The Stochastic Power Law Logistic Model: Derivations of Asymptotic Approximations of the first 3 Cumulants of the QSD, with s=9.

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The procedure **cumf** is used to determine the cumulant of order kk as a function of the (raw) moments of orders up to kk, while the procedure **muf** determines the (raw) moment of order kk as a function of the cumulants of orders up to kk. The procedure **ODEcum3** derives ODEs for the first 3 cumulants.
More details are given in the Maple work-sheet DeriveODECum3.

```
> restart;
```

```
> cumf:=proc(kk)
   local M,K,Ks,cum;
   description "Determines the cumulant of order kk as function of
  the raw moments of orders up to kk";
   M:=1+add(cat(mu,k)*theta^k/k!,k=1..kk);
   K:=log(M);
   Ks:=convert(series(K,theta,kk+1),polynom);
   cum:=sort(simplify(coeff(Ks,theta,kk)*kk!),[seq(cat(mu,j),j=1..
  kk)],plex);
  end proc:
```

```
> muf:=proc(kk)
   local K,M,Ms,mu;
   description "Determines the (raw) moment of order kk as function
  of the cumulants of orders up to kk";
   K:=add(cat(kappa,k)*theta^k/k!,k=1..kk);
   M:=exp(K);
   Ms:=convert(series(M,theta,kk+1),polynom);
   mu:=sort(coeff(Ms,theta,kk)*kk!,[seq(cat(kappa,j),j=1..kk)],
  plex);
  end proc:
```

```
> ODEcum3:=proc(s)
    local Dmu1,Dmu2,Dmu3,sub1,sub2,Dcum1,Dcum2,Dcum3,a,c,d,b;
    description "Derive ODEs for the first 3 cumulants";
    Dmu1:=a*mu1-b*cat(mu,s+1);
    Dmu2:=2*a*mu2+c*mu1-d*cat(mu,s+1)-2*b*cat(mu,s+2);
    Dmu3:=a*(mu1+3*mu3)+3*c*mu2-3*d*cat(mu,s+2)-b*(cat(mu,s+1)+3*
  cat(mu,s+3));
    sub1:=D(mu1)=Dmu1,D(mu2)=Dmu2,D(mu3)=Dmu3;
    sub2:=seq(cat(mu,i)=muf(i),i=1..3+s);
    Dcum1:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
  (1)))),[a,c,d,b])));
```

```
    Dcum2:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
  (2)))),[a,c,d,b])));
    Dcum3:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
  (3)))),[a,c,d,b])));
    a:=mu*(R0-1);
    c:=mu*(R0+1);
    d:=mu*(R0-alpha)/N^s;
    b:=mu*(R0+alpha)/N^s;
    [eval(Dcum1),eval(Dcum2),eval(Dcum3)];
  end proc:
```

Put s=9 and denote the derivatives of the first 3 cumulants by A, B, C.

```
> s:=9;
```

$$s := 9 \quad (1)$$

```
> A:=ODEcum3(s)[1];
  B:=ODEcum3(s)[2];
  C:=ODEcum3(s)[3];
```

