Nonlinear Observers for Unicycle Robots with Range Sensors

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***Abstract*— For nonlinear mobile systems equipped with exteroceptive sensors, the observability does not only depend on the initial conditions, but also on the control and the environment. This presents an interesting issue: how to design an observer together with the exciting control. In this paper the problem of designing an observer based on range sensor readings is studied. A design method based on periodic excitations is proposed for unicycle robotic systems.**

***Index Terms*— Nonlinear observers, active sensing, mobile robots.**

I. Introduction

For nonlinear systems in general, the observability does not only depend on the initial conditions, but also on the control. Moreover, in mobile robotic systems, exteroceptive sensors such as lasers and video cameras are typically used. Thus for such systems the observability (of the state of the system) does not only depend on the system itself, but also on the environment. Due to the nature of the sensors used, how to actively design an exciting control in order to gain observability has been a very important issue in the field of active perception in robotics and computer vision [2]. Although there have been several papers on the design of observers for non-uniformly observable nonlinear systems [6], [10], [11], further studies from the systems and control point of view are still needed [8]. As we know, many mobile robotic systems are not uniformly observable. Although there have been several case studies such as [7] where localizing a point feature via active vision is studied and [9] where a state-feedback controller for a unicycle robot is combined with an observer that estimates the orientation error, observer design for mobile systems that interact with the environment remains largely an open issue, in particular when the environment consists of terrains, rather than just point features.

In this note we will use a case study to reveal some of the unique aspects in designing observers for mobile robots equipped with exteroceptive sensors. Such a system can be modeled as follows.

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x, s_e(x)) \tag{2}$$

$$x_e = \phi(s), \tag{3}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and $s \in \mathbb{R}^q$. The output mapping h is a mapping of both the state x and the sensor interaction with the environment. Here we assume that exteroceptive sensors such as range sensors are used. $x_e = \phi(s)$ defines a r-dimensional smooth manifold in $\mathbb{R}^n$ where $r \le n$, which models terrains (walls) in the environment. $s_e(x)$ denotes the "perceived" point in the environment by the sensor mounted on the robot located at x. Naturally we assume that once ϕ is given, $s_e(x)$ is uniquely determined.

In this paper we study the so-called relocalization problem in robotics. Namely, we assume that the map $\phi(s)$ is already given, and we would like to observe the state of the system based on range sensor readings. Although we will use a specific system model for the problem, we hope that the discussion would nevertheless raise some general interest on active nonlinear observers.

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II. The system setup

We consider an oriented robot in the plane, using the so-called unicycle model. Namely, the state of the system is described by $(\,(x_1, x_2)\,, \theta) \in \mathbb{R}^2 \times S^1$ and it has two control inputs, the translational velocity v and the angular velocity ω. The governing kinematics is

$$\dot{x_1} = v \cos\theta \tag{4}$$

$$\dot{x_2} = v \sin\theta \tag{5}$$

$$\dot{\theta} = \omega. \tag{6}$$

Furthermore, we assume that the robot is equipped with two range-measuring sensors, oriented at angles φ_1, φ_2 with respect to the orientation of the robot frame. These sensors measure the distances ρ_1, ρ_2, to some smooth curve $r : S^1 \to \mathbb{R}^2$ or $r : \mathbb{R} \to \mathbb{R}^2$, along the ray originating at the point x of the robot and making an angle $\theta + \varphi_i$, $i = 1, 2$, with the x_1-axis. Thus we have two outputs for the system

$$y_1 = \rho_1(x, \theta, s_1) \tag{7}$$

$$y_2 = \rho_2(x, \theta, s_2), \tag{8}$$

where both ρ_1 and ρ_2 depend naturally also on the terrain manifold, which is defined by

$$x_{e1} = r_1(s) \tag{9}$$

$$x_{e2} = r_2(s). \tag{10}$$

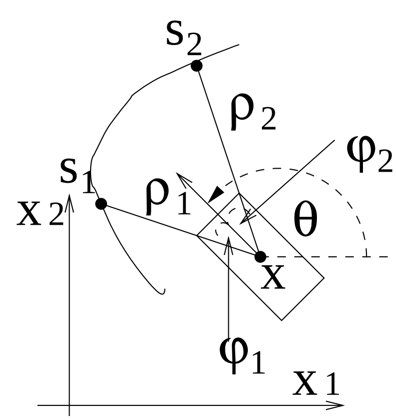

Fig. 1. The system setup

In this paper we assume that the curve $r = (r_1(s), r_2(s))^T$ is closed, i.e. $r : S^1 \to \mathbb{R}^2$, and that it encircles a convex domain D. The robot is restricted to move in the interior of this domain, and thus the outputs ρ_1, ρ_2 are well defined.

