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Brief paper

Active state estimation of nonlinear systems☆

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Abstract

In this paper, state observers for control systems with nonlinear outputs are studied. For such systems, the observability does not only depend on the initial conditions, but also on the exciting control used. Thus, for such systems, design of active control is an integral part of the design for state observers. Here some sufficient conditions are given for the convergence of an observer. It is also discussed, via a camera example, how to actively excite a system in order to improve the observability.

Keywords: Active sensing; Nonlinear observers; Mobile systems

1. Introduction

Consider a nonlinear system:

$$\dot{x}=f(x,t)+Bu(t)$$
$$y=h(x) \tag{1}$$

where $x \in R^n$, $u \in R^p$ is the input and $y \in R^m$ is the output one can measure, both f and h arc C^1 in a neighborhood of the origin and $f(0,t)=0\ \forall t\ h(0)=0$.

Since 1970s there has been an extensive study on the design of observers for such systems, (Kou, Elliot, & Tarn, 1975; Krener & Respondek, 1985; van der Schaft, 1986) etc. Most current methods lead to the design of an exponential observer. As a necessary condition for the existence of an exponential observer, the linearized pair $(\partial f(0,t)/\partial x, h(0))$ must be detectable. Under this condition, locally the different choices of input would barely affect the rate of convergence for an observer.

In this paper we consider nonlinear systems in the following form:

$$\dot{x}=f(x,t)+Bu(t)$$
$$y_i=\frac{p_i(x)}{q_i(x)}, \quad i=1,\ldots,m \tag{2}$$

where $x \in R^n$, $u \in R^p$ and each y_i is a scalar. All the mappings f, p_i, q_i are assumed to be C^1, $f(0,t)=0\ \forall t$ and B is constant. We will try to understand under what conditions one can construct an exponential observer for (2). In Matveev, Hu, Frezza, and Rehbinder (2000), such a result for a class of systems was given. However, that result only gives the condition in terms of the output $y(t)$. In this paper we will focus on the existence of input signals $u(t)$ such that the observer converges.

We should point out that here for any point x_e such that $f(x_e,t)=0$, $q_i(x_e)$ may be zero. Therefore, this case is different from the classical case (1). If $q_i(x)$ is constant and nonzero, the problem becomes the classical one. In this study we focus on the case where $q_i(x)$ is not constant and $q_i(0)=0$ (without loss of generality, we assume $x_e=0$). Obviously one can not apply linearization technique in this case and one needs to design the exciting control $u(t)$ such that $q_i(x(t)) \neq 0\ \forall t \geqslant 0$. A typical example of such a

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system is using a mounted video camera to localize a mobile robot.

As we will see later, for this kind of systems, the observability does not only depend on the initial conditions, but also on the control. This presents an interesting issue: how to design an exciting control to "maximize" the observability, namely how to design an active observer. This has been a very important issue in the field of active perception in robotics and computer vision (Eklundh, Uhlin, Nordlund, & Maki, 1996). However, study from the systems and control point of view is still lacking, except for bilinear systems. This is witnessed in Lin, Baillieul, and Bloch (2002) where it is pointed out that one of the key questions in nonlinear control is "how to design a nonlinear observer for nonlinear systems whose linearization is neither observable nor detectable".

In this paper we present some initial results regarding this design.

Example 1.1. Here we consider the problem of using video images to identify the relative location and orientation of a robot. We suppose its mounted camera can recognize a corner of a room, namely that the images of a corner point and at least two line directions can be observed continuously. We want to use these measurements to recover the 3D location of the camera in the room and the orientation of the camera's image plane in the room.

Suppose $p=(p_x, p_y, p_z)^{\mathrm{T}}$ is the position of a point feature in the camera-fixed coordinates, where p_z is the depth of the point. If Ω is defined according to

$$\Omega = \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix} \tag{3}$$

where $\boldsymbol{\omega}$ is the angular velocity measured in the camera frame, then it is well known that the evolution of the point is described by

$$\dot{p} = \Omega p + v$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} p_x/p_z \\ p_y/p_z \end{pmatrix}, \quad p_z \neq 0. \tag{4}$$

The outputs y_1 and y_2 represent the image of the point, and v is the translational velocity. Here we assume that the focal length f of the camera is 1. The line equations are more complex and will be discussed later.

This paper is organized as follows. In Section 2 we formulate the problem and briefly summarize the existing results. Section 3 is where we present our main theoretical results. In Section 4 we revisit the camera example and in Section 5 we show some simulation results for our example.