$$A := \mu\,(R0-1)\,\kappa 1 - \frac{1}{N^9}\Bigg(10\Bigg(\frac{\kappa 1^{10}}{10} + \frac{9\,\kappa 1^8\,\kappa 2}{2} + 12\,\kappa 1^7\,\kappa 3 + \big(63\,\kappa 2^2 + 21\,\kappa 4\big)\,\kappa 1^6 + \left(252\,\kappa 2\,\kappa 3 + \frac{126\,\kappa 5}{5}\right)\kappa 1^5 + \frac{189\,\kappa 2^5}{2} + \big(315\,\kappa 2^3 + 315\,\kappa 2\,\kappa 4 + 210\,\kappa 3^2 + 21\,\kappa 6\big)\,\kappa 1^4 + 315\,\kappa 2^3\,\kappa 4 + \big(1260\,\kappa 2^2\,\kappa 3 + 252\,\kappa 2\,\kappa 5 + 420\,\kappa 3\,\kappa 4 + 12\,\kappa 7\big)\,\kappa 1^3 + 210\,\kappa 3^2\,\kappa 4 + \left(\frac{945\,\kappa 2^4}{2} + 945\,\kappa 2^2\,\kappa 4 + 252\,\kappa 3\,\kappa 5 + \frac{315\,\kappa 4^2}{2} + \big(1260\,\kappa 3^2 + 126\,\kappa 6\big)\,\kappa 2 + \frac{9\,\kappa 8}{2}\right)\kappa 1^2 + \big(630\,\kappa 3^2 + 63\,\kappa 6\big)\,\kappa 2^2 + 12\,\kappa 3\,\kappa 7 + 21\,\kappa 4\,\kappa 6 + \frac{63\,\kappa 5^2}{5} + \big(1260\,\kappa 2^3\,\kappa 3 + 378\,\kappa 2^2\,\kappa 5 + 280\,\kappa 3^3 + 84\,\kappa 3\,\kappa 6 + 126\,\kappa 4\,\kappa 5 + \big(1260\,\kappa 3\,\kappa 4 + 36\,\kappa 7\big)\,\kappa 2 + \kappa 9\big)\,\kappa 1 + \frac{\kappa 10}{10} + \left(252\,\kappa 3\,\kappa 5 + \frac{315\,\kappa 4^2}{2} + \frac{9\,\kappa 8}{2}\right)\kappa 2\Bigg)\,\mu\,(R0+\alpha)\Bigg)$$

$$B := 2\,\mu\,(R0-1)\,\kappa 2 + \mu\,(R0+1)\,\kappa 1 - \frac{1}{N^9}\Bigg(10\Bigg(\frac{\kappa 1^{10}}{10} + \frac{9\,\kappa 1^8\,\kappa 2}{2} + 12\,\kappa 1^7\,\kappa 3 + \big(63\,\kappa 2^2 + 21\,\kappa 4\big)\,\kappa 1^6 + \left(252\,\kappa 2\,\kappa 3 + \frac{126\,\kappa 5}{5}\right)\kappa 1^5 + \frac{189\,\kappa 2^5}{2} + \big(315\,\kappa 2^3 + 315\,\kappa 2\,\kappa 4 + 210\,\kappa 3^2 + 21\,\kappa 6\big)\,\kappa 1^4 + 315\,\kappa 2^3\,\kappa 4 + \big(1260\,\kappa 2^2\,\kappa 3 + 252\,\kappa 2\,\kappa 5$$

$$+ 420\,\kappa_{3}\,\kappa_{4} + 12\,\kappa_{7})\,\kappa_{1}^{3} + 210\,\kappa_{3}^{2}\,\kappa_{4} + \Big(\frac{945\,\kappa_{2}^{4}}{2} + 945\,\kappa_{2}^{2}\,\kappa_{4} + 252\,\kappa_{3}\,\kappa_{5}$$

$$+ \frac{315\,\kappa_{4}^{2}}{2} + (1260\,\kappa_{3}^{2} + 126\,\kappa_{6})\,\kappa_{2} + \frac{9\,\kappa_{8}}{2}\Big)\,\kappa_{1}^{2} + (630\,\kappa_{3}^{2} + 63\,\kappa_{6})\,\kappa_{2}^{2}$$

$$+ 12\,\kappa_{3}\,\kappa_{7} + 21\,\kappa_{4}\,\kappa_{6} + \frac{63\,\kappa_{5}^{2}}{5} + (1260\,\kappa_{2}^{3}\,\kappa_{3} + 378\,\kappa_{2}^{2}\,\kappa_{5} + 280\,\kappa_{3}^{3}$$

$$+ 84\,\kappa_{3}\,\kappa_{6} + 126\,\kappa_{4}\,\kappa_{5} + (1260\,\kappa_{3}\,\kappa_{4} + 36\,\kappa_{7})\,\kappa_{2} + \kappa_{9})\,\kappa_{1} + \frac{\kappa_{10}}{10}$$

$$+ \Big(\Big(252\,\kappa_{3}\,\kappa_{5} + \frac{315\,\kappa_{4}^{2}}{2} + \frac{9\,\kappa_{8}}{2}\Big)\,\kappa_{2}\Big)\,\mu\,(R0 - \alpha)\Big) - \frac{1}{N^{9}}\Big(90\,\Big(\frac{2\,\kappa_{1}^{9}\,\kappa_{2}}{9}$$

