Computational examples of relative Betti diagrams

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The picture so far...

- We consider functors $M\colon I \to \mathbf{vect_k}$ from a finite poset $(I, \leq)$ to $\mathbf{k}$-vector spaces.
- We study their homological algebra relative to collections of functors and derive expressions for the **relative Betti diagrams** using **Koszul complexes**.

Betti diagrams from Koszul complexes

Poset terminology

Let a and b be elements of the poset $(I, \leq)$.

- A **parent of a** is a maximal element smaller than a. We denote by $\mathcal{U}(a)$ the set of parents of a.
- The **join of a and b**, if it exists, is the unique minimal upper bound $a \vee b \geq a, b$.
- The **meet of a and b**, if it exists, is the unique maximal lower bound $a \wedge b \leq a, b$.

Koszul complexes

- Suppose that $(I, \leq)$ is an **upper semilattice**: every two elements a and b have a join $a \vee b$.
- For a functor $M\colon I \to \mathbf{vect}_{\mathsf{k}}$ and a in I, we define the **Koszul complex of M at a** as the chain complex $\mathcal{K}_a M$ where, for all $d \geq 0$,

$$(\mathcal{K}_a M)_d := \bigoplus_{\substack{S \subseteq \mathcal{U}(a) \\ |S| = d \\ S \text{ has lower bound}}} M\Big(\bigwedge\nolimits_{(I \leq a)} S\Big),$$

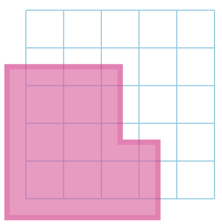
and differential maps of $\mathcal{K}_a M$ are induced from the structure maps of M.

Theorem [Chachólski-Jin-Tombari 2021]

- Let $(I, \leq)$ be an upper semilattice.
- For all functors $M\colon I \to \mathbf{vect_k}$, elements a in I, and $d \geq 0$,

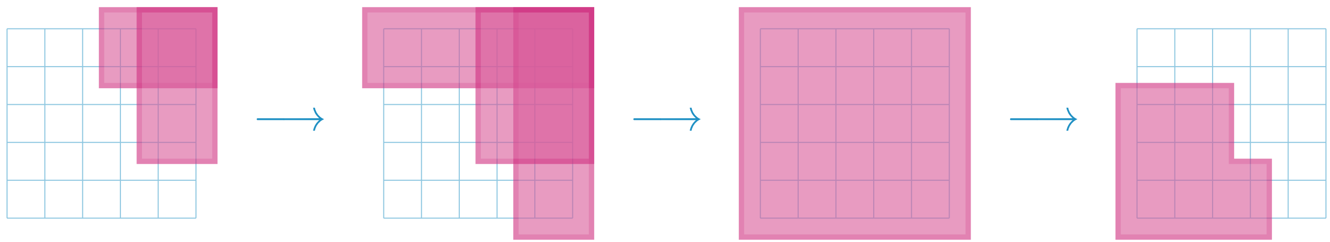
$$\beta^d M(a) = \dim H_d(\mathcal{K}_a M).$$

Example



Betti diagrams from Koszul complexes

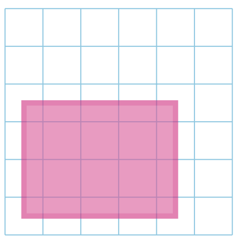
Example



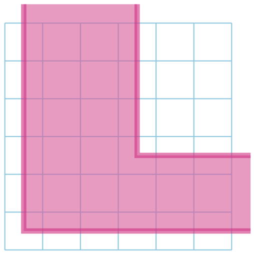
Computation of relative Betti diagrams

Non-free functors

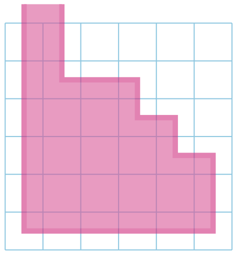
- Instead of resolving with free functors, we can try other shapes:



rectangles



lower hooks
[BOO2021]



single-source spread modules
[BBH2021]

