Relative homological algebra and Koszul complexes for multiparameter persistence

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Motivation

- Consider a topological space X and n continuous real-valued functions $f_i \colon X \to \mathbf{R}$, $i \in \{1, \dots, n\}$.
- For all a in $\mathbf{R}^n$, define $X_a := \{x \in X \mid \forall i \in \{1, \dots, n\},\ f_i(x) \leq a_i\}$.
- For all $d \geq 0$, we can study the d^{th} homology of the X_a's.
  - Moreover, if $a \leq b$ in $\mathbf{R}^n$ for the product order, then the containment $X_a \subseteq X_b$ induces a linear map $H_d(X_a) \to H_d(X_b)$.
- **Question:** What simple invariants can we compute from $H_d(X_\bullet) \colon \mathbf{R}^n \to \mathbf{vect_k}$?

Today’s talk

- We can approximate persistence modules by simpler modules using **relative projective resolutions**.
- Under certain conditions, we can explicitly compute the **Betti diagrams** of these resolutions using **Koszul complexes**.

Single-parameter persistence

Persistence modules as functors

- Consider the poset $(\mathbf{N}, \leq)$.
- We study persistence modules as **functors** $M\colon \mathbf{N} \to \mathbf{vect_k}$:
  - For each natural number a in $\mathbf{N}$, we associate a $\mathbf{k}$-vector space $M(a)$.
  - For each pair $a \leq b$ in $\mathbf{N}$, we associate a linear map $M(a \leq b)\colon M(a) \to M(b)$ called a **transition map**.
  - For each triple $a \leq b \leq c$ in $\mathbf{N}$, $M(a \leq c) = M(b \leq c)M(a \leq b)$.

Natural transformations between functors

- A **natural transformation** $f\colon M \to N$ between two functors M and N is
  - the data, for each a in $\mathbf{N}$, of a linear map $f(a)\colon M(a) \to N(a)$,
  - such that, for each pair $a \leq b$ in $\mathbf{N}$, $f(b)M(a \leq b) = N(a \leq b)f(a)$:

$$\begin{array}{ccc} M(a) & \xrightarrow{M(a\leq b)} & M(b) \\ {\scriptstyle f(a)}\downarrow & & \downarrow{\scriptstyle f(b)} \\ N(a) & \xrightarrow[N(a\leq b)]{} & N(b) \end{array}$$

- We denote by $\mathrm{Nat}(M, N)$ the space of natural transformations from M to N.

Free functors

- For a in $\mathbf{N}$, the **free functor at a** is the functor $\mathbf{k}[a,\infty)\colon \mathbf{N} \to \mathbf{vect_k}$ such that

$$\mathbf{k}[a,\infty)(b) = \begin{cases} \mathbf{k} & \text{if } b \geq a, \\ 0 & \text{otherwise,} \end{cases}$$

with identity transition maps:

$$0 \to 0 \to \cdots \to \underset{a-1}{0} \to \underset{a}{\mathbf{k}} \xrightarrow{\text{id}} \underset{a+1}{\mathbf{k}} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \mathbf{k} \xrightarrow{\text{id}} \cdots$$

Free presentations

- Every functor M can be presented as the quotient of two **free functors**:

$$0 \longrightarrow \bigoplus_{b \in \mathbf{N}} \mathbf{k}[b, \infty)^{\beta^1(b)} \longrightarrow \bigoplus_{a \in \mathbf{N}} \mathbf{k}[a, \infty)^{\beta^0(a)} \longrightarrow M \longrightarrow 0.$$

- This sequence is **minimal** if all of its endomorphisms are isomorphisms.
  - **Fact:** the minimal presentation is unique.

Barcodes

We can do better than free presentations:

- For $a < b$ in $\mathbf{N}$, the **bar from a to b** is the functor $\mathbf{k}[a, b) \colon \mathbf{N} \to \mathbf{vect}_{\mathbf{k}}$ such that

$$\mathbf{k}[a,b)(c) = \begin{cases} \mathbf{k} & \text{if } a \le c < b, \\ 0 & \text{otherwise,} \end{cases}$$

  with identity transition maps:

$$0 \to 0 \to \cdots \to \underset{a-1}{0} \to \underset{a}{\mathbf{k}} \xrightarrow{\text{id}} \underset{a+1}{\mathbf{k}} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \underset{b-1}{\mathbf{k}} \to \underset{b}{0} \to \cdots$$

Theorem [Zomorodian-Carlsson 2005]

- Every functor in $\mathrm{Fun}(\mathbf{N}, \mathbf{vect_k})$ is isomorphic to a unique direct sum of bars.

