The Stochastic Power Law Logistic Model: Derivations of Asymptotic Approximations of the first 3 Cumulants of the QSD, with s=5.

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The procedure **cumf** is used to determine the cumulant of order kk as a function of the (raw) moments of orders up to kk, while the procedure **muf** determines the (raw) moment of order kk as a function of the cumulants of orders up to kk. The procedure **ODEcum3** derives ODEs for the first 3 cumulants.
More details are given in the Maple work-sheet DeriveODECum3.

```
> restart;
```

```
> cumf:=proc(kk)
   local M,K,Ks,cum;
   description "Determines the cumulant of order kk as function of
  the raw moments of orders up to kk";
   M:=1+add(cat(mu,k)*theta^k/k!,k=1..kk);
   K:=log(M);
   Ks:=convert(series(K,theta,kk+1),polynom);
   cum:=sort(simplify(coeff(Ks,theta,kk)*kk!),[seq(cat(mu,j),j=1..
  kk)],plex);
  end proc:
```

```
> muf:=proc(kk)
   local K,M,Ms,mu;
   description "Determines the (raw) moment of order kk as function
  of the cumulants of orders up to kk";
   K:=add(cat(kappa,k)*theta^k/k!,k=1..kk);
   M:=exp(K);
   Ms:=convert(series(M,theta,kk+1),polynom);
   mu:=sort(coeff(Ms,theta,kk)*kk!,[seq(cat(kappa,j),j=1..kk)],
  plex);
  end proc:
```

```
> ODEcum3:=proc(s)
    local Dmu1,Dmu2,Dmu3,sub1,sub2,Dcum1,Dcum2,Dcum3,a,c,d,b;
    description "Derive ODEs for the first 3 cumulants";
    Dmu1:=a*mu1-b*cat(mu,s+1);
    Dmu2:=2*a*mu2+c*mu1-d*cat(mu,s+1)-2*b*cat(mu,s+2);
    Dmu3:=a*(mu1+3*mu3)+3*c*mu2-3*d*cat(mu,s+2)-b*(cat(mu,s+1)+3*
  cat(mu,s+3));
    sub1:=D(mu1)=Dmu1,D(mu2)=Dmu2,D(mu3)=Dmu3;
    sub2:=seq(cat(mu,i)=muf(i),i=1..3+s);
    Dcum1:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
  (1)))),[a,c,d,b])));
```

```
    Dcum2:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
  (2)))),[a,c,d,b])));
    Dcum3:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
  (3)))),[a,c,d,b])));
    a:=mu*(R0-1);
    c:=mu*(R0+1);
    d:=mu*(R0-alpha)/N^s;
    b:=mu*(R0+alpha)/N^s;
    [eval(Dcum1),eval(Dcum2),eval(Dcum3)];
  end proc:
```

Put s=5 and denote the derivatives of the first 3 cumulants by A, B, C.

```
> s:=5;
```

$$s := 5 \quad (1)$$

```
> A:=ODEcum3(s)[1];
  B:=ODEcum3(s)[2];
  C:=ODEcum3(s)[3];
```

$$A := \mu\,(R0-1)\,\kappa1 - \frac{1}{N^5}\Big(\big(\kappa1^6 + 15\,\kappa1^4\,\kappa2 + 20\,\kappa1^3\,\kappa3 + 15\,\kappa2^3 + \big(45\,\kappa2^2 + 15\,\kappa4\big)\,\kappa1^2 + 15\,\kappa2\,\kappa4 + 10\,\kappa3^2 + \big(60\,\kappa2\,\kappa3 + 6\,\kappa5\big)\,\kappa1 + \kappa6\big)\,\mu\,(R0+\alpha)\Big)$$

$$B := 2\,\mu\,(R0-1)\,\kappa2 + \mu\,(R0+1)\,\kappa1 - \frac{1}{N^5}\Big(\big(\kappa1^6 + 15\,\kappa1^4\,\kappa2 + 20\,\kappa1^3\,\kappa3 + 15\,\kappa2^3 + \big(45\,\kappa2^2 + 15\,\kappa4\big)\,\kappa1^2 + 15\,\kappa2\,\kappa4 + 10\,\kappa3^2 + \big(60\,\kappa2\,\kappa3 + 6\,\kappa5\big)\,\kappa1 + \kappa6\big)\,\mu\,(R0-\alpha)\Big) - \frac{1}{N^5}\Big(2\,\big(6\,\kappa1^5\,\kappa2 + 15\,\kappa1^4\,\kappa3 + 60\,\kappa1^3\,\kappa2^2 + 20\,\kappa1^3\,\kappa4 + 150\,\kappa1^2\,\kappa2\,\kappa3 + 90\,\kappa1\,\kappa2^3 + 15\,\kappa1^2\,\kappa5 + 90\,\kappa1\,\kappa2\,\kappa4 + 60\,\kappa1\,\kappa3^2 + 105\,\kappa2^2\,\kappa3 + 6\,\kappa1\,\kappa6 + 21\,\kappa2\,\kappa5 + 35\,\kappa3\,\kappa4 + \kappa7\big)\,\mu\,(R0+\alpha)\Big)$$

