Periodic and Recursive Control Theoretic Smoothing Splines

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Abstract

In this paper a recursive control theoretic smoothing spline approach is proposed for reconstructing a closed contour. Periodic splines are generated through minimizing a cost function subject to constraints imposed by a linear control system. The optimal control problem is shown to be proper and sufficient optimality conditions are derived for a special case of the problem using Hamilton-Jacobi-Bellman theory.

The filtering effect of the smoothing splines allows for usage of noisy sensor data. An important feature of the method is that several data sets for the same closed contour can be processed recursively so that the accuracy can be improved stepwise as new data becomes available.

1 Introduction

In this paper we focus on the problem of reconstructing closed contours from noisy and sparse samples.

Data smoothing is a classical problem in system and control history [1,3]. Smoothing splines were introduced in the 1960s. As opposed to interpolating splines, smoothing splines pass close to, rather than through interpolation points, providing a filtering or smoothing effect. A comprehensive overview is given in [4] and [5], where splines are studied in a statistical setting. In mathematical statistics, the aim of the smoothing spline is generally to fit a curve to a data set so that the error between the curve and the data has nice statistical properties, for instance that it is normally distributed.

Control theoretic smoothing splines were first introduced in [10], and are further studied in for instance [11, 12, 15]. In [16] the smoothing spline problem is studied using Hilbert space methods. [6] is, to the extent of the authors' knowledge, the first book to give a complete overview of the field. The aim of control theoretic smoothing splines is to find a trade-off between faithfulness to a given data set and control gain. This is motivated by for instance trajectory tracking applications, where exact tracking of way points often calls for undesirable large accelerations.

A nice property of smoothing splines is robustness. [13] observes that smoothing splines are in some sense band limited so that small changes in one data point will mainly affect the spline in a neighborhood of that point.

The focus of this paper is control theoretic smoothing splines with a periodicity constraint. A recursive approach is developed, where the estimate of the underlying, closed curve is improved stepwise as new data sets are recovered.

2 Related Work and Contributions

We note that recursive approaches to constructing splines have been investigated in [23] and [29]. In the field of robotics, recursive cubic B-spline methods for path planning have been presented in [21] and [22]. However, little work has been done in this direction with control theoretic splines. A notable contribution is presented in [6]. While the recursive smoothing spline problem formulated in [6] includes previous curve estimates, the formulation in the present paper only includes them implicitly.

Control theoretic smoothing splines may be viewed as point-to-point LQ optimal control problems with dynamic constraints. Point-to-point LQ optimal control problems have been investigated in for instance [24] and [25], where [24] treats the optimal output-transition problem for linear systems while [25] considers LTI continuous-time systems with affine constraints in initial and terminal states.

[14] investigates a point-to-point LQ optimal control problem under the assumption of dynamic constraints with a stochastic uncertainty. This paper examines a similar LQ problem, with the important distinction that we optimize over all periodic solutions. Again, results for periodic LQ problems can be found in papers dating back to the 1970s, [8, 9], but these do not cover the point-to-point problem.

In the field of mathematical statistics, early contributions on periodic smoothing splines include [26]. Until recently, little work had been done on periodic, control theoretic smoothing splines. Notable contributions have however emerged during the past few years. [28] studies applications of control theoretic smoothing splines to mobile robotics, and poses a problem where the periodicity constraint depends on input data. In [27] periodic, control theoretic smoothing spline problems are solved using Hilbert Space methods. In the current paper, optimality conditions for such splines are examined using Hamilton-Jacobi-Bellman theory for optimal control problems.

3 Outline

The paper is organized as follows. In Section 4, we state the contour estimation problem formally and propose a closed form and a recursive point-to-point LQ formulation for estimation of closed contours. In Section 5 we discuss optimality conditions for the periodic smoothing spline problem. Some simulation results are reported in Section 6, and a concluding summary is provided in Section 7.

4 Problem statement and motivation

Consider the problem of reconstructing continuous, smooth, closed curves in $\mathbb{R}^2$ from noisy and sparse measurement data. This problem arises for instance in mapping applications and trajectory tracking for mobile robots. In this section we pose two optimal control problems that aim to find the best estimate of an underlying, closed curve given noise contaminated samples from the curve. First, we introduce a closed form optimal control problem that yields a first estimate of the underlying curve and then a modified, recursive problem. A formal problem statement follows.

