The Stochastic Power Law Logistic Model: Derivations of Asymptotic Approximations of the first 3 Cumulants of the QSD, with s=7.

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The procedure **cumf** is used to determine the cumulant of order kk as a function of the (raw) moments of orders up to kk, while the procedure **muf** determines the (raw) moment of order kk as a function of the cumulants of orders up to kk. The procedure **ODEcum3** derives ODEs for the first 3 cumulants.
More details are given in the Maple work-sheet DeriveODECum3.

```
> restart;
```

```
> cumf:=proc(kk)
   local M,K,Ks,cum;
   description "Determines the cumulant of order kk as function of
  the raw moments of orders up to kk";
   M:=1+add(cat(mu,k)*theta^k/k!,k=1..kk);
   K:=log(M);
   Ks:=convert(series(K,theta,kk+1),polynom);
   cum:=sort(simplify(coeff(Ks,theta,kk)*kk!),[seq(cat(mu,j),j=1..
  kk)],plex);
  end proc:
```

```
> muf:=proc(kk)
   local K,M,Ms,mu;
   description "Determines the (raw) moment of order kk as function
  of the cumulants of orders up to kk";
   K:=add(cat(kappa,k)*theta^k/k!,k=1..kk);
   M:=exp(K);
   Ms:=convert(series(M,theta,kk+1),polynom);
   mu:=sort(coeff(Ms,theta,kk)*kk!,[seq(cat(kappa,j),j=1..kk)],
  plex);
  end proc:
```

```
> ODEcum3:=proc(s)
    local Dmu1,Dmu2,Dmu3,sub1,sub2,Dcum1,Dcum2,Dcum3,a,c,d,b;
    description "Derive ODEs for the first 3 cumulants";
    Dmu1:=a*mu1-b*cat(mu,s+1);
    Dmu2:=2*a*mu2+c*mu1-d*cat(mu,s+1)-2*b*cat(mu,s+2);
    Dmu3:=a*(mu1+3*mu3)+3*c*mu2-3*d*cat(mu,s+2)-b*(cat(mu,s+1)+3*
  cat(mu,s+3));
    sub1:=D(mu1)=Dmu1,D(mu2)=Dmu2,D(mu3)=Dmu3;
    sub2:=seq(cat(mu,i)=muf(i),i=1..3+s);
    Dcum1:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
  (1)))),[a,c,d,b])));
```

```
  Dcum2:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
(2)))),[a,c,d,b])));
  Dcum3:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
(3)))),[a,c,d,b])));
  a:=mu*(R0-1);
  c:=mu*(R0+1);
  d:=mu*(R0-alpha)/N^s;
  b:=mu*(R0+alpha)/N^s;
  [eval(Dcum1),eval(Dcum2),eval(Dcum3)];
end proc:
```

Put s=7 and denote the derivatives of the first 3 cumulants by A, B, C.

```
> s:=7;
```

$$s := 7 \tag{1}$$

```
> A:=ODEcum3(s)[1];
  B:=ODEcum3(s)[2];
  C:=ODEcum3(s)[3];
```

$$A := \mu\,(R0-1)\,\kappa 1 - \frac{1}{N^7}\Big(\big(\kappa 1^8 + 28\,\kappa 1^6\,\kappa 2 + 56\,\kappa 1^5\,\kappa 3 + \big(210\,\kappa 2^2 + 70\,\kappa 4\big)\,\kappa 1^4 + 105\,\kappa 2^4 + \big(560\,\kappa 2\,\kappa 3 + 56\,\kappa 5\big)\,\kappa 1^3 + 210\,\kappa 2^2\,\kappa 4 + \big(420\,\kappa 2^3 + 420\,\kappa 2\,\kappa 4 + 280\,\kappa 3^2 + 28\,\kappa 6\big)\,\kappa 1^2 + 56\,\kappa 3\,\kappa 5 + 35\,\kappa 4^2 + \big(840\,\kappa 2^2\,\kappa 3 + 168\,\kappa 2\,\kappa 5 + 280\,\kappa 3\,\kappa 4 + 8\,\kappa 7\big)\,\kappa 1 + \big(280\,\kappa 3^2 + 28\,\kappa 6\big)\,\kappa 2 + \kappa 8\big)\,\mu\,(R0+\alpha)\Big)$$

