The Stochastic Power Law Logistic Model: Derivations of Asymptotic Approximations of the first 3 Cumulants of the QSD, with s=10.

Ingemar Nåsell

The procedure **cumf** is used to determine the cumulant of order kk as a function of the (raw) moments of orders up to kk, while the procedure **muf** determines the (raw) moment of order kk as a function of the cumulants of orders up to kk. The procedure **ODEcum3** derives ODEs for the first 3 cumulants.
More details are given in the Maple work-sheet DeriveODECum3.

```
> restart;
```

```
> cumf:=proc(kk)
   local M,K,Ks,cum;
   description "Determines the cumulant of order kk as function of
  the raw moments of orders up to kk";
   M:=1+add(cat(mu,k)*theta^k/k!,k=1..kk);
   K:=log(M);
   Ks:=convert(series(K,theta,kk+1),polynom);
   cum:=sort(simplify(coeff(Ks,theta,kk)*kk!),[seq(cat(mu,j),j=1..
  kk)],plex);
  end proc:
```

```
> muf:=proc(kk)
   local K,M,Ms,mu;
   description "Determines the (raw) moment of order kk as function
  of the cumulants of orders up to kk";
   K:=add(cat(kappa,k)*theta^k/k!,k=1..kk);
   M:=exp(K);
   Ms:=convert(series(M,theta,kk+1),polynom);
   mu:=sort(coeff(Ms,theta,kk)*kk!,[seq(cat(kappa,j),j=1..kk)],
  plex);
  end proc:
```

```
> ODEcum3:=proc(s)
    local Dmu1,Dmu2,Dmu3,sub1,sub2,Dcum1,Dcum2,Dcum3,a,c,d,b;
    description "Derive ODEs for the first 3 cumulants";
    Dmu1:=a*mu1-b*cat(mu,s+1);
    Dmu2:=2*a*mu2+c*mu1-d*cat(mu,s+1)-2*b*cat(mu,s+2);
    Dmu3:=a*(mu1+3*mu3)+3*c*mu2-3*d*cat(mu,s+2)-b*(cat(mu,s+1)+3*
  cat(mu,s+3));
    sub1:=D(mu1)=Dmu1,D(mu2)=Dmu2,D(mu3)=Dmu3;
    sub2:=seq(cat(mu,i)=muf(i),i=1..3+s);
    Dcum1:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
  (1)))),[a,c,d,b])));
```

```
  Dcum2:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
(2)))),[a,c,d,b])));
  Dcum3:=map(sort,map(simplify,collect(subs(sub2,subs(sub1,D(cumf
(3)))),[a,c,d,b])));
  a:=mu*(R0-1);
  c:=mu*(R0+1);
  d:=mu*(R0-alpha)/N^s;
  b:=mu*(R0+alpha)/N^s;
  [eval(Dcum1),eval(Dcum2),eval(Dcum3)];
end proc:
```

Put s=10 and denote the derivatives of the first 3 cumulants by A, B, C.

```
> s:=10;
```

$$s := 10 \quad (1)$$

```
> A:=ODEcum3(s)[1];
  B:=ODEcum3(s)[2];
  C:=ODEcum3(s)[3];
```