Given a data set $D = \{(t_i, z_i) : i = 1, ..., N\}$, where $z_i = z(t_i)$, $t_i \in [0, T]$ and $z(T) = z(0)$. If $z_i = y(t_i) + \xi_i$ are noise contaminated samples from a closed continuous curve, where ξ_i is a symmetric, zero-mean white noise, how to find the curve $y(t)$ that best represents the data?

We view this as an optimal control problem, where $y(t)$ is the output of a dynamic system whose control input $u(t)$ should be properly designed.

4.1 Closed Form

The following closed form L_2 smoothing problem yields an estimate of $y(t)$ while minimizing the control effort $u(t)$.

Problem 4.1

$$\min_{u \in L_2[0,T]} J(u,x) = \frac{1}{2}\int_0^T u(t)'Q^{-1}u(t)dt + \frac{1}{2}\sum_{i=1}^{N}(t_i - t_{i-1})(z_i - Cx(t_i))'R^{-1}(z_i - Cx(t_i)) \tag{1}$$

$$\text{s.t.} \quad \dot{x} = Ax + Bu \tag{2}$$

$$x(0) = x(T), \tag{3}$$

where $t_0 = t_N - T$, and $t_i > t_{i-1}$ for $i = 1, \ldots, N$. The constraints (2) - (3) consist of an n-dimensional ODE with relative degree n and periodic boundary condition. The resulting smoothing spline is given by $y(t) = Cx(t)$. The system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{4}$$

is referred to as the spline generator of (1). Thus the sought curve y is the output of an n-tuple integrator with state variable x, controlled by an input $u = x_n = x_1^{(n)}$, the n^{th} derivative of x_1. The dimension of the spline generator determines on which derivative to impose the smoothing penalty. As the magnitude of the second derivative of $x_1(t)$ is proportional to the curvature of $x_1(t)$, $n = 2$ is a natural choice. Throughout the paper, the following numerical values are used.

$$\begin{aligned} &A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ &Q = 1, \qquad R_0 = 1/\varepsilon^2, \quad T = 2\pi. \end{aligned} \tag{5}$$

Let us have a closer look at the cost function (1). The integral imposes a penalty on large magnitude of the input $u(t)$, corresponding to the curvature of the curve $y(t) = Cx(t) = x_1(t)$. The sum punishes large deviations of the curve $y(t)$ from the data (t_i, z_i). In other words, the solution of Problem 4.1 is in some sense the optimal compromise between smoothness of the output curve and faithfulness to the data set. The magnitude of the smoothing parameter $\varepsilon > 0$ determines how much credibility is given to measurement data. In the next section it is shown how a modification of Problem 4.1 allows for a recursive approach to the contour estimation problem.

4.2 Recursive Form

As the data is noise contaminated, the resulting spline from one data set D may give a poor estimate of the underlying curve. If new data becomes available over time, improvements of the estimate can be made by solving a recursive form of Problem 4.1. Here, the optimal control $u^{k-1}(t)$ from the previous iteration is used in iteration k together with new data (t_i^k, z_i^k).

Problem 4.2

$$\min_{u^k \in L_2[0,T]} J^k(u^k, x^k) = \frac{1}{2}\int_0^T (u^k(t) - u^{k-1}(t))' Q^{-1} (u^k(t) - u^{k-1}(t)) dt + \frac{1}{2}\sum_{i=1}^N (t_i^k - t_{i-1}^k)(z_i^k - Cx^k(t_i^k))' R^{-1} (z_i^k - Cx^k(t_i^k)) \quad (6)$$

$$\text{s.t.} \quad \dot{x}^k = Ax^k + Bu^k \quad (7)$$

$$x^k(0) = x^k(T). \quad (8)$$

Introduce the notation

$$\begin{aligned} \tilde{z}_i^k &= z_i^k - x_i^{k-1}(t_i^k) \\ \tilde{x}^k(t) &= x^k(t) - x^{k-1}(t) \\ \tilde{u}^k(t) &= u^k(t) - u^{k-1}(t). \end{aligned} \quad (9)$$