$$B := 2\,\mu\,(R0-1)\,\kappa 2 + \mu\,(R0+1)\,\kappa 1 - \frac{1}{N^7}\Big(\big(\kappa 1^8 + 28\,\kappa 1^6\,\kappa 2 + 56\,\kappa 1^5\,\kappa 3 + \big(210\,\kappa 2^2 + 70\,\kappa 4\big)\,\kappa 1^4 + 105\,\kappa 2^4 + \big(560\,\kappa 2\,\kappa 3 + 56\,\kappa 5\big)\,\kappa 1^3 + 210\,\kappa 2^2\,\kappa 4 + \big(420\,\kappa 2^3 + 420\,\kappa 2\,\kappa 4 + 280\,\kappa 3^2 + 28\,\kappa 6\big)\,\kappa 1^2 + 56\,\kappa 3\,\kappa 5 + 35\,\kappa 4^2 + \big(840\,\kappa 2^2\,\kappa 3 + 168\,\kappa 2\,\kappa 5 + 280\,\kappa 3\,\kappa 4 + 8\,\kappa 7\big)\,\kappa 1 + \big(280\,\kappa 3^2 + 28\,\kappa 6\big)\,\kappa 2 + \kappa 8\big)\,\mu\,(R0-\alpha)\Big) - \frac{1}{N^7}\Big(2\,\big(8\,\kappa 1^7\,\kappa 2 + 28\,\kappa 1^6\,\kappa 3 + \big(168\,\kappa 2^2 + 56\,\kappa 4\big)\,\kappa 1^5 + \big(700\,\kappa 2\,\kappa 3 + 70\,\kappa 5\big)\,\kappa 1^4 + 1260\,\kappa 2^3\,\kappa 3 + \big(840\,\kappa 2^3 + 840\,\kappa 2\,\kappa 4 + 560\,\kappa 3^2 + 56\,\kappa 6\big)\,\kappa 1^3 + 378\,\kappa 2^2\,\kappa 5 + 280\,\kappa 3^3 + \big(2940\,\kappa 2^2\,\kappa 3 + 588\,\kappa 2\,\kappa 5 + 980\,\kappa 3\,\kappa 4 + 28\,\kappa 7\big)\,\kappa 1^2 + 84\,\kappa 3\,\kappa 6 + 126\,\kappa 4\,\kappa 5 + \big(840\,\kappa 2^4 + 1680\,\kappa 2^2\,\kappa 4 + 448\,\kappa 3\,\kappa 5 + 280\,\kappa 4^2 + \big(2240\,\kappa 3^2 + 224\,\kappa 6\big)\,\kappa 2 + 8\,\kappa 8\big)\,\kappa 1 + \big(1260\,\kappa 3\,\kappa 4 + 36\,\kappa 7\big)\,\kappa 2 + \kappa 9\big)\,\mu\,(R0+\alpha)\Big)$$

$$C := (\kappa 1 + 3\,\kappa 3)\,\mu\,(R0-1) + 3\,\mu\,(R0+1)\,\kappa 2 - \frac{1}{N^7}\Big(3\,\big(8\,\kappa 1^7\,\kappa 2 + 28\,\kappa 1^6\,\kappa 3 + \big(168\,\kappa 2^2 + 56\,\kappa 4\big)\,\kappa 1^5 + \big(700\,\kappa 2\,\kappa 3 + 70\,\kappa 5\big)\,\kappa 1^4 + 1260\,\kappa 2^3\,\kappa 3 + \big(840\,\kappa 2^3 + 840\,\kappa 2\,\kappa 4 + 560\,\kappa 3^2 + 56\,\kappa 6\big)\,\kappa 1^3 + 378\,\kappa 2^2\,\kappa 5 + 280\,\kappa 3^3 + \big(2940\,\kappa 2^2\,\kappa 3 + 588\,\kappa 2\,\kappa 5 + 980\,\kappa 3\,\kappa 4 + 28\,\kappa 7\big)\,\kappa 1^2 + 84\,\kappa 3\,\kappa 6 + 126\,\kappa 4\,\kappa 5 + \big(840\,\kappa 2^4 \tag{2}$$