$$+ \kappa_{1}^{8}\,\kappa_{3} + \Big(8\,\kappa_{2}^{2} + \frac{8\,\kappa_{4}}{3}\Big)\,\kappa_{1}^{7} + \Big(\frac{140\,\kappa_{2}\,\kappa_{3}}{3} + \frac{14\,\kappa_{5}}{3}\Big)\,\kappa_{1}^{6} + \Big(84\,\kappa_{2}^{3}$$

$$+ 84\,\kappa_{2}\,\kappa_{4} + 56\,\kappa_{3}^{2} + \frac{28}{5}\,\kappa_{6}\Big)\,\kappa_{1}^{5} + 385\,\kappa_{2}^{4}\,\kappa_{3} + \Big(490\,\kappa_{2}^{2}\,\kappa_{3} + 98\,\kappa_{2}\,\kappa_{5}$$

$$+ \frac{490}{3}\,\kappa_{3}\,\kappa_{4} + \frac{14}{3}\,\kappa_{7}\Big)\,\kappa_{1}^{4} + 154\,\kappa_{2}^{3}\,\kappa_{5} + \Big(280\,\kappa_{2}^{4} + 560\,\kappa_{2}^{2}\,\kappa_{4}$$

$$+ \frac{448\,\kappa_{3}\,\kappa_{5}}{3} + \frac{280\,\kappa_{4}^{2}}{3} + \Big(\frac{2240\,\kappa_{3}^{2}}{3} + \frac{224\,\kappa_{6}}{3}\Big)\,\kappa_{2} + \frac{8\,\kappa_{8}}{3}\Big)\,\kappa_{1}^{3}$$

$$+ \frac{308\,\kappa_{3}^{2}\,\kappa_{5}}{3} + (1260\,\kappa_{2}^{3}\,\kappa_{3} + 378\,\kappa_{2}^{2}\,\kappa_{5} + 280\,\kappa_{3}^{3} + 84\,\kappa_{3}\,\kappa_{6} + 126\,\kappa_{4}\,\kappa_{5}$$

$$+ (1260\,\kappa_{3}\,\kappa_{4} + 36\,\kappa_{7})\,\kappa_{2} + \kappa_{9})\,\kappa_{1}^{2} + (770\,\kappa_{3}\,\kappa_{4} + 22\,\kappa_{7})\,\kappa_{2}^{2} + \frac{22\,\kappa_{4}\,\kappa_{7}}{3}$$

$$+ \frac{154\,\kappa_{5}\,\kappa_{6}}{15} + \Big(210\,\kappa_{2}^{5} + 700\,\kappa_{2}^{3}\,\kappa_{4} + \frac{1400\,\kappa_{3}^{2}\,\kappa_{4}}{3} + (1400\,\kappa_{3}^{2}$$

$$+ 140\,\kappa_{6})\,\kappa_{2}^{2} + \frac{80\,\kappa_{3}\,\kappa_{7}}{3} + \frac{140\,\kappa_{4}\,\kappa_{6}}{3} + 28\,\kappa_{5}^{2} + \frac{2\,\kappa_{10}}{9} + (560\,\kappa_{3}\,\kappa_{5}$$

$$+ 350\,\kappa_{4}^{2} + 10\,\kappa_{8})\,\kappa_{2}\Big)\,\kappa_{1} + \frac{\kappa_{11}}{45} + \Big(\frac{3080}{9}\,\kappa_{3}^{3} + \frac{308}{3}\,\kappa_{3}\,\kappa_{6} + 154\,\kappa_{4}\,\kappa_{5}$$

$$+ \frac{11}{9}\,\kappa_{9}\Big)\,\kappa_{2} + \Big(\frac{385\,\kappa_{4}^{2}}{3} + \frac{11\,\kappa_{8}}{3}\Big)\,\kappa_{3}\Big)\,\mu\,(R0 + \alpha)\Big)$$

$$C := (\kappa_{1} + 3\,\kappa_{3})\,\mu\,(R0 - 1) + 3\,\mu\,(R0 + 1)\,\kappa_{2} - \frac{1}{N^{9}}\Big(135\,\Big(\frac{2\,\kappa_{1}^{9}\,\kappa_{2}}{9} + \kappa_{1}^{8}\,\kappa_{3} \qquad \textbf{(2)}$$