2. Preliminaries

We consider again system (1)

$$\dot{x}=f(x,t)+Bu$$
$$y=h(x). \tag{5}$$

In general, an observer for (5) should take the following form:

$$\dot{\hat{x}} = p(\hat{x}, h(x(t)), t, u), \tag{6}$$

with the corresponding error dynamics asymptotically stable. Consequently, $\|x(t)-\hat{x}(t)\| \to 0$ as $t \to \infty$.

It is fairly common in the literature that the following form of observer is considered:

$$\dot{\hat{x}} = f(\hat{x}, t) + l(h(x) - h(\hat{x})) + Bu. \tag{7}$$

where l is a constant gain matrix. Obviously, it is easier to design an observer in the above form (7). Unfortunately it has been shown (Hu, 1991) that for a nonlinear system it is not always possible to design an observer in the form of (7) whenever it is possible to have observers for (5).

With the type of outputs we consider in (2), it is natural that we should consider a more general class of observers than those defined by (7). For an observer (6), it must hold that if $\hat{x}_0 = x_0$, then $\hat{x}(t) = x(t)$ for $t \geqslant 0$. It can easily be seen that $p(\hat{x}, h(x), t, u)$ of the following form meets this requirement

$$p(\hat{x}, h(x), t, u) = f(\hat{x}, t) + l(\hat{x}, h(x) - h(\hat{x}), t) + Bu \tag{8}$$

where $l(\hat{x}, 0, t)=0$. Therefore, we define a general nonlinear observer as follows:

Definition 2.1. Let $x(x_0, t)$ be the solution of (5). A dynamic system described by

$$\dot{\hat{x}} = f(\hat{x}, t) + l(\hat{x}, h(x) - h(\hat{x}), t) + Bu \tag{9}$$

is said to be a (local) observer of (5), if both the following statements hold:

(I) $l(x, y, t)$ is C^1 and $l(x, 0, t) = 0$.

(II) For any initial condition $x_0 \in N$, the domain of interest, the zero point of the error $e = \hat{x}(t) - x(x_0, t)$, is asymptotically stable.

If additionally

$$\|x(t) - \hat{x}(t)\| \leqslant K \mathrm{e}^{-at}, \tag{10}$$

where $a > 0$, then (9) is said to be an exponential observer. This definition was first introduced in Kou et al. (1975).

Now we consider the system (2). Then the problem becomes under what conditions there exists an observer in the following form:

$$\dot{\hat{x}}=f(\hat{x},t)+Bu+l\left(\hat{x}, y_1(t)-\frac{p_1(\hat{x})}{q_1(\hat{x})}, \ldots, y_m(t) - \frac{p_m(\hat{x})}{q_m(\hat{x})}, t\right). \tag{11}$$

Here we do not attempt to solve the general case, but rather focus on a couple of special cases that are motivated by some applications.

3. Main results

In this paper we focus on the case where the dynamics is linear and the outputs are linear projections and we will only use an example to discuss the nonlinear case. So we represent (2) as

$$\dot{x}=A(t)x+Bu(t)$$
$$y=\frac{Px}{q^{\mathrm{T}}x}, \tag{12}$$

where $A(\cdot)$ is a C^1 $n\times n$ matrix and both $\|A(t)\|$ and $\|\dot{A}(t)\|$ are uniformly bounded, P is a $m\times n$ matrix and q is $n\times 1$ matrix. This case fits the camera example presented in the introduction. Although the state equations are still linear, the observability condition apparently will depend not only on $(A(t), P, q)$, but also on $u(t)$ (setting $u=0$ may make the system unobservable). This is in sharp contrast to the classical linear time-varying case, where the observability grammian does not depend on u.

Rewrite the output equation in (12) as

$$(P-yq^{\mathrm{T}})x=0.$$

This can be interpreted as if the output is defined implicitly. For this reason we call an observer for such a system an implicit observer. Using the equation above we consider the following class of observers

$$\dot{\hat{x}}=A(t)\hat{x}+Bu(t)+L(t)(P-y(t)q^{\mathrm{T}})\hat{x}. \tag{13}$$

This is similar to observers for linear time-varying systems, but the output enters the error dynamics directly. It is well known (Kalman, 1963; McGarty, 1974) that the error dynamics is asymptotically stable if some uniform observability condition is satisfied. Here we only present a more conservative version of the result (Matveev et al., 2000).