$$A := \mu\,(R0-1)\,\kappa 1 - \frac{1}{N^{10}}\Bigg(55\Bigg(\frac{\kappa 1^{11}}{55} + \kappa 1^{9}\,\kappa 2 + 3\,\kappa 1^{8}\,\kappa 3 + \left(18\,\kappa 2^{2} + 6\,\kappa 4\right)\kappa 1^{7} + \left(84\,\kappa 2\,\kappa 3 + \frac{42\,\kappa 5}{5}\right)\kappa 1^{6} + \left(126\,\kappa 2^{3} + 126\,\kappa 2\,\kappa 4 + 84\,\kappa 3^{2} + \frac{42}{5}\,\kappa 6\right)\kappa 1^{5} + 315\,\kappa 2^{4}\,\kappa 3 + \left(630\,\kappa 2^{2}\,\kappa 3 + 126\,\kappa 2\,\kappa 5 + 210\,\kappa 3\,\kappa 4 + 6\,\kappa 7\right)\kappa 1^{4} + 126\,\kappa 2^{3}\,\kappa 5 + \left(315\,\kappa 2^{4} + 630\,\kappa 2^{2}\,\kappa 4 + 168\,\kappa 3\,\kappa 5 + 105\,\kappa 4^{2} + \left(840\,\kappa 3^{2} + 84\,\kappa 6\right)\kappa 2 + 3\,\kappa 8\right)\kappa 1^{3} + 84\,\kappa 3^{2}\,\kappa 5 + \left(1260\,\kappa 2^{3}\,\kappa 3 + 378\,\kappa 2^{2}\,\kappa 5 + 280\,\kappa 3^{3} + 84\,\kappa 3\,\kappa 6 + 126\,\kappa 4\,\kappa 5 + \left(1260\,\kappa 3\,\kappa 4 + 36\,\kappa 7\right)\kappa 2 + \kappa 9\right)\kappa 1^{2} + \left(630\,\kappa 3\,\kappa 4 + 18\,\kappa 7\right)\kappa 2^{2} + 6\,\kappa 4\,\kappa 7 + \frac{42\,\kappa 5\,\kappa 6}{5} + \Bigg(189\,\kappa 2^{5} + 630\,\kappa 2^{3}\,\kappa 4 + 420\,\kappa 3^{2}\,\kappa 4 + \left(1260\,\kappa 3^{2} + 126\,\kappa 6\right)\kappa 2^{2} + 24\,\kappa 3\,\kappa 7 + 42\,\kappa 4\,\kappa 6 + \frac{126\,\kappa 5^{2}}{5} + \frac{\kappa 10}{5} + \left(504\,\kappa 3\,\kappa 5 + 315\,\kappa 4^{2} + 9\,\kappa 8\right)\kappa 2\Bigg)\kappa 1 + \frac{\kappa 11}{55} + \left(280\,\kappa 3^{3} + 84\,\kappa 3\,\kappa 6 + 126\,\kappa 4\,\kappa 5 + \kappa 9\right)\kappa 2 + \left(105\,\kappa 4^{2} + 3\,\kappa 8\right)\kappa 3\Bigg)\mu\,(R0+\alpha)\Bigg)$$

$$B := 2\,\mu\,(R0-1)\,\kappa 2 + \mu\,(R0+1)\,\kappa 1 - \frac{1}{N^{10}}\Bigg(55\Bigg(\frac{\kappa 1^{11}}{55} + \kappa 1^{9}\,\kappa 2 + 3\,\kappa 1^{8}\,\kappa 3$$

$$+\left(18\,\kappa 2^{2}+6\,\kappa 4\right)\kappa 1^{7}+\left(84\,\kappa 2\,\kappa 3+\frac{42\,\kappa 5}{5}\right)\kappa 1^{6}+\left(126\,\kappa 2^{3}+126\,\kappa 2\,\kappa 4\right.$$

$$\left.+84\,\kappa 3^{2}+\frac{42}{5}\,\kappa 6\right)\kappa 1^{5}+315\,\kappa 2^{4}\,\kappa 3+\left(630\,\kappa 2^{2}\,\kappa 3+126\,\kappa 2\,\kappa 5+210\,\kappa 3\,\kappa 4\right.$$

$$\left.+6\,\kappa 7\right)\kappa 1^{4}+126\,\kappa 2^{3}\,\kappa 5+\left(315\,\kappa 2^{4}+630\,\kappa 2^{2}\,\kappa 4+168\,\kappa 3\,\kappa 5+105\,\kappa 4^{2}\right.$$

$$\left.+\left(840\,\kappa 3^{2}+84\,\kappa 6\right)\kappa 2+3\,\kappa 8\right)\kappa 1^{3}+84\,\kappa 3^{2}\,\kappa 5+\left(1260\,\kappa 2^{3}\,\kappa 3+378\,\kappa 2^{2}\,\kappa 5\right.$$

$$\left.+280\,\kappa 3^{3}+84\,\kappa 3\,\kappa 6+126\,\kappa 4\,\kappa 5+\left(1260\,\kappa 3\,\kappa 4+36\,\kappa 7\right)\kappa 2+\kappa 9\right)\kappa 1^{2}$$

$$+\left(630\,\kappa 3\,\kappa 4+18\,\kappa 7\right)\kappa 2^{2}+6\,\kappa 4\,\kappa 7+\frac{42\,\kappa 5\,\kappa 6}{5}+\left(189\,\kappa 2^{5}+630\,\kappa 2^{3}\,\kappa 4\right.$$

