Computing relative Betti diagrams using Koszul complexes

Isaac Ren with W. Chachólski, A. Guidolin, M. Scolamiero, and F. Tombari

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Summary

- In multiparameter persistence, we do not have a simple decomposition theorem like **barcode decompositions**.
- Instead, we approximate persistence modules by simpler modules using **relative projective resolutions**.
- Under certain conditions, we explicitly compute the **Betti diagrams** of these resolutions using **Koszul complexes**.

$$\underbrace{\text{Computing}}_{4}\ \underbrace{\text{relative}}_{3}\ \underbrace{\text{Betti diagrams}}_{1}\ \text{using}\ \underbrace{\text{Koszul complexes}}_{2}$$

Betti diagrams

Persistence modules as functors

- We consider functors $M\colon I \to \mathrm{vect}_{\mathbf{k}}$ where $(I, \leq)$ is an arbitrary poset.
- We denote by $\mathrm{Fun}(I, \mathrm{vect}_{\mathbf{k}})$ the category of **functors indexed by** I.

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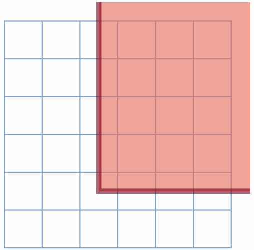
- We denote by $\mathrm{Fun}(I, \mathrm{vect}_{\mathbf{k}})$ the category of **functors indexed by** I.
- For a in I, the **free functor at** a is the functor $\mathbf{k}[a,\infty)\colon I \to \mathrm{vect}_{\mathbf{k}}$ such that

$$\mathbf{k}[a,\infty)(b) = \begin{cases} \mathbf{k} & \text{if } b \geq a, \\ 0 & \text{otherwise,} \end{cases}$$

  with identity transition maps.

- For example, for $I = \mathbf{N}^2$, the free functor at $(3,2)$ is



Free resolutions

- A **free resolution** of a functor $M : I \to \text{vect}_{\mathbf{k}}$ is an exact sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where, for all $d \geq 0$, $F_d = \bigoplus_{a \in I} \mathbf{k}[a, \infty)^{\beta^d(a)}$.

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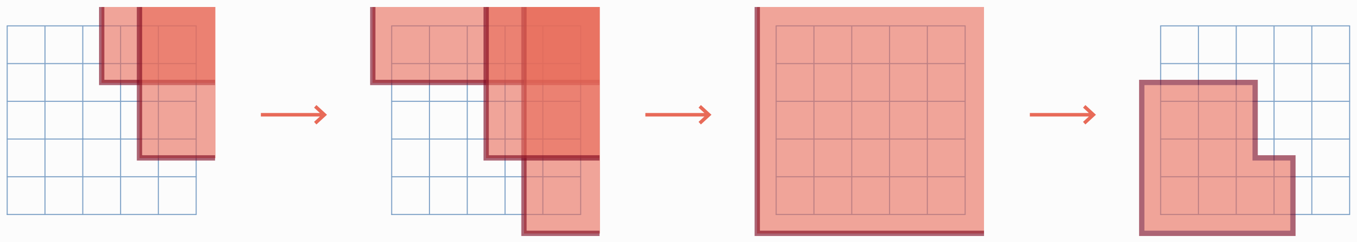
- **Example:**



Betti diagrams

- We are interested in the unique **minimal free resolution**:

$$\cdots \longrightarrow \bigoplus_{b \in I} \mathbf{k}[b,\infty)^{\beta^1(b)} \longrightarrow \bigoplus_{a \in I} \mathbf{k}[a,\infty)^{\beta^0(a)} \longrightarrow M \longrightarrow 0$$

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- For all $d \geq 0$, we collect these multiplicities in a function $\beta^d M : I \to \mathbf{N}$ called the d^{th} **Betti diagram of** M.
- **Problem:** In general, Betti diagrams require computing the entire minimal resolution. In particular, the differential maps are hard to compute.