$$C := \big(\kappa1 + 3\,\kappa3\big)\,\mu\,(R0-1) + 3\,\mu\,(R0+1)\,\kappa2 - \frac{1}{N^5}\Big(3\,\big(6\,\kappa1^5\,\kappa2 + 15\,\kappa1^4\,\kappa3 + 60\,\kappa1^3\,\kappa2^2 + 20\,\kappa1^3\,\kappa4 + 150\,\kappa1^2\,\kappa2\,\kappa3 + 90\,\kappa1\,\kappa2^3 + 15\,\kappa1^2\,\kappa5 + 90\,\kappa1\,\kappa2\,\kappa4 + 60\,\kappa1\,\kappa3^2 + 105\,\kappa2^2\,\kappa3 + 6\,\kappa1\,\kappa6 + 21\,\kappa2\,\kappa5 + 35\,\kappa3\,\kappa4 + \kappa7\big)\,\mu\,(R0-\alpha)\Big) - \frac{1}{N^5}\Big(\big(\kappa1^6 + 18\,\kappa1^5\,\kappa3 + \big(90\,\kappa2^2 + 15\,\kappa2 + 45\,\kappa4\big)\,\kappa1^4 + 270\,\kappa2^4 + \big(540\,\kappa2\,\kappa3 + 20\,\kappa3 + 60\,\kappa5\big)\,\kappa1^3 + 15\,\kappa2^3 + 585\,\kappa2^2\,\kappa4 + \big(540\,\kappa2^3 + 45\,\kappa2^2 + 630\,\kappa2\,\kappa4 + 450\,\kappa3^2 + 15\,\kappa4 + 45\,\kappa6\big)\,\kappa1^2 + 10\,\kappa3^2 + 168\,\kappa3\,\kappa5 + 105\,\kappa4^2 + \big(1710\,\kappa2^2\,\kappa3 + 630\,\kappa3\,\kappa4 + \big(60\,\kappa3 + 360\,\kappa5\big)\,\kappa2 + 6\,\kappa5 + 18\,\kappa7\big)\,\kappa1 + \big(810\,\kappa3^2 + 15\,\kappa4 + 81\,\kappa6\big)\,\kappa2 + \kappa6 + 3\,\kappa8\big)\,\mu\,(R0+\alpha)\Big) \quad (2)$$

Determine asymptotic approximations of the first 3 cumulants of the following forms. Assume also that kappa4 - kappa8 are O(N).

```
> kappa1:=x1*N + x2 + x3/N:
```

```
kappa2:=y1*N + y2:
kappa3:=z1*N:
kappa4:=u1*N:
kappa5:=u2*N:
kappa6:=u3*N:
kappa7:=u4*N:
kappa8:=u5*N:
```

Asymptotic expressions for A, B, C are written A=A1*N + A2 + A3/N, B=B1*N + B2, C =C1*N, where

```
> A1:=coeff(A,N,1); A2:=coeff(A,N,0); A3:=coeff(A,N,-1);
  B1:=coeff(B,N,1); B2:=coeff(B,N,0);
  C1:=coeff(C,N,1);
```

$$A1 := \mu\,(R0-1)\,x1 - x1^6\,\mu\,(R0+\alpha)$$

$$A2 := \mu\,(R0-1)\,x2 - \left(6\,x2\,x1^5 + 15\,x1^4\,y1\right)\mu\,(R0+\alpha)$$

$$A3 := \mu\,(R0-1)\,x3 - \left(60\,x2\,x1^3\,y1 + 15\,x1^4\,y2 + 20\,x1^3\,z1 + 45\,x1^2\,y1^2 + 2\,\left(x3\,x1^2 + 2\,x2^2\,x1 + x1\,\left(2\,x3\,x1 + x2^2\right)\right)x1^3 + 9\,x2^2\,x1^4\right)\mu\,(R0+\alpha)$$

$$B1 := 2\,\mu\,(R0-1)\,y1 + \mu\,(R0+1)\,x1 - x1^6\,\mu\,(R0-\alpha) - 12\,x1^5\,y1\,\mu\,(R0+\alpha)$$

$$B2 := 2\,\mu\,(R0-1)\,y2 + \mu\,(R0+1)\,x2 - \left(6\,x2\,x1^5 + 15\,x1^4\,y1\right)\mu\,(R0-\alpha) - 2\,\left(6\,x1^5\,y2 + 30\,x2\,x1^4\,y1 + 15\,x1^4\,z1 + 60\,x1^3\,y1^2\right)\mu\,(R0+\alpha)$$

$$C1 := (x1 + 3\,z1)\,\mu\,(R0-1) + 3\,\mu\,(R0+1)\,y1 - 18\,x1^5\,y1\,\mu\,(R0-\alpha) - \left(x1^6 + 18\,x1^5\,z1 + 90\,y1^2\,x1^4\right)\mu\,(R0+\alpha) \qquad \textbf{(3)}$$

6 equations can now be formed by setting each of A1, A2, A3, B1, B2, C1 equal to zero. These equations can be solved for the 6 unknowns x1, x2, x3, y1, y2, z1. The solutions are found sequentially, as follows:

First we solve A1=0 for x1.This equation has s+1=6 solutions. Among them, we exclude s=5 spurious solution.