$$+420\,\kappa 3^{2}\,\kappa 4+\left(1260\,\kappa 3^{2}+126\,\kappa 6\right)\kappa 2^{2}+24\,\kappa 3\,\kappa 7+42\,\kappa 4\,\kappa 6+\frac{126\,\kappa 5^{2}}{5}$$

$$\left.+\frac{\kappa 10}{5}+\left(504\,\kappa 3\,\kappa 5+315\,\kappa 4^{2}+9\,\kappa 8\right)\kappa 2\right)\kappa 1+\frac{\kappa 11}{55}+\left(280\,\kappa 3^{3}+84\,\kappa 3\,\kappa 6\right.$$

$$\left.\left.+126\,\kappa 4\,\kappa 5+\kappa 9\right)\kappa 2+\left(105\,\kappa 4^{2}+3\,\kappa 8\right)\kappa 3\right)\mu\left(R0-\alpha\right)\Big)$$

$$-\frac{1}{N^{10}}\left(330\left(\frac{\kappa 1^{10}\,\kappa 2}{15}+\frac{\kappa 1^{9}\,\kappa 3}{3}+\left(3\,\kappa 2^{2}+\kappa 4\right)\kappa 1^{8}+\left(20\,\kappa 2\,\kappa 3+2\,\kappa 5\right)\kappa 1^{7}\right.\right.$$

$$+\left(42\,\kappa 2^{3}+42\,\kappa 2\,\kappa 4+28\,\kappa 3^{2}+\frac{14}{5}\,\kappa 6\right)\kappa 1^{6}+63\,\kappa 2^{6}+\left(294\,\kappa 2^{2}\,\kappa 3\right.$$

$$\left.+\frac{294}{5}\,\kappa 2\,\kappa 5+98\,\kappa 3\,\kappa 4+\frac{14}{5}\,\kappa 7\right)\kappa 1^{5}+315\,\kappa 2^{4}\,\kappa 4+\left(210\,\kappa 2^{4}+420\,\kappa 2^{2}\,\kappa 4\right.$$

$$\left.+112\,\kappa 3\,\kappa 5+70\,\kappa 4^{2}+\left(560\,\kappa 3^{2}+56\,\kappa 6\right)\kappa 2+2\,\kappa 8\right)\kappa 1^{4}+\frac{280\,\kappa 3^{4}}{3}$$

$$+\left(1260\,\kappa 2^{3}\,\kappa 3+378\,\kappa 2^{2}\,\kappa 5+280\,\kappa 3^{3}+84\,\kappa 3\,\kappa 6+126\,\kappa 4\,\kappa 5+\left(1260\,\kappa 3\,\kappa 4\right.\right.$$

$$\left.\left.+36\,\kappa 7\right)\kappa 2+\kappa 9\right)\kappa 1^{3}+\left(840\,\kappa 3^{2}+84\,\kappa 6\right)\kappa 2^{3}+56\,\kappa 3^{2}\,\kappa 6+35\,\kappa 4^{3}$$

$$+\left(315\,\kappa 2^{5}+1050\,\kappa 2^{3}\,\kappa 4+700\,\kappa 3^{2}\,\kappa 4+\left(2100\,\kappa 3^{2}+210\,\kappa 6\right)\kappa 2^{2}+40\,\kappa 3\,\kappa 7\right.$$

$$\left.+70\,\kappa 4\,\kappa 6+42\,\kappa 5^{2}+\frac{\kappa 10}{3}+\left(840\,\kappa 3\,\kappa 5+525\,\kappa 4^{2}+15\,\kappa 8\right)\kappa 2\right)\kappa 1^{2}$$

$$+\left(504\,\kappa 3\,\kappa 5+315\,\kappa 4^{2}+9\,\kappa 8\right)\kappa 2^{2}+3\,\kappa 4\,\kappa 8+\frac{24\,\kappa 5\,\kappa 7}{5}+\frac{14\,\kappa 6^{2}}{5}$$

$$+\Big(1155\,\kappa2^{4}\,\kappa3+462\,\kappa2^{3}\,\kappa5+308\,\kappa3^{2}\,\kappa5+(2310\,\kappa3\,\kappa4+66\,\kappa7)\,\kappa2^{2}$$

$$+22\,\kappa4\,\kappa7+\frac{154\,\kappa5\,\kappa6}{5}+\frac{\kappa11}{15}+\Big(\frac{3080}{3}\,\kappa3^{3}+308\,\kappa3\,\kappa6+462\,\kappa4\,\kappa5$$