Lemma 3.1. *Suppose $A(\cdot)$ is uniformly Liapunov stable and let $\Phi(\tau,t)$ be the state transition matrix of* (12),

$$M(\tau):=(P-y(\tau)q^{\mathrm{T}})^{\mathrm{T}}(P-y(\tau)q^{\mathrm{T}}) \tag{14}$$

and suppose the system (12) *is uniformly observable, namely*

$$\mathcal{O}:=\int_t^{t+T}\Phi^{\mathrm{T}}(\tau,t)M(\tau)\Phi(\tau,t)\,\mathrm{d}\tau\geqslant\varepsilon I \tag{15}$$

for all $t\geqslant t_0$, $|x_0|\leqslant m$ and some positive T and ε. Then (13) *is an exponential observer, i.e. the error dynamics converges to zero exponentially for all $|x_0|\leqslant m$. In particular, one can choose $L=-W^{-1}(P-yq^{\mathrm{T}})^{\mathrm{T}}$, where $W>0$ satisfies*

$$\dot{W}+WA+A^{\mathrm{T}}W\leqslant 0. \tag{16}$$

However, this result only gives the condition in terms of $y(t)$ for our system and $y(t)$ depends on the exciting $u(t)$.

It would be much more convenient and practical if we could give conditions on $u(t)$ and the initial conditions. Furthermore, this would allow us to design u in order to enhance the observability.

Expanding $M(\tau)$ yields

$$M(\tau)=P^{\mathrm{T}}P-qy^{\mathrm{T}}(\tau)P-P^{\mathrm{T}}y(\tau)q^{\mathrm{T}}+\|y(\tau)\|^2qq^{\mathrm{T}}. \tag{17}$$

Defining

$$M_1(\tau)=P^{\mathrm{T}}P+\|y(\tau)\|^2qq^{\mathrm{T}}$$
$$M_2(\tau)=qy^{\mathrm{T}}(\tau)P+P^{\mathrm{T}}y(\tau)q^{\mathrm{T}} \tag{18}$$

means $M(\tau)=M_1(\tau)-M_2(\tau)$.

We can now expand the observability grammian as

$$\begin{aligned}\mathcal{O}&=\int_0^T\Phi^{\mathrm{T}}(t+\tau,t)M(\tau+t)\Phi(t+\tau,t)\,\mathrm{d}\tau\\&=\int_0^T\Phi^{\mathrm{T}}(t+\tau,t)M_1(\tau+t)\Phi(t+\tau,t)\,\mathrm{d}\tau\\&\quad-\int_0^T\Phi^{\mathrm{T}}(t+\tau,t)M_2(\tau+t)\Phi(t+\tau,t)\,\mathrm{d}\tau.\end{aligned} \tag{19}$$

It is reasonable to assume the following:

Assumption 3.1. For any scalar function $\sigma(\cdot)$ such that $\int_t^{t+T}|\sigma(s)|\,\mathrm{d}s\geqslant\varepsilon>0$ where $t\geqslant t_0$, we have for system (12)

$$\int_0^T\Phi^{\mathrm{T}}(t+\tau,t)M_\sigma(\tau+t)\Phi(t+\tau,t)\,\mathrm{d}\tau\geqslant\varepsilon_1 I>0, \tag{20}$$

where M_σ is defined as

$$M_\sigma(\tau+t)=P^{\mathrm{T}}P+\sigma^2(\tau+t)qq^{\mathrm{T}}. \tag{21}$$

The condition imposed on σ suggests that the excitement from the output should be persistent. The motivation of the assumption is that the system should be observable if the output can be persistently excited. In other words, the pair $(A(\cdot),(P^{\mathrm{T}}\sigma(\cdot)q)^{\mathrm{T}})$ is uniformly observable. In fact, in many cases it would suffice to assume that $(A(\cdot),(P^{\mathrm{T}}q)^{\mathrm{T}})$ is uniformly observable.

Eqs. (18) and (19) tell us that M_2 represents a possibly bad influence. Thus, a feasible approach for obtaining uniform observability is to solve the following optimal control problem:

$$\min_u\left|\int_0^T\langle\Phi^{\mathrm{T}}(t+\tau,t)q,\Phi^{\mathrm{T}}(t+\tau,t)P^{\mathrm{T}}y(\tau+t)\rangle\,\mathrm{d}\tau\right| \tag{22}$$

among all admissible controls such that $\|y(t)\|\leqslant C$. This will reduce the weight of the quadratic form $z^{\mathrm{T}}M_2z$.

By looking at (18) and (19) we also realize that choosing a suitable big oscillating y will increase the positive impact of M_1 while reducing the possibly damaging impact of M_2.

Now we need the following definition:

Definition 3.1. A linear system $\dot{x} = A(t)x$ is said to be q-persistent if for any x_0 such that $|q^{\mathrm{T}}x_0| > 0$, we have $|q^{\mathrm{T}}x(x_0, t)| \geqslant \gamma_{x_0} > 0$ for all $t \geqslant 0$.

If $\omega_1(t) = \omega_2(t) = 0$ in the camera example, then the system is q-persistent when v is set to zero.