Adjunction

- Let $\mathcal{P}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathbf{vect_k})$ be a parametrization functor.
- This induces and adjoint pair

$$\begin{array}{cccc} \mathcal{R}\colon \ \mathrm{Fun}(I, \mathbf{vect_k}) & \rightleftarrows & \mathrm{Fun}(J, \mathbf{vect_k}) & :\mathcal{L} \\ M & \mapsto & \mathrm{Nat}(\mathcal{P}(-), M) & \\ \mathcal{P}(a) & \leftarrow\!\shortmid & \mathbf{k}_{[a,\infty)} & \end{array}$$

Flat and thin parametrizations

- In particular, for all a in J, we have the morphism induced by the adjunction unit
$$\eta_a \colon \mathbf{k}_{[a,\infty)} \to \mathcal{R}\mathcal{L}\mathbf{k}_{[a,\infty)} = \mathrm{Nat}(\mathcal{P}(-), \mathcal{P}(a)).$$
- The parametrization $\mathcal{P}$ is
  - **flat** if η_a is an isomorphism for all a in J such that $\mathcal{P}(a) \neq 0$,
  - **thin** if η_a is an epimorphism for all a in J.

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Degeneracy locus

Let $\mathcal{P}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathbf{vect_k})$ be a thin parametrization.

- An element a in J is $\mathcal{P}$**-degenerate** if
  - $\mathcal{P}(a) = 0$, or
  - $\mathcal{P}(a) \neq 0$ and $\beta^d_{\mathcal{P}} M(\mathcal{P}(a)) \neq \beta^d \mathcal{R} M(a)$ for some $M\colon I \to \mathbf{vect_k}$ and $d \geq 0$.
- The set of $\mathcal{P}$-degenerate elements is the **degeneracy locus** of $\mathcal{P}$.

Theorem [CGRST 2022]

Suppose that $(J, \preccurlyeq)$ is a finite upper semilattice.

- If $\mathcal{P}$ is flat, then the degeneracy locus is contained in $\{a \in J \mid \mathcal{P}(a) = 0\}$.
- If $\mathcal{P}$ is thin, then the degeneracy locus is contained in

$$\bigcup_{\substack{a \in J \\ d \geq 0}} \operatorname{supp}(\beta^d \ker \eta_a).$$

Flat parametrizations

Upsets

- Consider the collection of **upset functors of $\mathbf{k}_I$**

$$\{\mathbf{k}_U \subseteq \mathbf{k}_I \mid U \in \mathrm{Up}(I)\}.$$

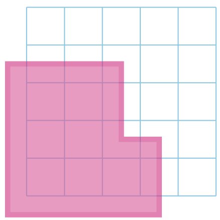
- If I has a unique maximal element, then the parametrization $\mathbf{k}_{-} \colon (\mathrm{Up}(I)^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Fun}(I, \mathbf{vect}_{\mathbf{k}})$ is flat.

Flat parametrizations

- The Koszul complex of a functor M at a nonempty upset $U \in \mathrm{Up}(I)$ is

$$(\mathcal{K}_F \mathcal{R} M)_d = \bigoplus_{\substack{S \subseteq \mathrm{Max}(U^c) \\ |S| = d}} \mathrm{Nat}(\mathbf{k}_{U \cup S}, M).$$

Example



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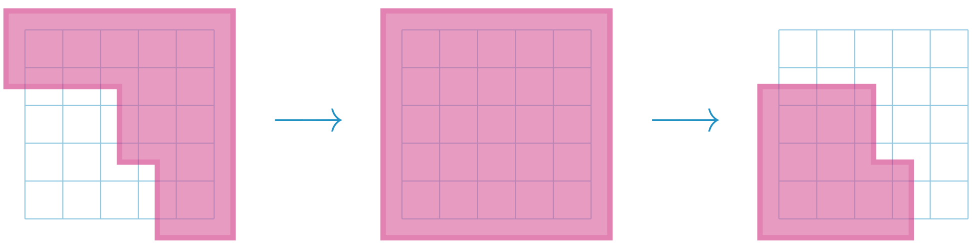