$$+\frac{11}{3}\,\kappa9\Big)\,\kappa2+(385\,\kappa4^{2}+11\,\kappa8)\,\kappa3\Big)\,\kappa1+\frac{\kappa12}{165}+\Big(840\,\kappa3^{2}\,\kappa4+48\,\kappa3\,\kappa7$$

$$+84\,\kappa4\,\kappa6+\frac{252}{5}\,\kappa5^{2}+\frac{2}{5}\,\kappa10\Big)\,\kappa2+\Big(168\,\kappa4\,\kappa5+\frac{4\,\kappa9}{3}\Big)\,\kappa3\Big)\,\mu\,(R0+\alpha)\Big)$$

$$C:=(\kappa1+3\,\kappa3)\,\mu\,(R0-1)+3\,\mu\,(R0+1)\,\kappa2-\frac{1}{N^{10}}\Big(495\,\Big(\frac{\kappa1^{10}\,\kappa2}{15}+\frac{\kappa1^{9}\,\kappa3}{3} \qquad \textbf{(2)}$$

$$+(3\,\kappa2^{2}+\kappa4)\,\kappa1^{8}+(20\,\kappa2\,\kappa3+2\,\kappa5)\,\kappa1^{7}+\Big(42\,\kappa2^{3}+42\,\kappa2\,\kappa4+28\,\kappa3^{2}$$

$$+\frac{14}{5}\,\kappa6\Big)\,\kappa1^{6}+63\,\kappa2^{6}+\Big(294\,\kappa2^{2}\,\kappa3+\frac{294}{5}\,\kappa2\,\kappa5+98\,\kappa3\,\kappa4+\frac{14}{5}\,\kappa7\Big)\,\kappa1^{5}$$

$$+315\,\kappa2^{4}\,\kappa4+(210\,\kappa2^{4}+420\,\kappa2^{2}\,\kappa4+112\,\kappa3\,\kappa5+70\,\kappa4^{2}+(560\,\kappa3^{2}$$

$$+56\,\kappa6)\,\kappa2+2\,\kappa8)\,\kappa1^{4}+\frac{280\,\kappa3^{4}}{3}+(1260\,\kappa2^{3}\,\kappa3+378\,\kappa2^{2}\,\kappa5+280\,\kappa3^{3}$$

$$+84\,\kappa3\,\kappa6+126\,\kappa4\,\kappa5+(1260\,\kappa3\,\kappa4+36\,\kappa7)\,\kappa2+\kappa9)\,\kappa1^{3}+(840\,\kappa3^{2}$$

$$+84\,\kappa6)\,\kappa2^{3}+56\,\kappa3^{2}\,\kappa6+35\,\kappa4^{3}+\Big(315\,\kappa2^{5}+1050\,\kappa2^{3}\,\kappa4+700\,\kappa3^{2}\,\kappa4$$

$$+(2100\,\kappa3^{2}+210\,\kappa6)\,\kappa2^{2}+40\,\kappa3\,\kappa7+70\,\kappa4\,\kappa6+42\,\kappa5^{2}+\frac{\kappa10}{3}$$

$$+(840\,\kappa3\,\kappa5+525\,\kappa4^{2}+15\,\kappa8)\,\kappa2\Big)\,\kappa1^{2}+(504\,\kappa3\,\kappa5+315\,\kappa4^{2}+9\,\kappa8)\,\kappa2^{2}$$

$$+3\,\kappa4\,\kappa8+\frac{24\,\kappa5\,\kappa7}{5}+\frac{14\,\kappa6^{2}}{5}+\Big(1155\,\kappa2^{4}\,\kappa3+462\,\kappa2^{3}\,\kappa5+308\,\kappa3^{2}\,\kappa5$$

$$+(2310\,\kappa3\,\kappa4+66\,\kappa7)\,\kappa2^{2}+22\,\kappa4\,\kappa7+\frac{154\,\kappa5\,\kappa6}{5}+\frac{\kappa11}{15}+\Big(\frac{3080}{3}\,\kappa3^{3}$$

$$+308\,\kappa3\,\kappa6+462\,\kappa4\,\kappa5+\frac{11}{3}\,\kappa9\Big)\,\kappa2+(385\,\kappa4^{2}+11\,\kappa8)\,\kappa3\Big)\,\kappa1+\frac{\kappa12}{165}$$