We note that if the robot stands still (both v and ω are set to zero), then the system is obviously unobservable.

Our problem is to construct, with the help of the input design, an observer for the full state of the robot. It is easy to see that this problem is equivalent to the reconstruction of orientation θ and the two parameter values $s_1, s_2 \in S^1$ corresponding to the points on the curve measured against. In order to do this we define a (new) state variable $p = (s_1, s_2, \theta) \in S \subset T^3$, which we call the *parameter configuration*. Since we will be concerned with the local properties only, we can consider p as an element of $\mathbb{R}^3$.

For the configuration $(x, \theta) \in \mathbb{R}^2 \times S^1$, the distances $\rho \in \mathbb{R}^2$ and the parameters $s \in \mathbb{R}^2$, we have the basic geometrical relationship

$$x + \rho_i R_\theta R_{\varphi_i} e_1 = r(s_i), \quad i = 1, 2, \tag{11}$$

and differentiating,

$$vR_\theta e_1 + \dot{\rho}_i R_\theta R_{\varphi_i} e_1 + \rho_i \omega R_\theta R_{\varphi_i} e_2 = r'(s_i)\dot{s}_i, \quad i = 1, 2, \tag{12}$$

where we use the prime "$'$" to denote partial derivative. Here we use the notation $e_1 = (1, 0)^T$, $e_2 = (0, 1)^T$, and

$$R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

For future use, we also define $M = R_{\pi/2}$, and note that $\partial R_\alpha / \partial\alpha = MR_\alpha = R_\alpha M$, $Me_1 = e_2$, $Me_2 = -e_1$.

Multiplying (12) with $e_2^T R_{-\varphi_i - \theta}$ from the left gives

$$\dot{s}_i = \frac{-v\sin\varphi_i + \rho_i\omega}{r'(s_i)^T R_\theta R_{\varphi_i} e_2}, \tag{13}$$

and multiplying (12) with $r'(s_i)^T M$, one obtains

$$\dot{\rho}_i = \frac{-vr'(s_i)^T R_\theta e_2 + \rho_i \omega r'(s_i)^T R_\theta R_{\varphi_i} e_1}{r'(s_i)^T R_\theta R_{\varphi_i} e_2}. \tag{14}$$

The above expressions are well-defined except when $r'(s_i)^T R_\theta R_{\varphi_i} e_2 = 0$, which occurs when a sensor measures tangentially to the wall. In the convex setting we have assumed, this will not happen.

Using only the values of ρ_1 and ρ_2 it is easy to see that we will, in general, only be able to determine the configuration of the robot up to some curve in $\mathbb{R}^2 \times S^1$. One can picture this by following a planar curve with the tips of two extended fingers. We call this curve the *statically unobservable submanifold*.

By also taking $\dot{\rho}$ into account, we find a similarly unobservable submanifold, also one–dimensional. The theoretical basis for active nonlinear observer design as we will show later is that under certain conditions the two different submanifolds are not parallel at the actual parameter configuration of the robot, and hence the system ought to be locally observable.

III. The statically unobservable submanifold

For a general parameter vector $p = (s_1, s_2, \theta)$ and two distances ρ_1, ρ_2, we define

$$z_1(p) = r(s_1) - \rho_1 R_\theta R_{\varphi_1} e_1, \tag{15}$$
$$z_2(p) = r(s_2) - \rho_2 R_\theta R_{\varphi_2} e_1, \tag{16}$$
$$z(p) = z_1(p) - z_2(p), \tag{17}$$
$$V_z = z^T z. \tag{18}$$