Translated functors

- Let $\mathcal{I} = \{0 < \cdots < n\}^r$ be a grid and fix U a nonempty upset of $\mathcal{I}$.
- Let $v_0 := \max\{v \in \mathcal{I} \mid v + \mathrm{Min}(U) \subseteq \mathcal{I}\}$ and consider the collection of **translated functors**

$$\{\mathbf{k}_{v+U} \mid v \in (\mathcal{I} \leq v_0)\}.$$

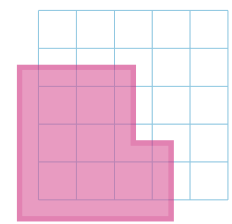
- The parametrization $\mathbf{k}_{-+U} \colon (\mathcal{I} \leq v_0)^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{I}, \mathbf{vect_k})$ is thin.

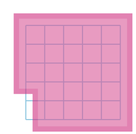
Flat parametrizations

- The Koszul complex of a functor M at $v \in (I \leq v_0)$ is

$$(\mathcal{K}_v \mathcal{R} M)_d = \bigoplus_{\substack{S \subseteq \mathcal{U}_{(I \leq v_0)}(v) \\ |S| = d}} \mathrm{Nat}(\mathbf{k}_{\bigwedge_{(I \leq v_0)} S + U}, M).$$

Example



where $U =$ 

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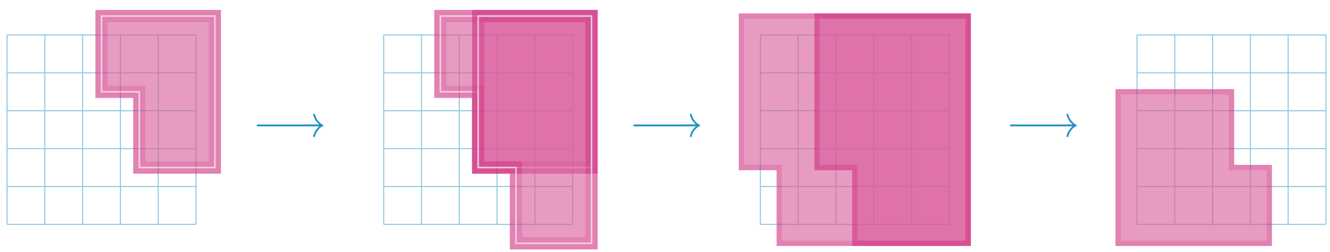
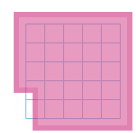


where $U =$ 

Thin parametrizations

Corollary [CGRST 2022]

- Suppose that $(J, \preccurlyeq)$ is a finite upper semilattice and $\mathcal{P}$ is thin.
- **Degeneracy condition:** suppose that, for all a in J, the induced sublattice $\langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle$ is contained in $\{b \in J \mid \mathcal{P}(b) = 0\}$.
- Then, for all functors $M\colon I \to \mathbf{vect_k}$, a in J such that $\mathcal{P}(a) \neq 0$, and $d \geq 0$,

$$\beta^d_{\mathcal{P}} M(a) = \dim H_d(\mathcal{K}_a \mathrm{Nat}(\mathcal{P}(-), M)).$$

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$$\beta^d_{\mathcal{P}} M(a) = \dim H_d(\mathcal{K}_a \mathrm{Nat}(\mathcal{P}(-), M)).$$

Assumption: in all of the following examples, I is an upper semilattice.