$$+\Big(840\,\kappa3^{2}\,\kappa4+48\,\kappa3\,\kappa7+84\,\kappa4\,\kappa6+\frac{252}{5}\,\kappa5^{2}+\frac{2}{5}\,\kappa10\Big)\,\kappa2+\Big(168\,\kappa4\,\kappa5$$

$$+\frac{4\,\kappa9}{3}\Big)\,\kappa3\Big)\,\mu\,(R0-\alpha)\Big)-\frac{1}{N^{10}}\Big(990\,\Big(\frac{\kappa1^{11}}{990}+\frac{\kappa1^{10}\,\kappa3}{30}+\Big(\frac{1}{3}\,\kappa2^{2}$$

$$+ \frac{1}{18} \kappa2 + \frac{1}{6} \kappa4) \kappa1^{9} + (\frac{9}{2} \kappa2 \kappa3 + \frac{1}{6} \kappa3 + \frac{1}{2} \kappa5) \kappa1^{8} + (12 \kappa2^{3} + \kappa2^{2}$$

$$+ 14 \kappa2 \kappa4 + 10 \kappa3^{2} + \frac{1}{3} \kappa4 + \kappa6) \kappa1^{7} + (133 \kappa2^{2} \kappa3 + 49 \kappa3 \kappa4 + (\frac{14 \kappa3}{3}$$

$$+ 28 \kappa5) \kappa2 + \frac{7 \kappa5}{15} + \frac{7 \kappa7}{5}) \kappa1^{6} + \frac{1533 \kappa2^{5} \kappa3}{2} + (126 \kappa2^{4} + 7 \kappa2^{3}$$

$$+ 273 \kappa2^{2} \kappa4 + \frac{14 \kappa3^{2}}{3} + \frac{392 \kappa3 \kappa5}{5} + 49 \kappa4^{2} + (378 \kappa3^{2} + 7 \kappa4$$

$$+ \frac{189 \kappa6}{5}) \kappa2 + \frac{7 \kappa6}{15} + \frac{7 \kappa8}{5}) \kappa1^{5} + (1155 \kappa2^{3} \kappa3 + 280 \kappa3^{3} + (35 \kappa3$$

$$+ 357 \kappa5) \kappa2^{2} + 126 \kappa4 \kappa5 + (1225 \kappa3 \kappa4 + 7 \kappa5 + 35 \kappa7) \kappa2 + (\frac{35 \kappa4}{3}$$

$$+ 84 \kappa6) \kappa3 + \frac{\kappa7}{3} + \kappa9) \kappa1^{4} + (\frac{35 \kappa3}{2} + \frac{777 \kappa5}{2}) \kappa2^{4} + \frac{1820 \kappa3^{3} \kappa4}{3}$$

$$+ (420 \kappa2^{5} + \frac{35 \kappa2^{4}}{2} + 1470 \kappa2^{3} \kappa4 + 1050 \kappa3^{2} \kappa4 + (3010 \kappa3^{2} + 35 \kappa4$$

$$+ 301 \kappa6) \kappa2^{2} + \frac{35 \kappa4^{2}}{6} + 105 \kappa4 \kappa6 + 63 \kappa5^{2} + \frac{\kappa10}{2} + (\frac{140}{3} \kappa3^{2}$$

$$+ 1232 \kappa3 \kappa5 + 770 \kappa4^{2} + \frac{14}{3} \kappa6 + 22 \kappa8) \kappa2 + (\frac{28 \kappa5}{3} + 60 \kappa7) \kappa3 + \frac{\kappa8}{6})$$

$$\kappa1^{3} + (2625 \kappa3 \kappa4 + 7 \kappa5 + 75 \kappa7) \kappa2^{3} + \frac{273 \kappa4^{2} \kappa5}{2} + (\frac{5355 \kappa2^{4} \kappa3}{2}$$

$$+ (70 \kappa3 + 1092 \kappa5) \kappa2^{3} + \frac{140 \kappa3^{3}}{9} + 770 \kappa3^{2} \kappa5 + (5565 \kappa3 \kappa4 + 21 \kappa5$$

$$+ 159 \kappa7) \kappa2^{2} + 77 \kappa5 \kappa6 + \frac{\kappa11}{6} + (2520 \kappa3^{3} + 1134 \kappa4 \kappa5 + (70 \kappa4$$