$$+ \Big(8\,\kappa_{2}^{2} + \frac{8\,\kappa_{4}}{3}\Big)\,\kappa_{1}^{7} + \Big(\frac{140\,\kappa_{2}\,\kappa_{3}}{3} + \frac{14\,\kappa_{5}}{3}\Big)\,\kappa_{1}^{6} + \Big(84\,\kappa_{2}^{3} + 84\,\kappa_{2}\,\kappa_{4}$$

$$+ 56\,\kappa_{3}^{2} + \frac{28}{5}\,\kappa_{6}\Big)\,\kappa_{1}^{5} + 385\,\kappa_{2}^{4}\,\kappa_{3} + \Big(490\,\kappa_{2}^{2}\,\kappa_{3} + 98\,\kappa_{2}\,\kappa_{5} + \frac{490}{3}\,\kappa_{3}\,\kappa_{4}$$

$$+ \frac{14}{3} \kappa7 \Big) \kappa1^{4} + 154 \kappa2^{3} \kappa5 + \Big(280 \kappa2^{4} + 560 \kappa2^{2} \kappa4 + \frac{448 \kappa3 \kappa5}{3} + \frac{280 \kappa4^{2}}{3}$$

$$+ \Big(\frac{2240 \kappa3^{2}}{3} + \frac{224 \kappa6}{3} \Big) \kappa2 + \frac{8 \kappa8}{3} \Big) \kappa1^{3} + \frac{308 \kappa3^{2} \kappa5}{3} + \big(1260 \kappa2^{3} \kappa3$$

$$+ 378 \kappa2^{2} \kappa5 + 280 \kappa3^{3} + 84 \kappa3 \kappa6 + 126 \kappa4 \kappa5 + (1260 \kappa3 \kappa4 + 36 \kappa7) \kappa2$$

$$+ \kappa9 \big) \kappa1^{2} + (770 \kappa3 \kappa4 + 22 \kappa7) \kappa2^{2} + \frac{22 \kappa4 \kappa7}{3} + \frac{154 \kappa5 \kappa6}{15} + \Big(210 \kappa2^{5}$$

$$+ 700 \kappa2^{3} \kappa4 + \frac{1400 \kappa3^{2} \kappa4}{3} + (1400 \kappa3^{2} + 140 \kappa6) \kappa2^{2} + \frac{80 \kappa3 \kappa7}{3}$$

$$+ \frac{140 \kappa4 \kappa6}{3} + 28 \kappa5^{2} + \frac{2 \kappa10}{9} + (560 \kappa3 \kappa5 + 350 \kappa4^{2} + 10 \kappa8) \kappa2 \Big) \kappa1$$

$$+ \frac{\kappa11}{45} + \Big(\frac{3080}{9} \kappa3^{3} + \frac{308}{3} \kappa3 \kappa6 + 154 \kappa4 \kappa5 + \frac{11}{9} \kappa9 \Big) \kappa2 + \Big(\frac{385 \kappa4^{2}}{3}$$

$$+ \frac{11 \kappa8}{3} \Big) \kappa3 \Big) \mu (R0 - \alpha) \Big) - \frac{1}{N^{9}} \Big(360 \Big(\frac{\kappa1^{10}}{360} + \frac{\kappa1^{9} \kappa3}{12} + \Big(\frac{3}{4} \kappa2^{2} + \frac{1}{8} \kappa2$$

$$+ \frac{3}{8} \kappa4 \Big) \kappa1^{8} + \Big(9 \kappa2 \kappa3 + \frac{1}{3} \kappa3 + \kappa5 \Big) \kappa1^{7} + \Big(21 \kappa2^{3} + \frac{7}{4} \kappa2^{2} + \frac{49}{2} \kappa2 \kappa4$$

$$+ \frac{35}{2} \kappa3^{2} + \frac{7}{12} \kappa4 + \frac{7}{4} \kappa6 \Big) \kappa1^{6} + \frac{315 \kappa2^{6}}{4} + \Big(\frac{399 \kappa2^{2} \kappa3}{2} + \frac{147 \kappa3 \kappa4}{2}$$