Koszul complexes

Koszul complexes

- Suppose that $(I, \leq)$ is an **upper semilattice** (all joins exist).
- For $M : I \to \mathrm{vect}_{\mathbf{k}}$ a functor and a in I, we define the **Koszul complex of M at a** as the chain complex $\mathcal{K}_a M$

$$\cdots \longrightarrow \bigoplus_{\substack{b,c \text{ covers of } a \\ b \wedge c \text{ exists}}} M(b \wedge c) \longrightarrow \bigoplus_{b \text{ cover of } a} M(b) \longrightarrow M(a).$$

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- More formally, for all $d \geq 0$,

$$(\mathcal{K}_a M)_d := \bigoplus_{\substack{S \text{ subset of covers of } a \\ |S|=d \\ S \text{ has lower bound}}} M\Big(\bigwedge\nolimits_{(I \leq a)} S\Big),$$

and the differential maps of $\mathcal{K}_a M$ are induced from the transition maps of M.

Theorem [Chachólski-Jin-Tombari 2021]

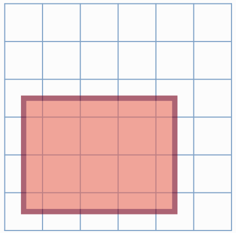
- Let $(I, \leq)$ be a finite upper semilattice.
- For all functors $M : I \to \mathrm{vect}_{\mathbf{k}}$, elements a in I, and $d \geq 0$,

$$\beta^d M(a) = \dim H_d(\mathcal{K}_a M).$$

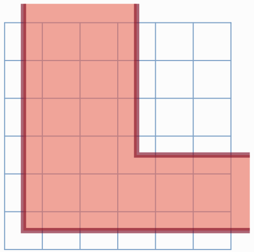
Relative homological algebra

Relative resolutions

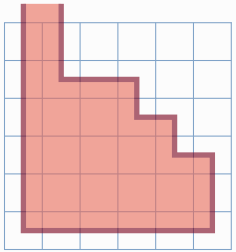
- Instead of resolving with free functors, we can try recreating bars.
- When $I = \mathbf{N}^2$, we can try:



rectangles



lower hooks
[BOO2022]



single-source spread modules
[BBH2022]

Parameterization

- Let $(J, \leqslant)$ be a poset.
- Let $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_{\mathbf{k}})$ be a **parameterization** functor associating to each element a of J a functor $\mathcal{T}(a)\colon I \to \mathrm{vect}_{\mathbf{k}}$.
- The collection of functors of interest is $\mathcal{P} \coloneqq \{\mathcal{T}(a) \mid a \in J,\ \mathcal{T}(a) \neq 0\}$.
- $\mathcal{T}$ is **thin** if, for all a, b in J, $\dim \mathrm{Nat}(\mathcal{T}(a), \mathcal{T}(b)) \leq 1$.