$$+1680\,\kappa2^2\,\kappa4+448\,\kappa3\,\kappa5+280\,\kappa4^2+\left(2240\,\kappa3^2+224\,\kappa6\right)\kappa2+8\,\kappa8\right)\kappa1$$

$$+\left(1260\,\kappa3\,\kappa4+36\,\kappa7\right)\kappa2+\kappa9\big)\,\mu\,(R0-\alpha)\big)-\frac{1}{N^7}\left(24\left(\frac{\kappa1^8}{24}+\kappa1^7\,\kappa3\right.\right.$$

$$+\left(7\,\kappa2^2+\frac{7}{6}\,\kappa2+\frac{7}{2}\,\kappa4\right)\kappa1^6+\left(63\,\kappa2\,\kappa3+\frac{7}{3}\,\kappa3+7\,\kappa5\right)\kappa1^5+105\,\kappa2^5$$

$$+\left(105\,\kappa2^3+\frac{35}{4}\,\kappa2^2+\frac{245}{2}\,\kappa2\,\kappa4+\frac{175}{2}\,\kappa3^2+\frac{35}{12}\,\kappa4+\frac{35}{4}\,\kappa6\right)\kappa1^4$$

$$+\frac{35\,\kappa2^4}{8}+\frac{735\,\kappa2^3\,\kappa4}{2}+\left(665\,\kappa2^2\,\kappa3+245\,\kappa3\,\kappa4+\left(\frac{70\,\kappa3}{3}+140\,\kappa5\right)\kappa2\right.$$

$$\left.+\frac{7\,\kappa5}{3}+7\,\kappa7\right)\kappa1^3+\frac{525\,\kappa3^2\,\kappa4}{2}+\left(315\,\kappa2^4+\frac{35\,\kappa2^3}{2}+\frac{1365\,\kappa2^2\,\kappa4}{2}\right.$$

$$+\frac{35\,\kappa3^2}{3}+196\,\kappa3\,\kappa5+\frac{245\,\kappa4^2}{2}+\left(945\,\kappa3^2+\frac{35\,\kappa4}{2}+\frac{189\,\kappa6}{2}\right)\kappa2+\frac{7\,\kappa6}{6}$$

$$\left.+\frac{7\,\kappa8}{2}\right)\kappa1^2+\left(\frac{1505\,\kappa3^2}{2}+\frac{35\,\kappa4}{4}+\frac{301\,\kappa6}{4}\right)\kappa2^2+\frac{35\,\kappa4^2}{24}+\frac{105\,\kappa4\,\kappa6}{4}$$

$$+\frac{63\,\kappa5^2}{4}+\left(1155\,\kappa2^3\,\kappa3+280\,\kappa3^3+\left(35\,\kappa3+357\,\kappa5\right)\kappa2^2+126\,\kappa4\,\kappa5\right.$$

$$\left.+\left(1225\,\kappa3\,\kappa4+7\,\kappa5+35\,\kappa7\right)\kappa2+\left(\frac{35\,\kappa4}{3}+84\,\kappa6\right)\kappa3+\frac{\kappa7}{3}+\kappa9\right)\kappa1$$

$$+\frac{\kappa10}{8}+\left(\frac{35}{3}\,\kappa3^2+308\,\kappa3\,\kappa5+\frac{385}{2}\,\kappa4^2+\frac{7}{6}\,\kappa6+\frac{11}{2}\,\kappa8\right)\kappa2+\left(\frac{7\,\kappa5}{3}\right.$$

$$\left.\left.+15\,\kappa7\right)\kappa3+\frac{\kappa8}{24}\right)\mu\,(R0+\alpha)\Bigg)$$