Theorem 3.1. *For system* (12), *suppose Assumption* 3.1 *is satisfied*, $A(t)$ *is both uniformly stable and q-persistent and the reachable subspace of the pair* $(A(\cdot), B)$ *contains* $\ker q^{\mathrm{T}}$. *Then there exists a control* $u(t)$ *such that the system is uniformly observable from all initial conditions satisfying* $|q^{\mathrm{T}}x_0| > 0$.

Proof. Let $x_1 = q^{\mathrm{T}}x$ and complete the coordinates with x_2, we can rewrite the system as

$$\dot{x}_1 = a_1x_1 + a_2x_2 + b_1u$$
$$\dot{x}_2 = A_1x_1 + A_2x_2 + B_2u.$$

q-persistence implies that both $q^{\mathrm{T}}x > 0$ and $q^{\mathrm{T}}x < 0$ are invariant under $\dot{x} = A(t)x$. Thus $q^{\mathrm{T}}x = 0$ is also invariant under $\dot{x} = A(t)x$. The invariance of $q^{\mathrm{T}}x = 0$ implies $a_2 = 0$, thus the system becomes

$$\dot{x}_1 = a_1x_1 + b_1u$$
$$\dot{x}_2 = A_1x_1 + A_2x_2 + B_2u,$$

where the assumption of stability and q-persistence of $A(t)$ implies for some $m, M > 0$, $m \leqslant \phi_1(t, t_0) = e^{\int_{t_0}^{t} a_1(s)\,\mathrm{d}s} \leqslant M \;\forall t \geqslant t_0$.

Without loss of generality, we assume that q^{T} is linearly independent of the rows of P. Otherwise it would imply that (P, A) is already observable and in this case proof of the theorem is straight forward. Under this assumption we can have $P = [0\bar{P}]$ and the output can be rewritten as

$$y = \frac{\bar{P}x_2}{x_1}.$$

$M(\tau)$ in (17) becomes

$$M(\tau) = \begin{pmatrix} y^{\mathrm{T}}(\tau)y(\tau) & -y^{\mathrm{T}}(\tau)\bar{P} \\ -\bar{P}^{\mathrm{T}}y(\tau) & \bar{P}^{\mathrm{T}}\bar{P} \end{pmatrix}.$$

It is obvious that Assumption 3.1 is satisfied if and only if $(A_2, \bar{P})$ is uniformly observable.

It is observed in Matthies, Kanade, and Szeliski (1989) that quick translational movements in planes (roughly x_2 in our case) parallel to the perception plane provide very rich depth (roughly x_1) information. Based on this observation, we can construct an open-loop exciting control as follows:

1. If $b_1 \neq 0$, let $\mathrm{sgn}(x_1(t_0))b_1u = k > 0$ for some duration T_1, to drive x_1 further away from 0; otherwise go directly to Step 2.
2. Let $u = D(\omega)\sin(\omega t)$, where D is chosen to satisfy $\|D(\omega)\| < \frac{m\omega}{M\|b_1\|}kT_1$. Thus $|x_1(t)| \geqslant \delta > 0 \;\forall t \geqslant T_1$. If $b_1 = 0$, we can simply take $D(\omega) = \omega[1, \ldots, 1]^{\mathrm{T}}$, for example.

With this control, we have

$$x_2(t) = \phi_2(t, t_0)x_2(t_0) + \phi_{21}(t, t_0)x_1(t_0) + \int_{t_0}^{t} \phi_{21}b_1u(s)\,\mathrm{d}s + \int_{t_0}^{t} \phi_2(t, s)B_2u(s)\,\mathrm{d}s,$$

where $\phi_{21}(t, t_0) = \int_{t_0}^{t} \phi_2(t, s)A_1(s)\phi_1(s, t_0)\,\mathrm{d}s$.

The stability assumption on $A(t)$ implies that $A_2(\cdot)$ is also critically stable (since $x_1 = 0$ is invariant when u is set to zero). Plug in the control in Step 2, we have

$$\int_{t_0}^{t} \phi_2(t, s)B_2u(s)\,\mathrm{d}s = -\frac{B_2D}{\omega}\cos(\omega t) + \phi_2(t, t_0)\frac{B_2D}{\omega}\cos(\omega t_0) - \frac{1}{\omega}\int_{t_0}^{t} \phi_2(t, s)A_2(s)B_2D\cos(\omega s)\,\mathrm{d}s.$$