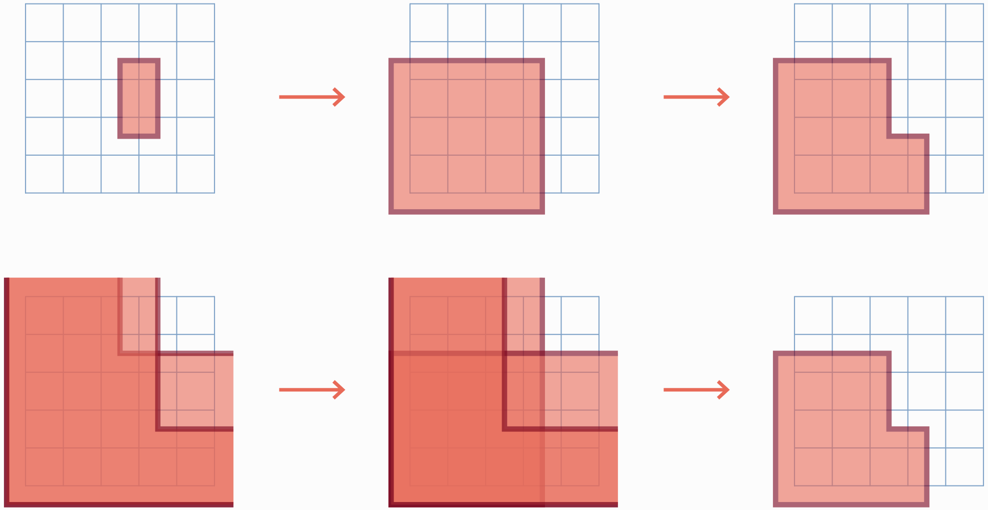
Relative projective resolutions

- Suppose that $\mathcal{T} : J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_{\mathbf{k}})$ is thin.
- $\mathcal{P}$**-projective resolutions** are $\mathcal{P}$**-exact** sequences of direct sums of elements of $\mathcal{P}$:

$$\cdots \longrightarrow \bigoplus_{b \in J} \mathcal{T}(b)^{\beta^1_{\mathcal{P}}(b)} \longrightarrow \bigoplus_{a \in J} \mathcal{T}(a)^{\beta^0_{\mathcal{P}}(a)} \longrightarrow M \longrightarrow 0.$$

- **Example:** with $J = I$ and $\mathcal{T}(a) = \mathbf{k}[a, \infty)$, we recover standard free resolutions.

Examples



Relative Betti diagrams and Koszul complexes

- Similarly to the standard case, we collect the multiplicities of elements of $\mathcal{P}$ in the minimal $\mathcal{P}$-projective resolution in **$\mathcal{P}$-Betti diagrams** $\beta^d_{\mathcal{P}} M \colon J \to \mathbf{N}$.
- **Problem:** we want to compute the $\mathcal{P}$-Betti diagrams of a functor $M \colon I \to \mathrm{vect}_{\mathbf{k}}$.
- **Solution:** we compute the standard Betti diagrams of the functor

$$\mathrm{Nat}(\mathcal{T}(-), M) \colon \begin{cases} J & \to & \mathrm{vect}_{\mathbf{k}} \\ a & \mapsto & \mathrm{Nat}(\mathcal{T}(a), M) \end{cases}$$

using Koszul complexes, and then transfer the diagrams to the relative side.

Theorem [Chachólski, Guidolin, R., Scolamiero, Tombari]

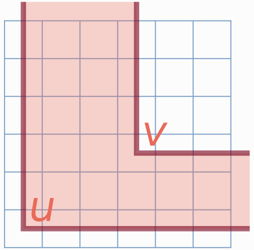
- Let $(J, \leqslant)$ be a finite upper semilattice and $\mathcal{T}$ a thin parameterization.
- Assume some other technical but realizable conditions.
- Then, for all functors $M \colon I \to \mathrm{vect}_{\mathbf{k}}$, all a in J such that $\mathcal{T}(a) \neq 0$, and all $d \geq 0$,

$$\beta^d_{\mathcal{P}} M(a) = \dim H_d(\mathcal{K}_a \operatorname{Nat}(\mathcal{T}(-), M)).$$