Clearly, if ρ_1 and ρ_2 are measured distances and p the actual parameter configuration of the robot, then $z_1 = z_2 = x$, and z and V_z vanish. Consider the differential of the map $z : p \mapsto \mathbb{R}^2$,

$$\begin{aligned} \partial_p z &= \begin{bmatrix} \frac{\partial z_x}{\partial s_1} & \frac{\partial z_x}{\partial s_2} & \frac{\partial z_x}{\partial \theta} \\ \frac{\partial z_y}{\partial s_1} & \frac{\partial z_y}{\partial s_2} & \frac{\partial z_y}{\partial \theta} \end{bmatrix} \\ &= \begin{bmatrix} r'(s_1) & -r'(s_2) & M(r(s_2) - r(s_1)) + Mz \end{bmatrix}. \end{aligned}$$

We see that if $z = 0$, this is onto (as a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$) as long as the slope of the curve is not parallel at $r(s_1)$ and $r(s_2)$ and simultaneously perpendicular to the line between these two points. By the Implicit Function Theorem, we then have the following,

Proposition 1: Assume that at a configuration (x, θ), where x lies in the interior of D, the robot measures distances ρ_1, ρ_2 against parameters s_1, s_2. Also assume that $r'(s_1)$ and $r'(s_2)$ are not parallel or, if they are, not perpendicular to the vector $r(s_1) - r(s_2)$. Then, locally around (s_1, s_2, θ), the equation $z(\hat{s}_1, \hat{s}_2, \hat{\theta}) = 0$ defines a one-dimensional surface (a curve) which passes through (s_1, s_2, θ). This is called the statically unobservable submanifold.

If we define $d = r(s_2) - r(s_1)$, then the kernel of $\partial_p z$ is spanned by

$$\nu_z = \begin{bmatrix} -r'(s_2)^T (d + z) \\ -r'(s_1)^T (d + z) \\ r'(s_1)^T M r'(s_2) \end{bmatrix}, \tag{19}$$

which vanishes as the rows of $\partial_p z$ become linearly dependent. In general, it might still be possible to find a continuation of the curve on which $V = 0$ through higher order terms. A clear case of when this is not possible would be when the sensors measures points on the boundary where there simply is no room to move the robot to a nearby point that would give the same measurements (e.g. measuring along the major axis of an ellipse).

It is also clear that if $V_z = 0$ (i.e. $z = 0$) for some $(\rho_1, \rho_2, \hat{s}_1, \hat{s}_2, \hat{\theta})$ and $z_1 = z_2 \in D$, then in the configuration $(z_1, \hat{\theta})$, the robot measures precisely the distances ρ_1, ρ_2 against the parameters $\hat{s}_1, \hat{s}_2$. That is, continuation of the level curves from Proposition 1 gives curves of *feasible* configurations which, from the information given by the measured distances, the robot could occupy.

Then it is clear that the gradient flow

$$\frac{d\hat{p}}{dt} = -k_s (\partial_{\hat{p}} z)^T z(\hat{p}), \quad k_s > 0$$

should be included in the observer design. It serves to localize the parameter configuration to the submanifold of feasible configurations.

Next, we will discuss what can be done in the statically unobservable direction.

IV. NONLINEAR OBSERVERS WITH ACTIVE CONTROL

It is clear that with any pair of fixed measurements $(\rho_1,\ \rho_2)$, the system will have a one dimensional unobservable submanifold. Thus it is necessary to design controls such that the outputs do not remain constant. In this section we will first discuss what are the constraints we need impose on the motion and the environment in order to make the system observable.

Assuming that we have obtained a point on the curve of feasible configurations by some optimization method, we would now like to localize the actual configuration of the robot by finding a point on the curve where the estimated output flow (time derivatives of the measured distances) coincide with the actual output flow.

From the basic relationship

$$x + \rho_i R_\theta R_{\varphi_i} e_1 = r(s_i), \quad i = 1, 2,$$

we obtain

$$r'(s_i)^T R_\theta (v e_2 + \dot\rho_i R_{\varphi_i} e_2 - \rho_i \omega R_{\varphi_i} e_1) = 0, \quad i = 1, 2, \tag{20}$$

by first differentiating with respect to time and then multiplying by $r'(s_i)^T M$ from the left. Now define $F : \mathbb{R}^3 \to \mathbb{R}^2$, for given $v, \omega, \rho_i, \dot\rho_i$, by

$$F(s_1, s_2, \theta) = \begin{bmatrix} r'(s_1)^T u_1 \\ r'(s_2)^T u_2, \end{bmatrix}. \tag{21}$$

where u_1, u_2 are given by

$$u_i = R_\theta (v e_2 + \dot\rho_i R_{\varphi_i} e_2 - \rho_i \omega R_{\varphi_i} e_1), \quad i = 1, 2. \tag{22}$$

Naturally one can use (14) to replace $\dot\rho_i$.