After this, B1=0 is solved for for y1, A2=0 is solved for x2, C1=0 is solved for z1, B2=0 is solved for y2, and A3=0 is solved for x3.

```
> x1a:=solve(A1,x1);
  x1:=((R0-1)/(R0+alpha))^(1/5);
```

$$x1a := 0,\ \frac{\left((R0-1)\,(R0+\alpha)^4\right)^{1/5}}{R0+\alpha},$$

$$\frac{\left(-\frac{1}{4} + \frac{\sqrt{5}}{4} + \frac{I\sqrt{2}\,\sqrt{5+\sqrt{5}}}{4}\right)\left((R0-1)\,(R0+\alpha)^4\right)^{1/5}}{R0+\alpha},$$

$$\frac{\left(-\frac{1}{4} - \frac{\sqrt{5}}{4} + \frac{I\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right)\left((R0-1)\,(R0+\alpha)^4\right)^{1/5}}{R0+\alpha},$$

$$\frac{\left(-\frac{1}{4}-\frac{\sqrt{5}}{4}-\frac{I\sqrt{2}\sqrt{5-\sqrt{5}}}{4}\right)\left((R0-1)(R0+\alpha)^4\right)^{1/5}}{R0+\alpha},$$

$$\frac{\left(-\frac{1}{4}+\frac{\sqrt{5}}{4}-\frac{I\sqrt{2}\sqrt{5+\sqrt{5}}}{4}\right)\left((R0-1)(R0+\alpha)^4\right)^{1/5}}{R0+\alpha}$$

$$x1 := \left(\frac{R0-1}{R0+\alpha}\right)^{1/5} \quad (4)$$

```
> y1:=factor(solve(B1,y1));
```

$$y1 := \frac{\left(\frac{R0-1}{R0+\alpha}\right)^{1/5} R0\,(\alpha+1)}{5\,(R0+\alpha)\,(R0-1)} \quad (5)$$

```
> x2:=solve(A2,x2);
```

$$x2 := -\frac{3\,R0\,(\alpha+1)}{5\,(R0+\alpha)\,(R0-1)} \quad (6)$$

```
> z1:=factor(solve(C1,z1));
```

$$z1 := -\frac{R0\left(\frac{R0-1}{R0+\alpha}\right)^{1/5}(\alpha+1)\,(5\,R0^2-R0\,\alpha-R0+5\,\alpha)}{25\,(R0+\alpha)^2\,(R0-1)^2} \quad (7)$$

```
> y2:=solve(B2,y2);
```

$$y2 := \frac{3\,(R0^2+\alpha)\,R0\,(\alpha+1)}{5\,(R0+\alpha)^2\,(R0-1)^2} \quad (8)$$

```
> x3:=factor(solve(A3,x3));
```

$$x3 := -\frac{(\alpha+1)\,(25\,R0^2+4\,R0\,\alpha+4\,R0+25\,\alpha)\left(\frac{R0-1}{R0+\alpha}\right)^{4/5} R0}{25\,(R0-1)^3\,(R0+\alpha)} \quad (9)$$

Sumarize the results for s=5:

```
> x1, x2, x3, y1, y2, z1;
```

$$\left(\frac{R0-1}{R0+\alpha}\right)^{1/5},\ -\frac{3\,R0\,(\alpha+1)}{5\,(R0+\alpha)\,(R0-1)}, \quad (10)$$

$$-\frac{(\alpha+1)\,(25\,R0^2+4\,R0\,\alpha+4\,R0+25\,\alpha)\left(\frac{R0-1}{R0+\alpha}\right)^{4/5} R0}{25\,(R0-1)^3\,(R0+\alpha)},$$

$$\frac{\left(\frac{R0-1}{R0+\alpha}\right)^{1/5} R0\,(\alpha+1)}{5\,(R0+\alpha)\,(R0-1)},\ \frac{3\,(R0^2+\alpha)\,R0\,(\alpha+1)}{5\,(R0+\alpha)^2\,(R0-1)^2},$$

$$-\frac{R0\left(\frac{R0-1}{R0+\alpha}\right)^{1/5}(\alpha+1)\left(5\,R0^2-R0\,\alpha-R0+5\,\alpha\right)}{25\,(R0+\alpha)^2\,(R0-1)^2}$$

>