The last term in the above equality tends to zero as ω tends to infinity. In fact, by the classical Riemann–Lebesgue theorem, we know that

$$\lim_{\omega\to\infty}\int_{t_1}^{t_2} f(t)\sin(\omega t)\,\mathrm{d}t = 0,$$
$$\lim_{\omega\to\infty}\int_{t_1}^{t_2} f(t)\cos(\omega t)\,\mathrm{d}t = 0$$

for any integrable function *f*, while

$$\lim_{\omega\to\infty}\int_{t_1}^{t_2} f^2(t)\sin^2(\omega t)\,\mathrm{d}t = \frac{1}{2}\int_{t_1}^{t_2} f^2(t)\,\mathrm{d}t$$

is bounded from below. Denote $x_{2u}(t) = -\frac{B_2D(\omega)}{\omega}\cos(\omega t)$, we can choose ω sufficiently large such that

$$\left|\int_{t}^{t+T} \phi_2^{\mathrm{T}}(s, t)\bar{P}^{\mathrm{T}}\bar{P}x_{2u}(s)\phi_1(s, t)\,\mathrm{d}s\right| \ll 1 \tag{23}$$

and

$$\left|\int_{t}^{t+T} \phi_{21}^{\mathrm{T}}(s, t)\bar{P}^{\mathrm{T}}\bar{P}x_{2u}(s)\phi_1(s, t)\,\mathrm{d}s\right| \ll 1, \tag{24}$$

while

$$\int_{t}^{t+T} \phi_1(s, t)|\bar{P}x_{2u}(s)|^2\phi_1(s, t)\,\mathrm{d}s \geqslant c > 0, \tag{25}$$

namely, it remains bounded from a positive number due to the reachability and observability assumption on the pair (A_2, B_2) and $(A_2, \bar{P})$.

We note that the integrand in (15) is always at least semi-definite, with inequalities similar to (23) and (24) one can show with straight forward calculations that in the matrix integration defined by (15) some of the diagonal terms (or sign indefinite terms) vanish (almost), while the corresponding item on the diagonal remains bounded from below due to (25). This extra positive item on the diagonal will make the matrix positive definite. Therefore the system is uniformly observable. □

Although the optimal control problem (22) would provide us with an alternative approach for finding a control, in general it is a nonconvex problem and difficult to solve analytically. Since the purpose of this paper is mainly to demonstrate the dependence of observer on control, rather than attempting to solve the problem of optimal design of active controls, we can limit the control to some special classes and try to find a suboptimal open-loop controller. By observing the functional to be maximized and from the proof of the theorem, it is reasonable to assume that the control is a periodic function, as discussed above.

Another interesting issue regarding the design of active observers for (12) is the robustness of algorithm with respect to measurement noise. The key requirement for (13) to converge is that (15) holds. Therefore (13) would still converge if the noise is small in comparison to the true measurement. However, when the noise is big, the error dynamics for (13) may become unstable, which is in sharp contrast to the linear case.

In the rest of this paper, we shall use Example 1.1 to illustrate the ideas presented so far.

4. The camera example

We should first point out that there has been a vast literature on *computer vision*, see for example Shapiro and Stockman (2001), Faugeras, Luong, and Papadopoulo (2001) and the references therein. The purpose of this section is to use Example 1.1 to illustrate the ideas of active perception presented in the paper. To focus on this aim, we do not incorporate measurement uncertainties and noises in our simulation.

For a robot moving around in a 3D environment lines and points are of interest when trying to figure out the position and orientation using vision. How many lines and points one needs to determine the rotation and translation depends on how the features are placed relative to the camera. If the camera for example moves along a straight line a detected line parallel to it does not add any depth information.

In this section, we first derive the model for line measurements. Then we use the point feature to illustrate the design of a linear observer, and the line feature to illustrate the possibility of nonlinear observers.

4.1. Measurement of a line

One way of describing a line in 3D is to specify a point p, as done in the introduction, plus a direction

$$d = \left(d_x, d_y, d_z\right)^{\mathrm{T}}. \tag{26}$$

Though not necessary, in this paper we choose to model d as a vector of norm one. If p is a fixed point, such as a corner (intersection with another line), one can, as discussed in the beginning of this paper, describe its dynamics by (4). Similarly we can write

$$\dot{d} = \Omega(t)d. \tag{27}$$

The projection of a line onto the image plane of a camera is well understood. We however still give an elementary derivation here in order to put the observation in the form we study in the paper. Let $P_{2D}(p)$ denote the projection of the point p onto the image plane. If the focal length f is assumed to be 1 the 2D projection can be written

$$P_{2D}(p) = \left(\frac{p_x}{p_z}, \frac{p_y}{p_z}\right)^{\mathrm{T}}. \tag{28}$$

To compute the 2D projection of the 3D direction we first consider the projection of two points lying on the line $p+td$:

$$P_{2D}(p + t_i d) = \frac{1}{p_z + t_i d_z}\begin{pmatrix} p_x + t_i d_x \\ p_y + t_i d_y \end{pmatrix} \tag{29}$$

where $i=1, 2$. The direction of the line projection can now be obtained by looking at the difference between these points.