The differential of F with respect to the parameter vector is given by

$$\partial_p F = \begin{bmatrix} r''(s_1)^T u_1 & 0 & r'(s_1)^T M u_1 \\ 0 & r''(s_2)^T u_2 & r'(s_2)^T M u_2 \end{bmatrix}, \tag{23}$$

The kernel of $\partial_p F$, as long as this has full rank, is spanned by

$$\nu_F = \begin{bmatrix} -(r''(s_2)^T u_2)(r'(s_1)^T M u_1) \\ -(r''(s_1)^T u_1)(r'(s_2)^T M u_2) \\ (r''(s_1)^T u_1)(r''(s_2)^T u_2) \end{bmatrix}. \tag{24}$$

Assume now $F = 0$ and $p_0 = (s_1, s_2, \theta)$ are the true parameters and state, then we have

$$\begin{aligned} u_i(p_0) &= R_\theta R_{\varphi_i} \begin{bmatrix} v \sin\varphi_i - \omega\rho_i \\ v\cos\varphi_i + \dot\rho_i \end{bmatrix} \\ &= (v\sin\varphi_i - \omega\rho_i) R_\theta R_{\varphi_i} \begin{bmatrix} 1 \\ -\frac{r'(s_i)^T R_\theta R_{\varphi_i} e_1}{r'(s_i)^T R_\theta R_{\varphi_i} e_2} \end{bmatrix}, \\ & \quad i = 1, 2, \end{aligned}$$

or

$$M u_i(p_0) = \frac{(v\sin\varphi_i - \omega\rho_i)}{r'(s_i)^T R_\theta R_{\varphi_i} e_2} r'(s_i), \quad i = 1, 2. \tag{25}$$

Hence, we have

$$\nu_F(p_0) = \frac{(v\sin\varphi_1 - \omega\rho_1)(v\sin\varphi_2 - \omega\rho_2)}{(r'(s_1)^T R_\theta R_{\varphi_1} e_2)(r'(s_2)^T R_\theta R_{\varphi_2} e_2)} \times \\ \times \begin{bmatrix} (r''(s_2)^T M r'(s_2))(r'(s_1)^T r'(s_1)) \\ (r''(s_1)^T M r'(s_1))(r'(s_2)^T r'(s_2)) \\ (r''(s_1)^T M r'(s_1))(r''(s_2)^T M r'(s_2)) \end{bmatrix}.$$

What we should consider, though, is

$$\partial_p F \nu_z|_{p_0} = \begin{bmatrix} \frac{v\sin\varphi_1 - \omega\rho_1}{r'(s_1)^T R_\theta R_{\varphi_1} e_2} D_1 \\ \frac{v\sin\varphi_2 - \omega\rho_2}{r'(s_2)^T R_\theta R_{\varphi_2} e_2} D_2 \end{bmatrix}, \tag{26}$$

where

$$D_1 = ((r''(s_1)^T M r'(s_1))(r'(s_2)^T d) + \\ (r'(s_1)^T r'(s_1))(r'(s_1)^T M r'(s_2))),$$

$$D_2 = ((r''(s_2)^T M r'(s_2))(r'(s_1)^T d) + \\ (r'(s_2)^T r'(s_2))(r'(s_1)^T M r'(s_2))).$$

As long as this is non-zero, the statically unobservable submanifold crosses the dynamically unobservable submanifold transversely, which implies that we are bound to have at least local observability through ρ and $\dot\rho$ or the output flow.

We discuss now what constraints we have to impose on the environment and the control so that $\partial_p F \nu_z|_{p_0}$ does not vanish.