Computing an example

Lower hooks

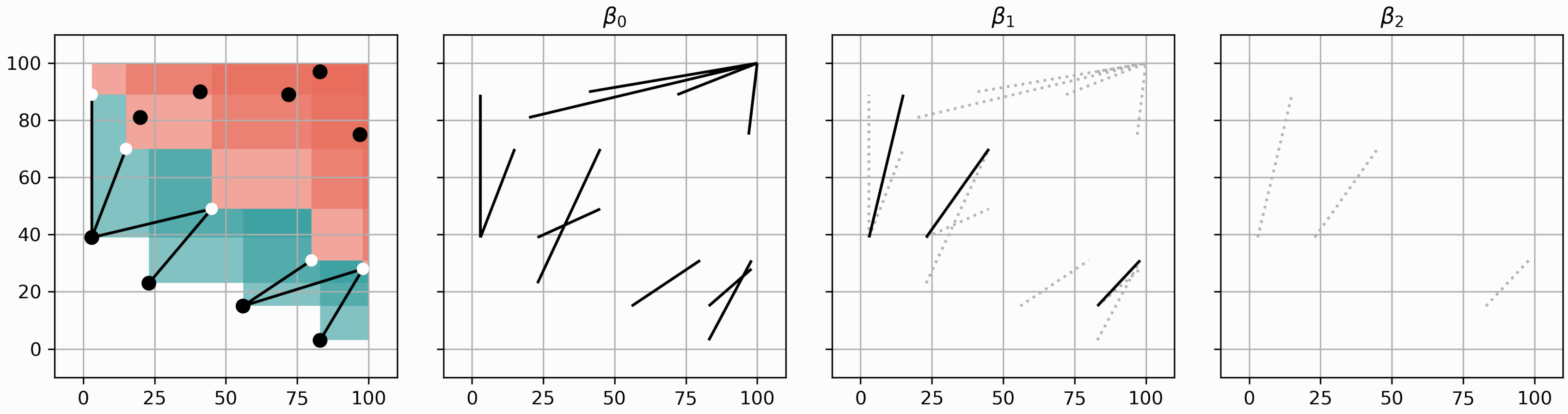
- We use the poset $J := \{(u, v) \mid u \leq v \in I\}$ with the product order, and the parameterization $\mathcal{T} : (u, v) \mapsto \operatorname{coker}(\mathbf{k}[v, \infty) \to \mathbf{k}[u, \infty))$:

$\mathcal{T}(u, v) =$ 

- **Fact:** For all (u, v), we have $\operatorname{Nat}(\mathcal{T}(u, v), M) = \ker(M(u) \to M(v))$.

Implementation

- We take as input a free presentation of a persistence module in the field of two elements $\mathbf{F}_2$.



Complexity

- Time complexity is $O(2^{6r}n^{2r+3})$, where r is the dimension of the grid and n is the number of generators and relations in the input presentation.
- Space complexity is $O(rn^{2r+2})$.
- To compare, naively computing the free minimal resolution of $\mathrm{Nat}(\mathcal{T}(-), M)$ is $O(r^2 2^{2r} n^{8r})$ and computing the signed barcode **[BOO2022]** is $O(r^3 n^{2r+1})$.

Cokernels vs. kernels

Proposition (Guidolin, R.)

- Let $M\colon I \to \mathrm{vect}_{\mathbf{k}}$ be a functor.
- Consider the induced functors $\ker M \coloneqq (\ker(M(u) \to (v)))_{(u,v)\in J}$ and $\operatorname{coker} M \coloneqq (\operatorname{coker}(M(u) \to (v)))_{(u,v)\in J}$ on J.
- Then, for all (u, v) in J,

$$H_*(\mathcal{K}_{u,v}(\ker M)) \cong H_{*+2}(\mathcal{K}_{u,v}(\operatorname{coker} M)).$$

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- Then, for all (u, v) in J,

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- Time complexity stays the same but space complexity goes down to $O(n^{2r+1})$.

Conclusions

Future work

- **Stability** and **hierarchical stabilization** of relative Betti diagrams.
- Construction of new **computable metrics** for functors.
- Relative resolutions over **continuous** posets.
- Comparing with relative **injective** resolutions

References

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- B. Blanchette, T. Brüstle, and E. Hanson. *Homological approximations in persistence theory*, 2022.
- M. Botnan, S. Oppermann, and S. Oudot. *Signed barcodes for multi-parameter persistence via rank decompositions and rank-exact resolutions*, 2022.
- **Preprint:** *Koszul complexes and relative homological algebra of functors over posets*, arXiv:2209.05923.
- **Implementation:** `https://github.com/th-rtyf-re/lowerhooks`