$$P_{2D}(p_2) - P_{2D}(p_1) = \frac{t_2 - t_1}{(p_z + t_2 d_z)(p_z + t_1 d_z)}\begin{pmatrix} p_z d_x - p_x d_z \\ p_z d_y - p_y d_z \end{pmatrix} \tag{30}$$

A measurement of the point is given by (4). One such identifiable point could be the intersection of two coplanar lines. Based on (30) it is clear that a measurement of the 2D projection of the direction looks like

$$\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} = C\begin{pmatrix} p_z d_x - p_x d_z \\ p_z d_y - p_y d_z \end{pmatrix} = C d_{2\mathrm{D}}. \tag{31}$$

It is convenient to assume that $[y_3, y_4]^{\mathrm{T}}$ is a vector of length one. In that case $C = 1/||d_{2\mathrm{D}}||$. Then the system consisting of (4), (27) and (31) is in the form of (2).

Now note that

$$\begin{aligned}\begin{pmatrix} p_z d_x - p_x d_z \\ p_z d_y - p_y d_z \end{pmatrix} &= p_z\begin{pmatrix} d_x - (p_x/p_z)d_z \\ d_y - (p_y/p_z)d_z \end{pmatrix} \\ &= p_z\begin{pmatrix} d_x - y_1 d_z \\ d_y - y_2 d_z \end{pmatrix}.\end{aligned} \tag{32}$$

Using (32) in (31) gives

$$y_3 C p_z (d_y - y_2 d_z) = y_4 C p_z (d_x - y_1 d_z) \tag{33}$$

yielding

$$y_3(d_y - y_2 d_z) - y_4(d_x - y_1 d_z) = 0. \tag{34}$$

Eq. (34) can be rewritten as

$$\begin{pmatrix} -y_4 \\ y_3 \\ y_4 y_1 - y_2 y_3 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = B_d(y)d = 0. \tag{35}$$

The above expression is well suited for the design of an observer.

4.2. Localization

Now we construct an observer to estimate one of the lines and the intersecting point. Defining

$$B_p(y) = \begin{pmatrix} -1 & 0 & y_1 \\ 0 & -1 & y_2 \end{pmatrix}, \tag{36}$$

Matveev et al. (2000) shows that for a point p satisfying (4)

$$B_p(y)p = 0. \tag{37}$$

Comparing (4) and (37) to (13) and (16) yields the observer

$$\dot{\hat{p}} = \Omega\hat{p} + v - W^{-1}B_p(y)^{\mathrm{T}}B_p(y)\hat{p}. \tag{38}$$

If (27) and (35) are matched against (13) and (16) one gets

$$\dot{\hat{d}} = \Omega\hat{d} - W^{-1}B_d(y)^{\mathrm{T}}B_d(y)\hat{d}. \tag{39}$$

However, when using (39) $\hat{d}$ is not guaranteed to converge to a vector of norm one. This could be a problem since we model d as a vector of unit length. In order to remedy this, we can add an extra forcing term. We observe that for the following system:

$$\dot{x} = -x(x^{\mathrm{T}}x - 1), \tag{40}$$

the direction of $x(t)$ is not affected by the dynamic equation above, but the norm of $x(t)$ is driven towards one. By adding this extra feature to (39), we get

$$\dot{\hat{d}} = \Omega\hat{d} - W^{-1}B_d(y)^{\mathrm{T}}B_d(y)\hat{d} - \hat{d}(\hat{d}^{\mathrm{T}}\hat{d} - 1). \tag{41}$$

Then, the observer becomes quite nonlinear.

5. Simulation

One could imagine a robot operating in an environment containing polygonal objects. In such a case points and directions can be extracted from the boundaries of these polygons. The example below could originate from such a situation.

When simulating a combined point and direction estimation the following setting was used. Let $\hat{p}$ and $\hat{d}$ denote the estimates of p and d, $\Delta p = \hat{p} - p$ and $\Delta d = \hat{d} - d$. The parameters $\omega_1 = 0$, $\omega_2 = \sin(\pi t)$, $\omega_3 = 0$, $W = 0.2I$ were chosen and the initial conditions used were

$$p(0) = [1, -1, 2]^{\mathrm{T}}, \quad \hat{p}(0) = 1.3p(0),$$
$$\theta = \pi/13, \quad \phi = 2\pi/3, \quad \hat{\theta} = 1.2\theta, \quad \hat{\phi} = 0.8\phi,$$
$$d(0) = [\cos(\theta)\,\cos(\phi), \sin(\theta)\,\cos(\phi), \sin(\phi)],$$
$$\hat{d}(0) = [\cos(\hat{\theta})\,\cos(\hat{\phi}), \sin(\hat{\theta})\,\cos(\hat{\phi}), \sin(\hat{\phi})].$$