Determine asymptotic approximations of the first 3 cumulants of the following forms. Assume also that kappa4 - kappa10 are O(N).

```
> kappa1:=x1*N + x2 + x3/N:
  kappa2:=y1*N + y2:
  kappa3:=z1*N:
  kappa4:=u1*N:
  kappa5:=u2*N:
  kappa6:=u3*N:
  kappa7:=u4*N:
  kappa8:=u5*N:
  kappa9:=u6*N:
  kappa10:=u7*N:
```

Asymptotic expressions for A, B, C are written A = A1*N + A2 + A3/N, B = B1*N + B2, C = C1*N, where

```
> A1:=coeff(A,N,1); A2:=coeff(A,N,0); A3:=coeff(A,N,-1);
  B1:=coeff(B,N,1); B2:=coeff(B,N,0);
  C1:=coeff(C,N,1);
```

$$A1 := \mu\,(R0-1)\,x1 - x1^8\,\mu\,(R0+\alpha)$$
$$A2 := \mu\,(R0-1)\,x2 - (8\,x2\,x1^7 + 28\,x1^6\,y1)\,\mu\,(R0+\alpha)$$
$$A3 := \mu\,(R0-1)\,x3 - (2\,(2\,(2\,x3\,x1 + x2^2)\,x1^2 + 4\,x2^2\,x1^2)\,x1^4 + 16\,x2^2\,x1^6 + 28\,x1^6\,y2 + 168\,x2\,x1^5\,y1 + 56\,x1^5\,z1 + 210\,y1^2\,x1^4)\,\mu\,(R0+\alpha)$$
$$B1 := 2\,\mu\,(R0-1)\,y1 + \mu\,(R0+1)\,x1 - x1^8\,\mu\,(R0-\alpha) - 16\,x1^7\,y1\,\mu\,(R0+\alpha)$$
$$B2 := 2\,\mu\,(R0-1)\,y2 + \mu\,(R0+1)\,x2 - (8\,x2\,x1^7 + 28\,x1^6\,y1)\,\mu\,(R0-\alpha) - 2\,(8\,x1^7\,y2 + 56\,x2\,x1^6\,y1 + 28\,x1^6\,z1 + 168\,y1^2\,x1^5)\,\mu\,(R0+\alpha)$$
$$C1 := (x1 + 3\,z1)\,\mu\,(R0-1) + 3\,\mu\,(R0+1)\,y1 - 24\,x1^7\,y1\,\mu\,(R0-\alpha) - 24\left(\frac{1}{24}\,x1^8 + x1^7\,z1 + 7\,y1^2\,x1^6\right)\mu\,(R0+\alpha) \tag{3}$$

6 equations can now be formed by setting each of A1, A2, A3, B1, B2, C1 equal to zero. These equations can be solved for the 6 unknowns x1, x2, x3, y1, y2, z1. The solutions are found sequentially, as follows:
First we solve A1=0 for x1.This equation has s+1=8 solutions. Among them, we exclude s=7 spurious solution.
After this, B1=0 is solved for for y1, A2=0 is solved for x2, C1=0 is solved for z1, B2=0 is solved for y2, and A3=0 is solved for x3.

```
> x1a:=solve(A1,x1);
  x1:=((R0-1)/(R0+alpha))^(1/7);
```