Spread modules [Blanchette-Brüstle-Hanson 2021]

- Let S and T be subsets of pairwise incomparable elements of I such that
  - every element s in S is bounded above by an element $t \geq s$ of T,
  - every element t in T is bounded below by an element $s \leq t$ of S.
- The **spread** with **sources** S and **sinks** T is the subset of I

$$[S, T] := \{v \in I \mid \exists s \in S, \exists t \in T, s \leq v \leq t\}.$$

- We then consider the collection of **spread** (or **general interval**) **modules**

$$\{\mathbf{k}_{[S,T]} \mid S, T \text{ as above}\}.$$

Thin parametrizations

- Spread modules are parametrized by the functor

$$\mathcal{Q}\colon \begin{cases} \{U, V \in \mathrm{Up}(I)^{\mathrm{op}} \mid V \supseteq U\}^{\mathrm{op}} & \to & \mathrm{Fun}(I, \mathbf{vect}_{\mathbf{k}}) \\ (U, V) & \mapsto & \mathbf{k}_{V \setminus U} = \mathrm{coker}(\mathbf{k}_U \to \mathbf{k}_V) \end{cases}.$$

- However, if I is not a total order, then no parametrization of spread modules can be thin.

Single-source spread modules [Blanchette-Brüstle-Hanson 2021]

- Instead, consider the subcollection of **single-source spread modules**

$$\{\mathbf{k}_{[\{s\},T]} \mid \{s\}, T \text{ as before}\}.$$

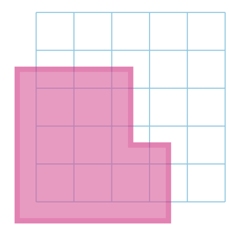
- The restricted (re)parametrization $\mathcal{Q}\colon \{(v, U) \in \mathcal{I} \times \mathrm{Up}(\mathcal{I})^{\mathrm{op}} \mid v \leq U\}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{I}, \mathbf{vect}_{\mathbf{k}})$ is thin.
- The poset $\{(v, U) \in \mathcal{I} \times \mathrm{Up}(\mathcal{I})^{\mathrm{op}} \mid v \leq U\}$ is an upper semilattice.
- The degeneracy condition is satisfied: in particular, $\langle \mathrm{supp}(\beta^0 \ker \eta_{v,U}) \rangle$ is generated by $(u, [u, \infty))$ for $u \in \mathrm{Min}(U)$.

Thin parametrizations

- The Koszul complex of a functor M at (v, U) is

$$(\mathcal{K}_{(v,U)}\mathcal{R}M)_d = \bigoplus_{\substack{S\subseteq\mathcal{U}_I(v),T\subseteq(v\leq\mathrm{Max}(U^c)) \\ |S|+|T|=d \\ S \text{ has lower bound}}} \bigcap_{u\in\mathrm{Min}(U\cup T)} \ker M(\textstyle\bigwedge_{(I\leq v)} S \leq u).$$

Example



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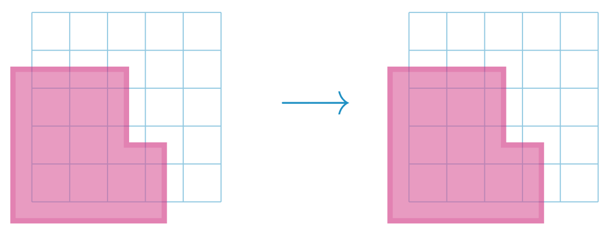


Lower hooks [Botnan-Oppermann-Oudot 2021]

- Consider the collection of **lower hooks**

$$\{\mathrm{coker}(\mathbf{k}_{[w,\infty)} \subseteq \mathbf{k}_{[v,\infty)}) \mid w \leq v \in \mathcal{I}\},$$

  parametrized by the poset $(J, \preccurlyeq) = \{(v, w) \in \mathcal{I}^2 \mid v \leq w\}$ equipped with the product order.

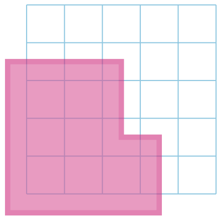
- The restricted (re)parametrization $\mathcal{Q}\colon J^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{I}, \mathbf{vect_k})$ is thin.

- J is an upper semilattice, and the degeneracy condition is satisfied: in particular, $\langle \mathrm{supp}(\beta^0 \ker \eta_{v,w}) \rangle$ is just $\{(w, w)\}$.