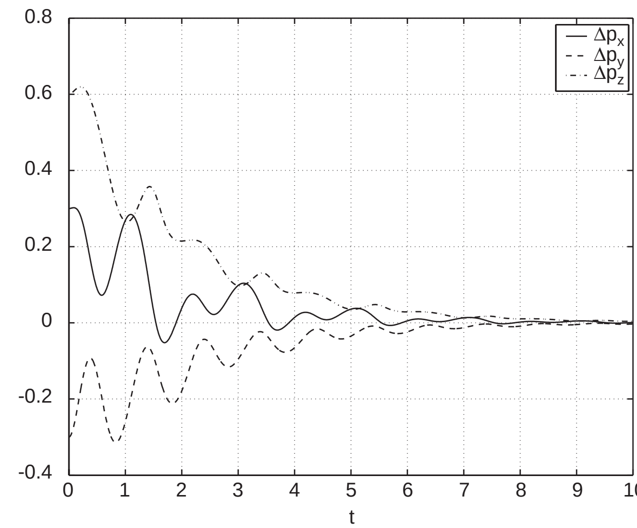

Fig. 1. Δp.

The velocity was chosen as a periodic function

$$v(t) = \begin{pmatrix} -2\pi\,\sin(2\pi rt) \\ 2\pi\,\cos(2\pi rt) \\ 0 \end{pmatrix},$$

where $r = 1$. With this control it can be verified that the p system, described by (4), is uniformly observable. And increasing the frequency of the velocity speeded up the convergence of the estimates. We note that with our choice of initial condition the depth ($|q^{\mathrm{T}}x_0|$) is already large enough, thus it is not necessary to use a constant control first to increase the depth further, as is suggested in the proof of Theorem 3.1.

Remark 5.1. On the other hand, for a v consisting of two zero components and one oscillating component, $\hat{p}$ and $\hat{d}$ failed to converge to p and d since the system is not uniformly observable in this case.

The result of applying Eq. (38) is shown in Fig. 1. As seen there $\hat{p}$ converged nicely to p. It should be pointed out that the point estimation is independent of the direction. The estimates of d will on the other hand depend on p via outputs. This means that a good choice of v is equally important for estimating d.

All simulations in this section were done in Matlab, using ode45 for integration.

5.1. Using a linear observer

First Eq. (39) without norm feedback for $\hat{d}$ was used.

When using (39) the norm of $\hat{d}$ is not guaranteed to converge to one. To evaluate how well the direction of $\hat{d}$ describes the one of d, d is compared to $\tilde{d} = \hat{d}/||\hat{d}||$. Furthermore we let $\tilde{\Delta}d = \tilde{d} - d$. Figs. 2 and 3 show results for this

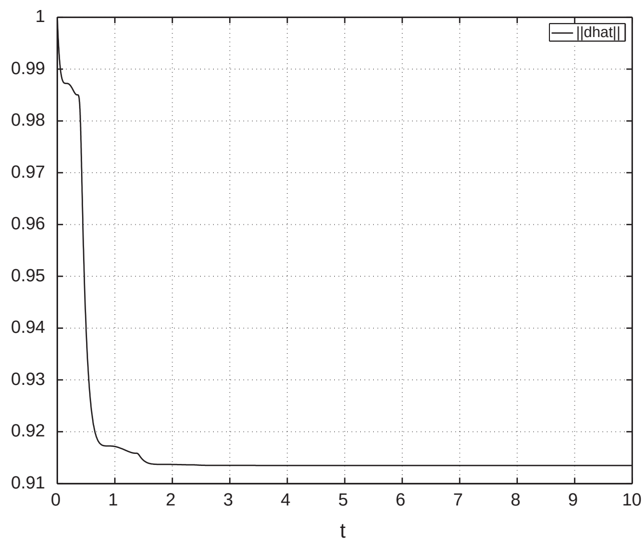

Fig. 2. Norm of $\hat{d}$ linear case.

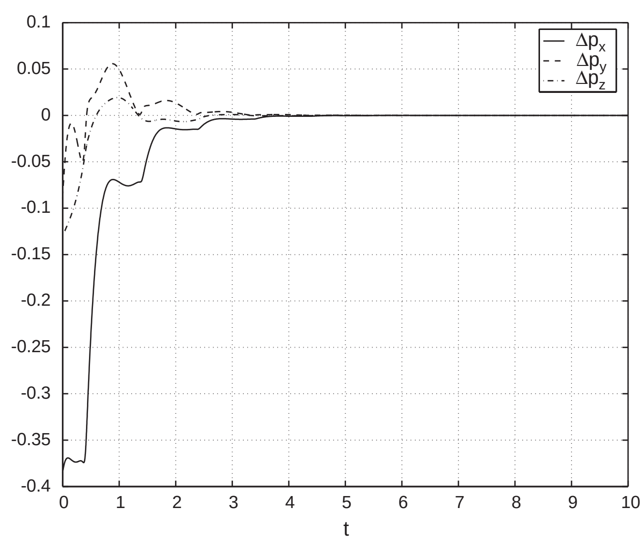

Fig. 3. $\tilde{\Delta} d$ linear case.

case. $\hat{d}$ converges to a vector parallel to d, but has a slightly smaller norm.