$$+ 756 \kappa6) \kappa3 + 2 \kappa7 + 9 \kappa9) \kappa2 + (\frac{1925 \kappa4^{2}}{2} + \frac{14 \kappa6}{3} + \frac{55 \kappa8}{2}) \kappa3 + (7 \kappa5$$

$$+ 55 \kappa7) \kappa4 + \frac{\kappa9}{18}) \kappa1^{2} + (\frac{5320 \kappa3^{3}}{3} + 798 \kappa4 \kappa5 + (35 \kappa4 + 532 \kappa6) \kappa3 + \kappa7$$

$$+ \frac{19 \kappa9}{3}) \kappa2^{2} + (\frac{14 \kappa5}{3} + 52 \kappa7) \kappa3^{2} + \frac{26 \kappa6 \kappa7}{5} + (315 \kappa2^{6} + \frac{21 \kappa2^{5}}{2}$$

$$+ \frac{3255 \kappa2^{4} \kappa4}{2} + \frac{1540 \kappa3^{4}}{3} + (4410 \kappa3^{2} + 35 \kappa4 + 441 \kappa6) \kappa2^{3} + \frac{385 \kappa4^{3}}{2}$$

$$+ (70 \kappa3^{2} + 2688 \kappa3 \kappa5 + 1680 \kappa4^{2} + 7 \kappa6 + 48 \kappa8) \kappa2^{2} + (\frac{70 \kappa4}{3}$$

$$+ 308\,\kappa 6\Big)\,\kappa 3^2 + \frac{7\,\kappa 5^2}{5} + \frac{132\,\kappa 5\,\kappa 7}{5} + \frac{77\,\kappa 6^2}{5} + \frac{\kappa 10}{90} + \frac{\kappa 12}{30} + \Big(4550\,\kappa 3^2\,\kappa 4$$

$$+ \frac{35\,\kappa 4^2}{2} + 455\,\kappa 4\,\kappa 6 + 273\,\kappa 5^2 + \frac{13\,\kappa 10}{6} + (28\,\kappa 5 + 260\,\kappa 7)\,\kappa 3 + \frac{\kappa 8}{2}\Big)\,\kappa 2$$

$$+ \left(\left(924\,\kappa 4\,\kappa 5 + \frac{4\,\kappa 7}{3} + \frac{22\,\kappa 9}{3}\right)\kappa 3 + \left(\frac{7\,\kappa 6}{3} + \frac{33\,\kappa 8}{2}\right)\kappa 4\right)\kappa 1 + \frac{\kappa 11}{990} + \frac{\kappa 13}{330}$$

$$+ \Big(\frac{140\,\kappa 3^3}{9} + 1078\,\kappa 3^2\,\kappa 5 + \frac{539\,\kappa 5\,\kappa 6}{5} + \frac{7\,\kappa 11}{30} + \Big(\frac{2695\,\kappa 4^2}{2} + \frac{14\,\kappa 6}{3}$$

$$+ \frac{77\,\kappa 8}{2}\Big)\,\kappa 3 + (7\,\kappa 5 + 77\,\kappa 7)\,\kappa 4 + \frac{\kappa 9}{18}\Big)\,\kappa 2 + \Big(\frac{35}{6}\,\kappa 4^2 + 182\,\kappa 4\,\kappa 6$$

$$+ \frac{546}{5}\,\kappa 5^2 + \frac{13}{15}\,\kappa 10 + \frac{1}{6}\,\kappa 8\Big)\,\kappa 3 + \left(\frac{\kappa 7}{3} + \frac{13\,\kappa 9}{6}\right)\kappa 4 + \Big(\frac{7\,\kappa 6}{15}$$

$$+ \frac{39\,\kappa 8}{10}\Big)\,\kappa 5\Big)\,\mu\,(R0 + \alpha)\Big)$$

Determine asymptotic approximations of the first 3 cumulants of the following forms. Assume also that kappa4 - kappa13 are O(N).

```
> kappa1:=x1*N + x2 + x3/N:
  kappa2:=y1*N + y2:
  kappa3:=z1*N:
  kappa4:=u1*N:
  kappa5:=u2*N:
  kappa6:=u3*N:
  kappa7:=u4*N:
  kappa8:=u5*N:
  kappa9:=u6*N:
  kappa10:=u7*N:
  kappa11:=u8*N:
  kappa12:=u9*N:
  kappa13:=u10*N:
```