We will assume that $\partial_p z$ has full rank, so that ν_z can be expressed as (19) , which is equivalent to assume

H 1: Denote the range of the sensors by R_s. There are no two points s_1 and s_2 on the curve with $\|r(s_2) - r(s_1)\| \le 2R_s \sin\frac{(\phi_1+\phi_2)}{2}$, such that $r'(s_1)$ and $r'(s_2)$ are parallel and at the same time perpendicular to $r(s_2) - r(s_1)$.

Should it happen that

$$\begin{bmatrix} \sin\varphi_1 & -\rho_1 \\ \sin\varphi_2 & -\rho_2 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} = 0,$$

then $\partial_p F \nu_z|_{p_0}$ vanishes completely. Thus we need the following assumption on the control design:

H 2: The exciting control $v,\ \omega$ should be designed such that $v\sin\varphi_i - \omega\rho_i \neq 0$, for $i = 1, 2$.

Having one of the components zero corresponds to moving in such a way that $\dot s_i = 0$ for that sensor. If the sensors are directed to different sides of the steering direction and the control input is not zero, this cannot happen since ρ_1 and ρ_2 both are positive.

Assumption *H*2 has actually suggested a way for designing the active control. Namely, the following criterion can be used:

$$\max_{|v|,|\omega|\le k} \| \begin{bmatrix} \sin\varphi_1 & -\rho_1 \\ \sin\varphi_2 & -\rho_2 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \|.$$

Under the above assumptions, $\partial_p F|_{p_0}$ will still not have full rank if r has zero curvature ($r''(s) = 0$) at both these points, and $\partial_p F \nu_z|_{p_0}$ will vanish if furthermore $r'(s_1)$ and $r'(s_2)$ are parallel.

The final way $\partial_p F \nu_z|_{p_0}$ can vanish is if

$$D_1 = 0 \text{ and } D_2 = 0.$$

To give a geometrical interpretation of this, consider when r is the unit circle, i.e. $r(\alpha) = R_\alpha e_1$. Then we have

$$\begin{aligned}&(r''(\alpha_1)^T M r'(\alpha_1))(r'(\alpha_2)^T d)+\\&(r'(\alpha_1)^T r'(\alpha_1))(r'(\alpha_1)^T M r'(\alpha_2))\\&= e_2^T R_{\alpha_2}^T (R_{\alpha_2} e_1 - R_{\alpha_1} e_1) + e_2^T R_{\alpha_1}^T M R_{\alpha_2} e_2\\&= 0,\end{aligned}$$

This corresponds to the observation that in a circular symmetry, neither the measured distances nor the time derivatives thereof change if we displace the robot in the ”symmetric direction”.

Based on the above discussion, we need to make the last assumption:

H 3: There are no two points s_1 and s_2 on the curve with $\|r(s_2) - r(s_1)\| \leq 2R_s \sin \frac{(\phi_1+\phi_2)}{2}$, such that
1. $\|r''(s_1)\|^2 + \|r''(s_2)\|^2 = 0$,

or,

2. $D_1 = 0$, $D_2 = 0$.

Design of observer

Under the above assumptions, we propose the following observer

$$\frac{d\hat{p}}{dt} = -k_s(\partial_{\hat{p}} z)^T z(\hat{p}) - k_a(\partial_{\hat{p}} F)^T F(\hat{p}), \tag{27}$$

where $k_s, k_a > 0$ are some suitably chosen feedback gains.

V. Convergence analysis

In this section we will only study the scenario where periodic exciting controls are used. We show that in this case (27) converges asymptotically to $p_0(t)$, the true parameters and state trajectory, with bounded errors.

Proposition 2: Suppose Assumptions *H*1-3 hold and the motion generated by the exciting control is periodic. Then, by using the observer (27), locally the estimation error is bounded and the bound can be made arbitrarily small asymptotically by tuning k_s and k_a.

Proof

It is easy to see that the true parameters and state trajectory $p_0(t)$ is indeed an equilibrium for (27), since $z(p_0(t)) = 0$ and $F(p_0(t)) = 0$.