$$x1a := 0,\ \frac{\left((R0-1)\,(R0+\alpha)^6\right)^{1/7}}{R0+\alpha},$$
$$\frac{\left(\cos\left(\frac{2\,\pi}{7}\right) + \mathrm{I}\cos\left(\frac{3\,\pi}{14}\right)\right)\left((R0-1)\,(R0+\alpha)^6\right)^{1/7}}{R0+\alpha},$$
$$\frac{\left(-\cos\left(\frac{3\,\pi}{7}\right) + \mathrm{I}\cos\left(\frac{\pi}{14}\right)\right)\left((R0-1)\,(R0+\alpha)^6\right)^{1/7}}{R0+\alpha},$$
$$\frac{\left(-\cos\left(\frac{\pi}{7}\right) + \mathrm{I}\cos\left(\frac{5\,\pi}{14}\right)\right)\left((R0-1)\,(R0+\alpha)^6\right)^{1/7}}{R0+\alpha},$$
$$\frac{\left(-\cos\left(\frac{\pi}{7}\right) - \mathrm{I}\cos\left(\frac{5\,\pi}{14}\right)\right)\left((R0-1)\,(R0+\alpha)^6\right)^{1/7}}{R0+\alpha},$$
$$\frac{\left(-\cos\left(\frac{3\,\pi}{7}\right) - \mathrm{I}\cos\left(\frac{\pi}{14}\right)\right)\left((R0-1)\,(R0+\alpha)^6\right)^{1/7}}{R0+\alpha},$$

$$\frac{\left(\cos\left(\frac{2\pi}{7}\right) - I\cos\left(\frac{3\pi}{14}\right)\right)\left((R0-1)\,(R0+\alpha)^6\right)^{1/7}}{R0+\alpha}$$

$$x1 := \left(\frac{R0-1}{R0+\alpha}\right)^{1/7} \quad \textbf{(4)}$$

```
> y1:=factor(solve(B1,y1));
```

$$y1 := \frac{\left(\frac{R0-1}{R0+\alpha}\right)^{1/7} R0\,(\alpha+1)}{7\,(R0+\alpha)\,(R0-1)} \quad \textbf{(5)}$$

```
> x2:=solve(A2,x2);
```

$$x2 := -\frac{4\,R0\,(\alpha+1)}{7\,(R0+\alpha)\,(R0-1)} \quad \textbf{(6)}$$

```
> z1:=factor(solve(C1,z1));
```

$$z1 := -\frac{\left(\frac{R0-1}{R0+\alpha}\right)^{1/7} R0\,(\alpha+1)\,(7\,R0^2 - R0\,\alpha - R0 + 7\,\alpha)}{49\,(R0+\alpha)^2\,(R0-1)^2} \quad \textbf{(7)}$$

```
> y2:=solve(B2,y2);
```

$$y2 := \frac{4\,(R0^2+\alpha)\,R0\,(\alpha+1)}{7\,(R0+\alpha)^2\,(R0-1)^2} \quad \textbf{(8)}$$

```
> x3:=factor(solve(A3,x3));
```

$$x3 := -\frac{2\,R0\,(\alpha+1)\left(\frac{R0-1}{R0+\alpha}\right)^{6/7}(28\,R0^2 + 3\,R0\,\alpha + 3\,R0 + 28\,\alpha)}{49\,(R0+\alpha)\,(R0-1)^3} \quad \textbf{(9)}$$

Summarize the rersults for s=7:

```
> x1, x2, x3, y1, y2, z1;
```

$$\left(\frac{R0-1}{R0+\alpha}\right)^{1/7}, -\frac{4\,R0\,(\alpha+1)}{7\,(R0+\alpha)\,(R0-1)}, \quad \textbf{(10)}$$

$$-\frac{2\,R0\,(\alpha+1)\left(\frac{R0-1}{R0+\alpha}\right)^{6/7}(28\,R0^2 + 3\,R0\,\alpha + 3\,R0 + 28\,\alpha)}{49\,(R0+\alpha)\,(R0-1)^3},$$

$$\frac{\left(\frac{R0-1}{R0+\alpha}\right)^{1/7} R0\,(\alpha+1)}{7\,(R0+\alpha)\,(R0-1)}, \frac{4\,(R0^2+\alpha)\,R0\,(\alpha+1)}{7\,(R0+\alpha)^2\,(R0-1)^2},$$

$$-\frac{\left(\frac{R0-1}{R0+\alpha}\right)^{1/7} R0\,(\alpha+1)\,(7\,R0^2 - R0\,\alpha - R0 + 7\,\alpha)}{49\,(R0+\alpha)^2\,(R0-1)^2}$$

```
>
```