Asymptotic expressions for A, B, C are written A = A1*N + A2 + A3/N, B = B1*N + B2, C = C1*N, where

```
> A1:=coeff(A,N,1); A2:=coeff(A,N,0); A3:=coeff(A,N,-1);
  B1:=coeff(B,N,1); B2:=coeff(B,N,0);
  C1:=coeff(C,N,1);
```

$$A1 := \mu\,(R0 - 1)\,x1 - x1^{11}\,\mu\,(R0 + \alpha)$$

$$A2 := \mu\,(R0 - 1)\,x2 - 55\left(\frac{1}{5}\,x2\,x1^{10} + x1^9\,y1\right)\mu\,(R0 + \alpha)$$

$$A3 := \mu\,(R0 - 1)\,x3 - 55\,\Big(\frac{x3\,x1^{10}}{55} + \frac{2\,x2^2\,x1^9}{11} + \frac{1}{55}\big(x1\,(2\,(x3\,x1^4 + 4\,x2^2\,x1^3$$

$$+ x1\,(2\,(2\,x3\,x1 + x2^2)\,x1^2 + 4\,x2^2\,x1^2))\,x1^5 + 25\,x2^2\,x1^8)\big) + x1^9\,y2$$

$$+ 9\, x2\, x1^8\, y1 + 3\, x1^8\, z1 + 18\, y1^2\, x1^7\Big)\, \mu\, (R0+\alpha)$$

$$B1 := 2\, \mu\, (R0-1)\, y1 + \mu\, (R0+1)\, x1 - x1^{11}\, \mu\, (R0-\alpha) - 22\, x1^{10}\, y1\, \mu\, (R0+\alpha)$$

$$B2 := 2\, \mu\, (R0-1)\, y2 + \mu\, (R0+1)\, x2 - 55\left(\frac{1}{5}\, x2\, x1^{10} + x1^9\, y1\right)\mu\, (R0-\alpha)$$

$$- 330\left(\frac{1}{15}\, x1^{10}\, y2 + \frac{2}{3}\, x2\, x1^9\, y1 + \frac{1}{3}\, x1^9\, z1 + 3\, y1^2\, x1^8\right)\mu\, (R0+\alpha)$$

$$C1 := (x1 + 3\, z1)\, \mu\, (R0-1) + 3\, \mu\, (R0+1)\, y1 - 33\, x1^{10}\, y1\, \mu\, (R0-\alpha) \qquad \textbf{(3)}$$

$$- 990\left(\frac{1}{990}\, x1^{11} + \frac{1}{3}\, y1^2\, x1^9 + \frac{1}{30}\, x1^{10}\, z1\right)\mu\, (R0+\alpha)$$

6 equations can now be formed by setting each of A1, A2, A3, B1, B2, C1 equal to zero. These equations can be solved for the 6 unknowns x1, x2, x3, y1, y2, z1. The solutions are found sequentially, as follows:
First we solve A1=0 for x1.This equation has s+1=11 solutions. Among them, we exclude s=10 spurious solution.
After this, B1=0 is solved for for y1, A2=0 is solved for x2, C1=0 is solved for z1, B2=0 is solved for y2, and A3=0 is solved for x3.

```
> x1a:=solve(A1,x1);
  x1:=((R0-1)/(R0+alpha))^(1/10);
```

$$x1a := 0,\ \frac{\left((R0-1)\,(R0+\alpha)^9\right)^{1/10}}{R0+\alpha},$$

$$\frac{\left(\frac{\sqrt{5}}{4} + \frac{1}{4} + \frac{I\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right)\left((R0-1)\,(R0+\alpha)^9\right)^{1/10}}{R0+\alpha},$$

$$\frac{\left(\frac{\sqrt{5}}{4} - \frac{1}{4} + \frac{I\sqrt{2}\,\sqrt{5+\sqrt{5}}}{4}\right)\left((R0-1)\,(R0+\alpha)^9\right)^{1/10}}{R0+\alpha},$$

$$\frac{\left(-\frac{\sqrt{5}}{4} + \frac{1}{4} + \frac{I\sqrt{2}\,\sqrt{5+\sqrt{5}}}{4}\right)\left((R0-1)\,(R0+\alpha)^9\right)^{1/10}}{R0+\alpha},$$