Let $\delta p = \hat{p} - p_0(t)$ and we define a candidate Lyapunov function as

$$V(\delta p) = \frac{1}{2}\|\delta p\|^2,$$

then,

$$\dot{V} = -\delta p^T [k_s(\partial_{\hat{p}} z)^T z(\hat{p}) + k_a(\partial_{\hat{p}} F)^T F(\hat{p})] - \delta p^T \dot{p}_0(t).$$

When δp is sufficiently small, we have

$$z(\hat{p}) \approx \partial_{p_0} z \delta p, \;\; F(\hat{p}) \approx \partial_{p_0} F \delta p.$$

Then,

$$\dot{V} = -\,\delta p^T Q(t) \delta p + o(\|\delta p\|^2) - \delta p^T \dot{p}_0(t),$$

where

$$Q(t) = k_s(\partial_{p_0} z)^T \partial_{p_0} z + k_a(\partial_{p_0} F)^T \partial_{p_0} F.$$

By the transversality assumption of the two submanifolds, we have $rank\ [(\partial_{p_0} z)^T\ (\partial_{p_0} F)^T]^T = 3$. Therefore

$$Q(t) = [\sqrt{k_s}(\partial_{p_0} z)^T\ \sqrt{k_a}(\partial_{p_0} F)^T] \begin{bmatrix} \sqrt{k_s}\partial_{p_0} z \\ \sqrt{k_a}\partial_{p_0} F \end{bmatrix}$$

is always of full rank, and thus positive definite for any fixed t.

The assumption of periodic motion implies that $Q(t)$ is periodic and $\dot{p}_0(t)$ is bounded. Thus when δp is sufficiently small and if we let $k_s > k,\ k_a > k$, we have

$$\dot{V} \leq -k\gamma\|\delta p\|^2 + P\|\delta p\|, \tag{28}$$

where $\gamma > 0,\ P > 0$. This shows that by tuning k we can make the tracking error arbitrarily small as $t \to \infty$. QED

Remark 1: Since the first term in (28) is quadratically negative definite, one can expect that the observer is reasonably robust with respect to measurement errors.

VI. Simulation

In this section we show some Matlab simulation results. In the simulation setup, the robot is placed inside an ellipse shaped closed wall.

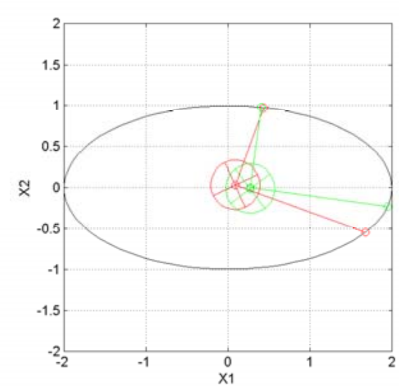

Fig. 2. The initial estimation

Figure 2 shows the initial guess of the state versus the true state. Figure 3 shows that the estimation of the state is converging to the true state. Figure 4 shows the history of estimation errors.

VII. Conclusion

In this paper we have studied the problem of designing an observer for a mobile robot together with the exciting control. For such systems, there is no obvious way to apply existing methods for the observer design. By discussing conditions we have to impose on the control and the environment in order to have observability, we have suggested a method for the observer design based on periodic excitations.

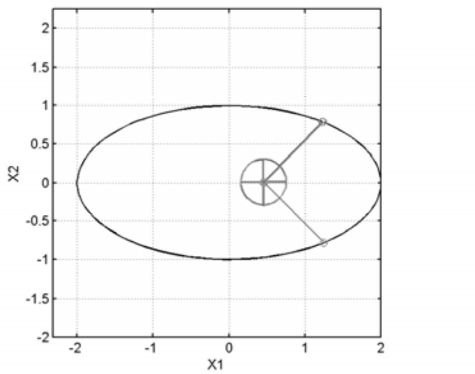

Fig. 3. The final estimation

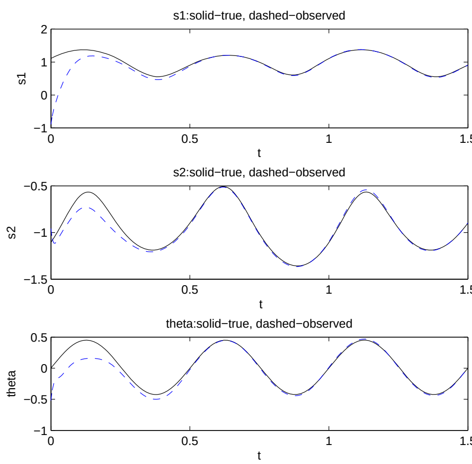

Fig. 4. The observation errors

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