- We show that each critical point is a regular value of the exponential map.

- We define

$$F^2 : \begin{cases} \mathscr{P}_{d_1} \times \mathscr{P}_{d_2} \times \mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \setminus \Delta) \times \mathbf{R}^{k+3} & \to J^1(\mathbf{R}^n, \mathbf{R}) \times {}_{k+1}J^2(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}^{k+3} \\ (p, q, x, \overline{y}, \overline{\lambda}, \mu, r) & \mapsto \begin{array}{c} (x, p(x), \nabla p(x), \overline{y}, q(\overline{y}), \\ \nabla q(\overline{y}), H(q)(\overline{y}), \overline{\lambda}, \mu, r) \end{array} \end{cases},$$

$$W^2 := \left\{ \begin{array}{rcl} (x, s, u) & \in & J^1(\mathbf{R}^n, \mathbf{R}), \\ (\overline{y}, \overline{t}, \overline{v}, \overline{H}) & \in & {}_{k+1}J^2(\mathbf{R}^n, \mathbf{R}), \\ \overline{\lambda} & \in & \mathbf{R}^{k+1}, \\ \mu, r & \in & \mathbf{R} \end{array} \middle| \begin{array}{l} (x, s, u, \overline{y}, \overline{t}, \overline{v}, \overline{\lambda}, \mu, r) \in W^1, \\ \forall i \in \{0, \ldots, k\}, \\ \quad \mathrm{rk}[(I_n + rH_i)(I_n - v_i v_i^\top), v_i] < n \end{array} \right\}.$$

- The extra condition on W^2 translates to being a regular value for the exponential map.

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Idea of proof:

- We define necessary algebraic equations for degenerateness.
- **Fact:** Degenerate critical points satisfy certain algebraic equations related to the second fundamental form of Y, which, at a point $y \in Y$, can be expressed as

$$\mathrm{II}_y : \begin{cases} T_yY \times T_yY & \to & N_yY = \mathbf{R}\nabla q(y) \\ (v, w) & \mapsto & -\frac{\nabla q(y)}{\|\nabla q(y)\|} v^\top H(q)(y) w. \end{cases}$$

- We define

$$F^3 : \left\{ \begin{array}{ccc} \mathscr{P}_{d_1} \times \mathscr{P}_{d_2} \times \mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \setminus \Delta) \times \mathbf{R}^{k+3} & \to & J^2(\mathbf{R}^n, \mathbf{R}) \times {}_{k+1}J^2(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}^{k+3} \\ (p, q, x, \overline{y}, \overline{\lambda}, \mu, r) & \mapsto & \begin{array}{c} (x, p(x), \nabla p(x), H(p)(x), \overline{y}, \\ q(\overline{y}), \nabla q(\overline{y}), H(q)(\overline{y}), \overline{\lambda}, \mu, r)' \end{array} \end{array} \right.$$

$$W^3 := \left\{ \begin{array}{rcl} (x, s, u, G) & \in & J^2(\mathbf{R}^n, \mathbf{R}), \\ (\overline{y}, \overline{t}, \overline{v}, \overline{H}) & \in & {}_{k+1}J^2(\mathbf{R}^n, \mathbf{R}), \\ \overline{\lambda} & \in & \mathbf{R}^{k+1}, \\ \mu, r & \in & \mathbf{R} \end{array} \middle| \begin{array}{l} (x, s, u, \overline{y}, \overline{t}, \overline{v}, \overline{\lambda}, \mu, r) \in W^1, \\ \forall i \in \{0, \ldots, k\},\ \nu_i := \frac{v_i}{\|v_i\|} \in \mathbf{R}^n, \\ \forall i,\ Q_i := (I_n - \nu_i \nu_i^\top) \frac{H_i}{\|v_i\|} (I_n - \nu_i \nu_i^\top) \in \mathbf{R}^{n \times n}, \\ \forall i,\ \tilde{H}_i := \frac{-Q_i}{1 - rQ_i} \in \mathbf{R}^{n \times n}, \\ V := [\frac{u}{\|u\|}, \nu_0, \ldots, \nu_k] \in \mathbf{R}^{n \times (k+2)}, \\ L := I_n - V(V^\top V)^{-1} V^\top \in \mathbf{R}^{n \times n}, \\ \operatorname{rk}\left(\mu L^\top G L + r \sum_{i=0}^{k} \lambda_i L^\top \tilde{H}_i L\right) \leq n - k - 3 \end{array} \right\}.$$

Summary

- We (re)develop Morse theory for distance functions between subsets of $\mathbf{R}^n$ using the notion of **continuous selections**.
- We establish that the nondegeneracy of distance functions between algebraic hypersurfaces is generic using a **multijet parametric transversality theorem**.
- We also compute bounds for the number of critical points of such functions, which **generalize bounds** on the bottleneck degree and the Euclidean distance degree.
    - Our results should hold in the complex case as well.

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