5.2. Using a nonlinear observer

The second case studied was the observer described by (41) with a norm feedback for $\hat{d}$. The outcome of the simulation is displayed in Figs. 4 and 5. They show how $\hat{d}$ converges nicely to d.

Let

$$\hat{A} = \Omega - W^{-1} B_d^{\mathrm{T}} B_d \tag{42}$$

and note that $B_d d = 0$. Now Eq. (41) can be rewritten

$$\dot{\hat{d}} = \hat{A}\hat{d} - \hat{d}(\hat{d}^{\mathrm{T}}\hat{d} - 1). \tag{43}$$

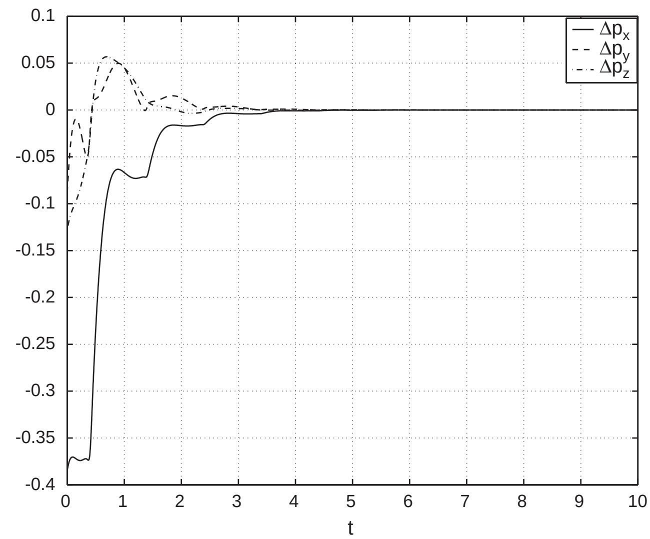

Fig. 4. Δd nonlinear case.

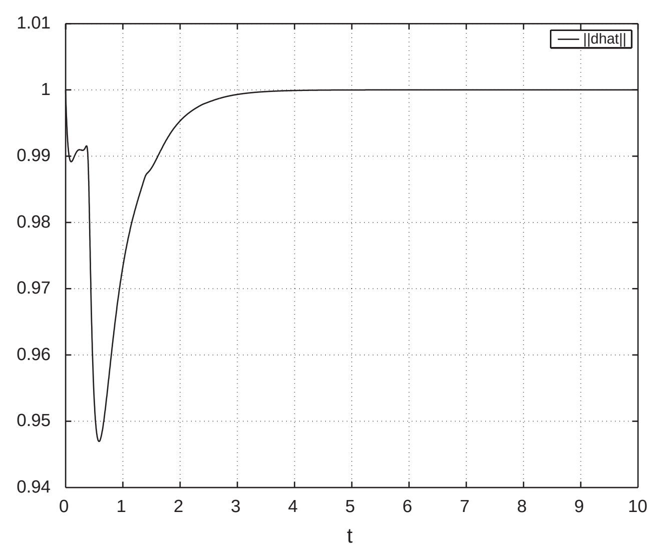

Fig. 5. Norm of $\hat{d}$ nonlinear case.

Letting $e = \Delta d = \hat{d} - d$ yields

$$\hat{d}^{\mathrm{T}}\hat{d} = e^{\mathrm{T}}e + 2d^{\mathrm{T}}e + 1. \tag{44}$$

The dynamics of e will be governed by

$$\dot{e} = \hat{A}e - (e + d)(e^{\mathrm{T}}e + 2d^{\mathrm{T}}e). \tag{45}$$

If $||e||$ is assumed to be small the above equation can locally be approximated by

$$\dot{e} = (\hat{A} - 2dd^{\mathrm{T}})e. \tag{46}$$

For a periodic $(\hat{A} - 2dd^{\mathrm{T}})$ a Floquet test (Khalil, 1996) of stability can be applied. When the states of our simulation were plugged into such a test stability was confirmed.

6. Conclusions

In this paper theoretical results for improving observability by active control design are presented. Though we do not present any method for optimizing observers the paper can be thought of a first step towards designing observers and feasible exciting input signals for better estimates.

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