$$+ (7 \kappa3 + 42 \kappa5) \kappa2 + \frac{7 \kappa5}{10} + \frac{21 \kappa7}{10} \Big) \kappa1^{5} + \frac{21 \kappa2^{5}}{8} + \frac{3255 \kappa2^{4} \kappa4}{8}$$

$$+ \Big(\frac{315 \kappa2^{4}}{2} + \frac{35 \kappa2^{3}}{4} + \frac{1365 \kappa2^{2} \kappa4}{4} + \frac{35 \kappa3^{2}}{6} + 98 \kappa3 \kappa5 + \frac{245 \kappa4^{2}}{4}$$

$$+ \Big(\frac{945 \kappa3^{2}}{2} + \frac{35 \kappa4}{4} + \frac{189 \kappa6}{4} \Big) \kappa2 + \frac{7 \kappa6}{12} + \frac{7 \kappa8}{4} \Big) \kappa1^{4} + \frac{385 \kappa3^{4}}{3}$$

$$+ \Big(1155 \kappa2^{3} \kappa3 + 280 \kappa3^{3} + (35 \kappa3 + 357 \kappa5) \kappa2^{2} + 126 \kappa4 \kappa5 + \big(1225 \kappa3 \kappa4$$

$$+ 7 \kappa5 + 35 \kappa7 \big) \kappa2 + \Big(\frac{35 \kappa4}{3} + 84 \kappa6 \Big) \kappa3 + \frac{\kappa7}{3} + \kappa9 \Big) \kappa1^{3} + \Big(\frac{2205 \kappa3^{2}}{2}$$

$$+ \frac{35 \kappa4}{4} + \frac{441 \kappa6}{4} \Big) \kappa2^{3} + \frac{385 \kappa4^{3}}{8} + \Big(315 \kappa2^{5} + \frac{105 \kappa2^{4}}{8} + \frac{2205 \kappa2^{3} \kappa4}{2}$$

$$+ \frac{1575 \kappa3^{2} \kappa4}{2} + \Big(\frac{4515 \kappa3^{2}}{2} + \frac{105 \kappa4}{4} + \frac{903 \kappa6}{4} \Big) \kappa2^{2} + \frac{35 \kappa4^{2}}{8}$$

$$+ \frac{315 \kappa4 \kappa6}{4} + \frac{189 \kappa5^{2}}{4} + \frac{3 \kappa10}{8} + \Big(35 \kappa3^{2} + 924 \kappa3 \kappa5 + \frac{1155}{2} \kappa4^{2} + \frac{7}{2} \kappa6$$

$$+\frac{33}{2}\,\kappa 8\Big)\,\kappa 2+(7\,\kappa 5+45\,\kappa 7)\,\kappa 3+\frac{\kappa 8}{8}\Big)\,\kappa 1^{2}+\Big(\frac{35}{2}\,\kappa 3^{2}+672\,\kappa 3\,\kappa 5$$

$$+420\,\kappa 4^{2}+\frac{7}{4}\,\kappa 6+12\,\kappa 8\Big)\,\kappa 2^{2}+\Big(\frac{35\,\kappa 4}{6}+77\,\kappa 6\Big)\,\kappa 3^{2}+\frac{7\,\kappa 5^{2}}{20}+\frac{33\,\kappa 5\,\kappa 7}{5}$$

$$+\frac{77\,\kappa 6^{2}}{20}+\Big(\frac{5355\,\kappa 2^{4}\,\kappa 3}{4}+(35\,\kappa 3+546\,\kappa 5)\,\kappa 2^{3}+\frac{70\,\kappa 3^{3}}{9}+385\,\kappa 3^{2}\,\kappa 5$$

$$+\Big(\frac{5565\,\kappa 3\,\kappa 4}{2}+\frac{21\,\kappa 5}{2}+\frac{159\,\kappa 7}{2}\Big)\,\kappa 2^{2}+\frac{77\,\kappa 5\,\kappa 6}{2}+\frac{\kappa 11}{12}+\Big(1260\,\kappa 3^{3}$$

$$+567\,\kappa 4\,\kappa 5+(35\,\kappa 4+378\,\kappa 6)\,\kappa 3+\kappa 7+\frac{9\,\kappa 9}{2}\Big)\,\kappa 2+\Big(\frac{1925\,\kappa 4^{2}}{4}+\frac{7\,\kappa 6}{3}$$