Standard homological algebra for multiparameter persistence

Multiparameter functors

- Instead of functors $\mathbf{N} \to \mathbf{vect_k}$, we can consider functors $\mathbf{N}^n \to \mathbf{vect_k}$.
  - The poset $(\mathbf{N}^n, \leq)$ is equipped with the product order.

Functors over arbitrary posets

- We now consider functors $M\colon I \to \mathbf{vect_k}$ where $(I, \leq)$ is an arbitrary poset.
- We denote by $\mathrm{Fun}(I, \mathbf{vect_k})$ the category of **functors indexed by I**.

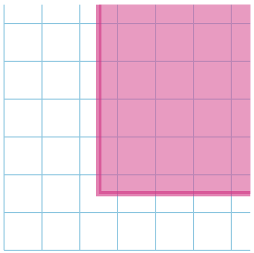
Free functors over posets

- For a in I, the **free functor at a** is the functor $\mathbf{k}[a,\infty)\colon \mathbf{N} \to \mathbf{vect}_{\mathbf{k}}$ such that

$$\mathbf{k}[a,\infty)(b) = \begin{cases} \mathbf{k} & \text{if } b \geq a, \\ 0 & \text{otherwise,} \end{cases}$$

  with identity transition maps.

- For example, if $I = \mathbf{N}^2$, then the free functor at $(3,2)$ is



Free resolutions

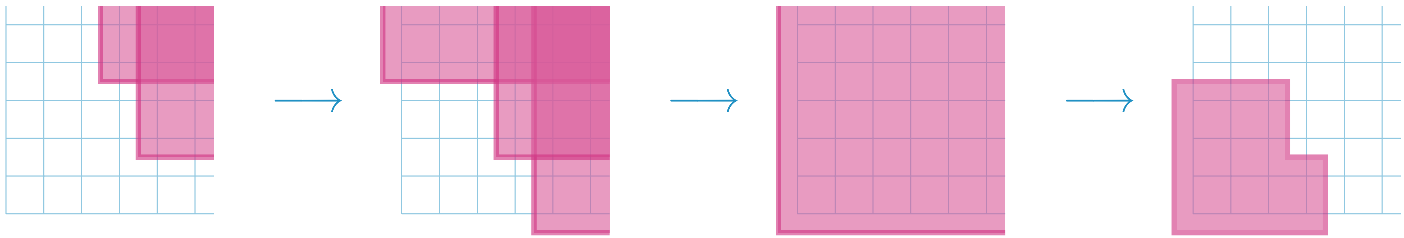
- A **free resolution** of a functor $M\colon I \to \mathbf{vect_k}$ is an exact sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

  where, for all $d \geq 0$, $F_d = \bigoplus_{a\in I} \mathbf{k}[a, \infty)^{\beta^d(a)}$.

  - We also have the notion of a unique **minimal free resolution**.

Example



Betti diagrams

- For a functor M, the multiplicities $\beta^d(a)$ of the unique minimal free resolution are of interest.
  - For all $d \geq 0$, we collect these multiplicities in a function $\beta^d M \colon I \to \mathbf{N}$ called the d^{th} **Betti diagram of M**.
  - **Problem:** In general, Betti diagrams require computing the entire minimal resolution. In particular, the differential maps are hard to compute.

Poset terminology

Let a and b be elements of the poset $(I, \leq)$.

- A **cover of a** is a maximal element smaller than a.
- The **join of a and b**, if it exists, is the unique minimal upper bound $a \vee b \geq a, b$.
- The **meet of a and b**, if it exists, is the unique maximal lower bound $a \wedge b \leq a, b$.

Upper semilattices

- The poset $(I, \leq)$ is an **upper semilattice** if every pair of elements a, b has a join $a \vee b$.
- If I is an upper semilattice, then we can individually compute values of Betti diagrams.