- The Koszul complex of a functor M at (v, w) is

$$(\mathcal{K}_{(v,w)}\mathcal{R}M)_d = \bigoplus_{\substack{S \subseteq \mathcal{U}_I(v),\ T \subseteq (v \leq \mathcal{U}_I(w)) \\ |S|+|T|=d \\ S \text{ has lower bound}}} \ker M\left(\bigwedge\nolimits_{(I \leq v)} S \leq \bigwedge\nolimits_{(I \leq w)} T\right).$$

Example



Thin parametrizations

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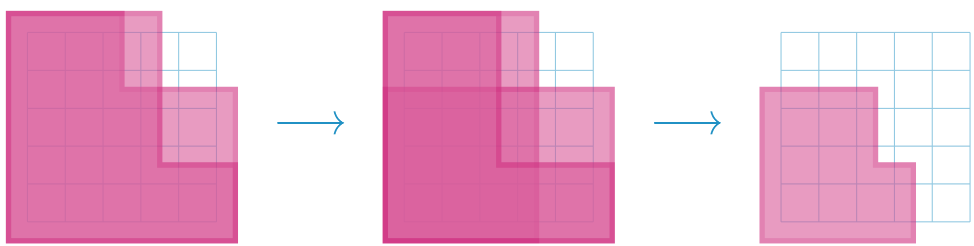

Non-example and partial solution: simple intervals

Simple intervals

- Consider the collection of **simple intervals**

$$\{\mathbf{k}_{[v,w]} \mid (v,w) \in J\}.$$

- The parametrization $\mathbf{k}_{[-,-]} \colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathbf{vect_k})$ is thin and the poset J is an upper semilattice whenever I is.
- However, the degeneracy locus is not well-behaved.
  - In particular, $k_{[v,w]}$ is never the zero functor, but sometimes $\ker \eta_{v,w}$ is nonzero.

Rectangles on a grid

- Let $I = \{0 < \cdots < n\}^r$ be a grid and consider the same collection as before.
- Now identify J with the subposet

$$\left\{\left(v, \bigcup_{i=1}^{r}(v + (w_i - v_i)e_i \leq I)\right) \,\middle|\, v, w \in I, v \leq w\right\}$$

of the poset parametrizing single-source spread modules.

- The parametrization $(v, w) \mapsto \operatorname{coker}\left(\bigoplus_{i=1}^{r} \mathbf{k}_{[v+(w_i-v_i)e_i,\infty)} \to \mathbf{k}_{[v,\infty)}\right)$ is thin, J is an upper semilattice, and the degeneracy condition is satisfied.

Non-example and partial solution: simple intervals

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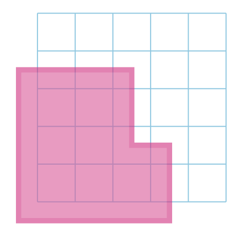

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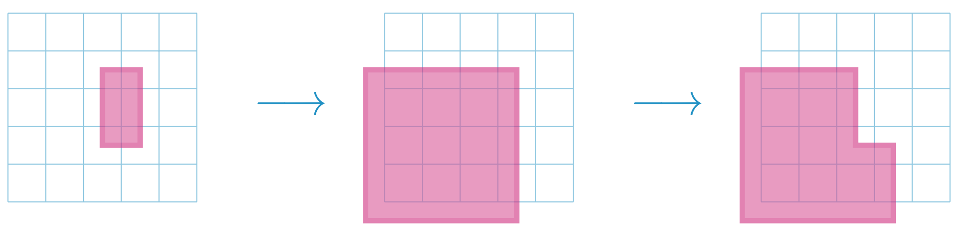

Outlook

- **Software implementation** of the computation of Betti diagrams relative to lower hooks.
- **Stability** and **hierarchical stabilization** of relative Betti diagrams.
- Construction of new **computable metrics** for functors.

Thank you for your attention :)

References

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〉 W. Chachólski, A. Jin, and F. Tombari. *Realisations of posets and tameness*, 2021.

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