$$+\frac{55\,\kappa 8}{4}\Big)\,\kappa 3+\Big(\frac{7\,\kappa 5}{2}+\frac{55\,\kappa 7}{2}\Big)\,\kappa 4+\frac{\kappa 9}{36}\Big)\,\kappa 1+\frac{\kappa 10}{360}+\frac{\kappa 12}{120}$$

$$+\Big(\frac{2275\,\kappa 3^{2}\,\kappa 4}{2}+\frac{35\,\kappa 4^{2}}{8}+\frac{455\,\kappa 4\,\kappa 6}{4}+\frac{273\,\kappa 5^{2}}{4}+\frac{13\,\kappa 10}{24}+(7\,\kappa 5$$

$$+65\,\kappa 7)\,\kappa 3+\frac{\kappa 8}{8}\Big)\,\kappa 2+\Big(231\,\kappa 4\,\kappa 5+\frac{\kappa 7}{3}+\frac{11\,\kappa 9}{6}\Big)\,\kappa 3+\Big(\frac{7\,\kappa 6}{12}$$

$$+\frac{33\,\kappa 8}{8}\Big)\,\kappa 4\Big)\,\mu\,(R0+\alpha)\Big)$$

Determine asymptotic approximations of the first 3 cumulants of the following forms. Assume also that kappa4 - kappa12 are O(N).

```
> kappa1:=x1*N + x2 + x3/N:
  kappa2:=y1*N + y2:
  kappa3:=z1*N:
  kappa4:=u1*N:
  kappa5:=u2*N:
  kappa6:=u3*N:
  kappa7:=u4*N:
  kappa8:=u5*N:
  kappa9:=u6*N:
  kappa10:=u7*N:
  kappa11:=u8*N:
  kappa12:=u9*N:
```

Asymptotic expressions for A, B, C are written A = A1*N + A2 + A3/N, B = B1*N + B2, C = C1*N, where

```
> A1:=coeff(A,N,1); A2:=coeff(A,N,0); A3:=coeff(A,N,-1);
  B1:=coeff(B,N,1); B2:=coeff(B,N,0);
  C1:=coeff(C,N,1);
```

$$A1 := \mu\,(R0-1)\,x1-x1^{10}\,\mu\,(R0+\alpha)$$

$$A2 := \mu\,(R0-1)\,x2-10\,\Big(x2\,x1^{9}+\frac{9}{2}\,x1^{8}\,y1\Big)\,\mu\,(R0+\alpha)$$

$$A3 := \mu\,(R0-1)\,x3 - 10\left(\frac{(x3\,x1^4 + 4\,x2^2\,x1^3 + x1\,(2\,(2\,x3\,x1 + x2^2)\,x1^2 + 4\,x2^2\,x1^2))\,x1^5}{5} + \frac{5\,x2^2\,x1^8}{2} + \frac{9\,x1^8\,y2}{2} + 36\,x2\,x1^7\,y1 + 12\,x1^7\,z1 + 63\,y1^2\,x1^6\right)\mu\,(R0+\alpha)$$

$$B1 := 2\,\mu\,(R0-1)\,y1 + \mu\,(R0+1)\,x1 - x1^{10}\,\mu\,(R0-\alpha) - 20\,x1^9\,y1\,\mu\,(R0+\alpha)$$

$$B2 := 2\,\mu\,(R0-1)\,y2 + \mu\,(R0+1)\,x2 - 10\left(x2\,x1^9 + \frac{9}{2}\,x1^8\,y1\right)\mu\,(R0-\alpha) - 90\left(\frac{2}{9}\,x1^9\,y2 + 2\,x2\,x1^8\,y1 + x1^8\,z1 + 8\,y1^2\,x1^7\right)\mu\,(R0+\alpha)$$

$$C1 := (x1 + 3\,z1)\,\mu\,(R0-1) + 3\,\mu\,(R0+1)\,y1 - 30\,x1^9\,y1\,\mu\,(R0-\alpha) - 360\left(\frac{3}{4}\,y1^2\,x1^8 + \frac{1}{12}\,x1^9\,z1 + \frac{1}{360}\,x1^{10}\right)\mu\,(R0+\alpha) \tag{3}$$