Koszul complexes for Betti diagrams

- Suppose that $(I, \leq)$ is an upper semilattice.
- For $M \colon I \to \mathbf{vect}_{\mathbf{k}}$ a functor and a in I, we define the **Koszul complex of M at a** as the chain complex $\mathcal{K}_a M$

$$\cdots \longrightarrow \bigoplus_{\substack{b,c \text{ covers of } a \\ b \wedge c \text{ exists}}} M(b \wedge c) \longrightarrow \bigoplus_{b \text{ cover of } a} M(b) \longrightarrow M(a).$$

- More formally, for all $d \geq 0$,

$$(\mathcal{K}_a M)_d := \bigoplus_{\substack{S \text{ subset of covers of } a \\ |S|=d \\ S \text{ has lower bound}}} M\left(\bigwedge_{(I \leq a)} S\right).$$

- The differential maps of $\mathcal{K}_a M$ are induced from the transition maps of M.

Theorem [Chachólski-Jin-Tombari 2021]

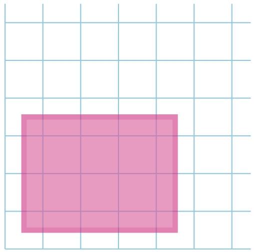
- Let $(I, \leq)$ be an upper semilattice.
- For all functors $M\colon I \to \mathbf{vect_k}$, elements a in I, and $d \geq 0$,

$$\beta^d M(a) = \dim H_d(\mathcal{K}_a M).$$

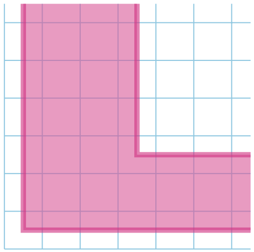
Relative homological algebra for multiparameter persistence

Non-free functors

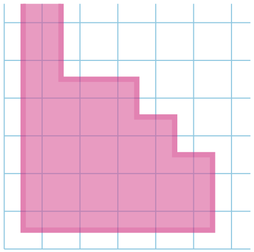
- Instead of resolving with free functors, we can try recreating bars.
  - When $I = \mathbf{N}^2$, we can try:



rectangles



lower hooks
[BOO2021]



single-source spread modules
[BBH2021]

Relative projectives

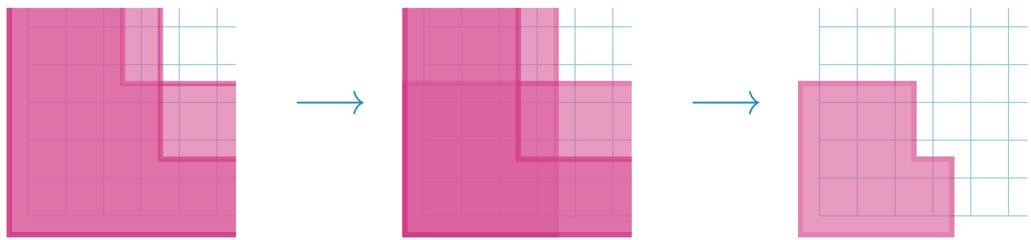
- We fix a collection $\mathcal{C}$ of functors in $\mathrm{Fun}(I, \mathbf{vect_k})$.
- A natural transformation $f\colon M \to N$ is a **$\mathcal{C}$-epimorphism** if, for all A in $\mathcal{C}$, the linear map $\mathrm{Nat}(A, f)\colon \mathrm{Nat}(A, M) \to \mathrm{Nat}(A, N)$ is surjective.
- A functor $A\colon I \to \mathbf{vect_k}$ is **$\mathcal{C}$-projective** if, for every $\mathcal{C}$-epimorphism $f\colon M \to N$, the linear map $\mathrm{Nat}(A, f)\colon \mathrm{Nat}(A, M) \to \mathrm{Nat}(A, N)$ is surjective.
- A short sequence $L \to M \to N$ is **$\mathcal{C}$-exact** if, for all A in $\mathcal{C}$, the short sequence $\mathrm{Nat}(A, L) \to \mathrm{Nat}(A, M) \to \mathrm{Nat}(A, N)$ is exact.