$$\frac{\left(-\frac{\sqrt{5}}{4} - \frac{1}{4} + \frac{I\sqrt{2}\,\sqrt{5-\sqrt{5}}}{4}\right)\left((R0-1)\,(R0+\alpha)^9\right)^{1/10}}{R0+\alpha},$$

$$-\frac{\left((R0-1)\,(R0+\alpha)^9\right)^{1/10}}{R0+\alpha},$$

$$\frac{\left(-\frac{\sqrt{5}}{4}-\frac{1}{4}-\frac{I\sqrt{2}\sqrt{5-\sqrt{5}}}{4}\right)\left((R0-1)(R0+\alpha)^9\right)^{1/10}}{R0+\alpha},$$

$$\frac{\left(-\frac{\sqrt{5}}{4}+\frac{1}{4}-\frac{I\sqrt{2}\sqrt{5+\sqrt{5}}}{4}\right)\left((R0-1)(R0+\alpha)^9\right)^{1/10}}{R0+\alpha},$$

$$\frac{\left(\frac{\sqrt{5}}{4}-\frac{1}{4}-\frac{I\sqrt{2}\sqrt{5+\sqrt{5}}}{4}\right)\left((R0-1)(R0+\alpha)^9\right)^{1/10}}{R0+\alpha},$$

$$\frac{\left(\frac{\sqrt{5}}{4}+\frac{1}{4}-\frac{I\sqrt{2}\sqrt{5-\sqrt{5}}}{4}\right)\left((R0-1)(R0+\alpha)^9\right)^{1/10}}{R0+\alpha}$$

$$x1 := \left(\frac{R0-1}{R0+\alpha}\right)^{1/10} \quad \text{(4)}$$

```
> y1:=factor(solve(B1,y1));
```

$$y1 := \frac{\left(\frac{R0-1}{R0+\alpha}\right)^{1/10} R0\,(\alpha+1)}{10\,(R0+\alpha)\,(R0-1)} \quad \text{(5)}$$

```
> x2:=solve(A2,x2);
```

$$x2 := -\frac{11\,R0\,(\alpha+1)}{20\,(R0+\alpha)\,(R0-1)} \quad \text{(6)}$$

```
> z1:=factor(solve(C1,z1));
```

$$z1 := -\frac{R0\left(\frac{R0-1}{R0+\alpha}\right)^{1/10}(\alpha+1)\,(10\,R0^2-R0\,\alpha-R0+10\,\alpha)}{100\,(R0+\alpha)^2\,(R0-1)^2} \quad \text{(7)}$$

```
> y2:=solve(B2,y2);
```

$$y2 := \frac{11\,(R0^2+\alpha)\,(\alpha+1)\,R0}{20\,(R0+\alpha)^2\,(R0-1)^2} \quad \text{(8)}$$

```
> x3:=factor(solve(A3,x3));
```

$$x3 := -\frac{11\,R0\,(\alpha+1)\,(100\,R0^2+7\,R0\,\alpha+7\,R0+100\,\alpha)}{800\left(\frac{R0-1}{R0+\alpha}\right)^{1/10}(R0+\alpha)^2\,(R0-1)^2} \quad \text{(9)}$$

Summarize the results for s=10:

```
> x1, x2, x3, y1, y2, z1;
```

$$\left(\frac{R0-1}{R0+\alpha}\right)^{1/10},\ -\frac{11\,R0\,(\alpha+1)}{20\,(R0+\alpha)\,(R0-1)}, \quad \text{(10)}$$

$$-\frac{11\, R0\, (\alpha+1)\, \left(100\, R0^2+7\, R0\, \alpha+7\, R0+100\, \alpha\right)}{800 \left(\frac{R0-1}{R0+\alpha}\right)^{1/10} (R0+\alpha)^2\, (R0-1)^2},$$

$$\frac{\left(\frac{R0-1}{R0+\alpha}\right)^{1/10} R0\, (\alpha+1)}{10\, (R0+\alpha)\, (R0-1)},\ \frac{11\, \left(R0^2+\alpha\right)\, (\alpha+1)\, R0}{20\, (R0+\alpha)^2\, (R0-1)^2},$$

$$-\frac{R0 \left(\frac{R0-1}{R0+\alpha}\right)^{1/10} (\alpha+1)\, \left(10\, R0^2-R0\, \alpha-R0+10\, \alpha\right)}{100\, (R0+\alpha)^2\, (R0-1)^2}$$

>