6 equations can now be formed by setting each of A1, A2, A3, B1, B2, C1 equal to zero. These equations can be solved for the 6 unknowns x1, x2, x3, y1, y2, z1. The solutions are found sequentially, as follows:
First we solve A1=0 for x1.This equation has s+1=10 solutions. Among them, we exclude s=9 spurious solution.
After this, B1=0 is solved for for y1, A2=0 is solved for x2, C1=0 is solved for z1, B2=0 is solved for y2, and A3=0 is solved for x3.

```
> x1a:=solve(A1,x1);
  x1:=((R0-1)/(R0+alpha))^(1/9);
```

$$x1a := 0,\ \frac{((R0-1)\,(R0+\alpha)^8)^{1/9}}{R0+\alpha},$$

$$\frac{\left(\cos\left(\frac{2\pi}{9}\right) + \mathrm{I}\cos\left(\frac{5\pi}{18}\right)\right)((R0-1)\,(R0+\alpha)^8)^{1/9}}{R0+\alpha},$$

$$\frac{\left(\cos\left(\frac{4\pi}{9}\right) + \mathrm{I}\cos\left(\frac{\pi}{18}\right)\right)((R0-1)\,(R0+\alpha)^8)^{1/9}}{R0+\alpha},$$

$$\frac{\left(-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)((R0-1)\,(R0+\alpha)^8)^{1/9}}{R0+\alpha},$$

$$\frac{\left(-\cos\left(\frac{\pi}{9}\right) + \mathrm{I}\cos\left(\frac{7\pi}{18}\right)\right)((R0-1)\,(R0+\alpha)^8)^{1/9}}{R0+\alpha},$$

$$\frac{\left(-\cos\left(\frac{\pi}{9}\right) - \mathrm{I}\cos\left(\frac{7\pi}{18}\right)\right)((R0-1)\,(R0+\alpha)^8)^{1/9}}{R0+\alpha},$$

$$\frac{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)\left((R0-1)\,(R0+\alpha)^{8}\right)^{1/9}}{R0+\alpha},$$

$$\frac{\left(\cos\left(\frac{4\,\pi}{9}\right)-I\cos\left(\frac{\pi}{18}\right)\right)\left((R0-1)\,(R0+\alpha)^{8}\right)^{1/9}}{R0+\alpha},$$

$$\frac{\left(\cos\left(\frac{2\,\pi}{9}\right)-I\cos\left(\frac{5\,\pi}{18}\right)\right)\left((R0-1)\,(R0+\alpha)^{8}\right)^{1/9}}{R0+\alpha}$$

$$x1 := \left(\frac{R0-1}{R0+\alpha}\right)^{1/9} \tag{4}$$

```
> y1:=factor(solve(B1,y1));
```

$$y1 := \frac{\left(\frac{R0-1}{R0+\alpha}\right)^{1/9} R0\,(\alpha+1)}{9\,(R0+\alpha)\,(R0-1)} \tag{5}$$

```
> x2:=solve(A2,x2);
```

$$x2 := -\frac{5\,R0\,(\alpha+1)}{9\,(R0+\alpha)\,(R0-1)} \tag{6}$$

```
> z1:=factor(solve(C1,z1));
```

$$z1 := -\frac{\left(\frac{R0-1}{R0+\alpha}\right)^{1/9} R0\,(\alpha+1)\,(9\,R0^{2}-R0\,\alpha-R0+9\,\alpha)}{81\,(R0+\alpha)^{2}\,(R0-1)^{2}} \tag{7}$$

```
> y2:=solve(B2,y2);
```

$$y2 := \frac{5\,(R0^{2}+\alpha)\,R0\,(\alpha+1)}{9\,(R0-1)^{2}\,(R0+\alpha)^{2}} \tag{8}$$

```
> x3:=factor(solve(A3,x3));
```

$$x3 := -\frac{5\,(63\,R0^{2}+5\,R0\,\alpha+5\,R0+63\,\alpha)\left(\frac{R0-1}{R0+\alpha}\right)^{8/9}(\alpha+1)\,R0}{243\,(R0+\alpha)\,(R0-1)^{3}} \tag{9}$$