Relative projective resolutions

- A $\mathcal{C}$**-projective resolution** of a functor $M\colon I \to \mathbf{vect_k}$ is a $\mathcal{C}$-exact sequence of functors
$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$
where the F_d are $\mathcal{C}$-projective.
  - We also have the notion of a unique **minimal** $\mathcal{C}$**-projective resolution**.

Example

- A lower hook resolution can look like



Parameterization by a poset

- Let $(J, \preccurlyeq)$ be a poset.
- Let $\mathcal{P}\colon J^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{I}, \mathbf{vect_k})$ be a functor associating to each element a of J a functor $\mathcal{P}(a)\colon \mathcal{I} \to \mathbf{vect_k}$.
  - The collection of functors is now $\mathcal{C} := \{\mathcal{P}(a) \mid a \in J,\ \mathcal{P}(a) \neq 0\}$.
  - $\mathcal{P}$ is **thin** if, for all a, b in J, $\dim \mathrm{Nat}(\mathcal{P}(a), \mathcal{P}(b)) \leq 1$.
    - **Fact:** in this case, all $\mathcal{C}$-projectives are direct sums of elements of $\mathcal{C}$.

Relative Betti diagrams

- Suppose that $\mathcal{P}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathbf{vect_k})$ is thin.
- Relative projective resolutions are then sequences of direct sums of elements of $\mathcal{C}$:

$$\cdots \longrightarrow \bigoplus_{b\in J} \mathcal{P}(b)^{\beta^1(b)} \longrightarrow \bigoplus_{a\in J} \mathcal{P}(a)^{\beta^0(a)} \longrightarrow M \longrightarrow 0.$$

  - Similarly to the standard case, we collect the multiplicities of elements of $\mathcal{C}$ in the minimal $\mathcal{C}$-projective resolution in $\mathcal{P}$-**Betti diagrams** $\beta^d_{\mathcal{P}} M\colon J \to \mathbf{N}$.

Relative Koszul complexes

- **Problem:** we want to compute the $\mathcal{P}$-Betti diagrams of a functor $M\colon I \to \mathbf{vect_k}$.
- **Solution:** we compute the standard Betti diagrams of the functor

$$\mathrm{Nat}(\mathcal{P}(-), M)\colon \begin{cases} J & \to & \mathbf{vect_k} \\ a & \mapsto & \mathrm{Nat}(\mathcal{P}(a), M) \end{cases}$$

using Koszul complexes, and then transfer the diagrams to the relative side.

Theorem

- Let $M\colon I \to \mathbf{vect}_{\mathsf{k}}$ be a functor and suppose that
  - $(J, \preccurlyeq)$ is an upper semilattice and $\mathcal{P}$ is thin,
  - $\{a \in J \mid \mathcal{P}(a) = 0\}$ is closed under joins,
  - for all a, b in J, if a is minimal $\succcurlyeq b$ such that $\mathrm{Nat}(\mathcal{P}(a), \mathcal{P}(b)) = 0$, then $\mathcal{P}(a) = 0$.
- Then, for all a in J such that $\mathcal{P}(a) \neq 0$ and $d \geq 0$,

$$\beta^d_{\mathcal{P}} M(a) = \dim H_d(\mathcal{K}_a \mathrm{Nat}(\mathcal{P}(-), M)).$$

Conclusions

Summary

Given a finite upper semilattice $(J, \preccurlyeq)$ and a thin functor $\mathcal{P}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathbf{vect}_{\mathbf{k}})$:

absolute poset	**relative poset**
$(I, \leq)$	$(J, \preccurlyeq)$
$M\colon I \to \mathbf{vect}_{\mathbf{k}}$	$\mathrm{Nat}(\mathcal{P}(-), M)\colon J \to \mathbf{vect}_{\mathbf{k}}$
copy of $\mathcal{P}(a)$ in the minimal $\mathcal{C}$-projective resolution	copy of $\mathbf{k}[a, -)$ in the minimal free resolution

Koszul complexes of $\mathrm{Nat}(\mathcal{P}(-), M)$ to compute multiplicities in the minimal resolution

Outlook

- **Software implementation** of the computation of Betti diagrams relative to lower hooks.
- **Stability** and **hierarchical stabilization** of relative Betti diagrams.
- Construction of new **computable metrics** for functors.

Thank you for your attention :)

References

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- Preprint soon!