At iteration k, the spline solution $\tilde{x}^k$ of Problem 4.2 is an adjustment of the curve x^{k-1} based on the new data (t_i^k, z_i^k). Substituting for the variables (9) in (6) and (7) - (8), we obtain

$$\min_{\tilde{u}^k \in L_2[0,T]} J^k(\tilde{u}^k, \tilde{x}^k) = \int_0^T \tilde{u}^k(t)' Q^{-1} \tilde{u}^k(t) dt + \frac{1}{2}\sum_{i=1}^N (t_i^k - t_{i-1}^k)(\tilde{z}_i^k - C\tilde{x}^k(t_i))' R^{-1} (\tilde{z}_i^k - C\tilde{x}^k(t_i)) \quad (10)$$

$$\dot{\tilde{x}}^k = A\tilde{x}^k + B\tilde{u}^k \quad (11)$$

$$\tilde{x}^k(0) = \tilde{x}^k(T), \quad (12)$$

which are identical to (1) - (3). Therefore, solution methods and optimality conditions for Problem 4.1 and Problem 4.2 are identical. The curve estimate at iteration k can now be written

$$x^k = x^1 + \sum_{j=2}^{k-1} \tilde{x}^j \quad (13)$$

where x^1 is the spline solution to Problem 4.1 for the first batch of data. In the next section, optimality conditions for this smoothing problem is discussed.

Remark 1 *It is intuitively easy to see that using the closed form (Problem 4.1) and increasing the number N of data points, the spline output should approach the underlying curve if the noise is symmetric. If such a data set is available, this may be an option. In many applications, however, new data may arrive at different points in time, calling for an update of the estimate. This is for instance the case when data is collected by teams of cooperating autonomous vehicles. As k increases, the error of the spline estimate with this recursive formulation decreases only slightly slower than when increasing N for the closed form. In the extreme, a further motivation for the recursive formulation is that as $N \to \infty$, Problem 4.1 may experience numerical instability in implementation.*

5 Properness and Optimality

Problem 4.1 is a continuous time problem with discrete data and periodic boundary conditions. Such problems, without the periodic constraint, have been widely studied in the literature, see for example the books by Bryson and Ho [1], Leondes [2] and Jazwinski [3]. However, as far as we know, it is difficult to find results concerning the periodic case. In this section, we investigate conditions for solving this problem. We begin by studying the proper periodicity conditions.

5.1 Proper periodicity conditions

In this section, we adopt the notations used in [7]. Let $\bar{J}^* = J(\bar{u}^*, \bar{x}^*)$ in Problem 4.1, where $\bar{u}^*$ is any constant function.

Definition 1 *The optimal control problem is **proper** if there exists an admissible control $\bar{u}(\cdot)$ such that*

$$J(\bar{u}(\cdot), \bar{x}(\cdot)) < \bar{J}^*. \tag{14}$$

In this context, proper periodicity conditions refers to conditions establishing whether an optimal, periodic control problem is proper or not. The following proposition establishes that Problem 4.1 is proper for all but a special case of data input.

Proposition 1 *For distinct sampling angles $[t_1, \ldots, t_N]$, Problem 4.1 is proper if and only if $\exists\, i, j \in [1, N]$ such that $z_i \neq z_j$.*

Proof The proof is constructed by showing that a particular, time varying, periodic control $\bar{u}(t) = \alpha \hat{u}(t)$ satisfies (14). For details, see Appendix A.

■

Next, we discuss optimality conditions for Problem 4.1.

5.2 Optimality Conditions and Hamilton-Jacobi-Bellman Theory

In the following, sufficient optimality conditions for Problem 4.1 are examined using a Hamilton-Jacobi-Bellman approach. A set of differential equations are derived leading to an expression of the optimal control u. It should be noted that for the particular choice of system matrices (5), straightforward and well known approaches for regular smoothing splines are applicable [5]. The purpose of this section is to analyze smoothing splines from a control perspective and suggest solutions for more complex spline generators.

First, a brief review of the Hamilton-Jacobi-Bellman theory is given in Section 5.2.1. Then, in Section 5.2.2, sufficient optimality conditions are given for the special case of Problem 4.1 where the input is a continuous curve. Finally, the discrete data case is discussed in Section 5.2.3.

5.2.1 Optimality Conditions for a General Periodic Control Problem

Consider the problem

Problem 5.1

$$\min_{u \in L_2[0,T]} J(u,x) = \int_0^T L(x(t),u(t))dt \tag{15}$$

$$\begin{aligned} \text{s.t.} \quad \dot{x} &= f(x,u) & (16) \\ x(0) &= x(T). & (17) \end{aligned}$$

Some definitions of useful concepts follow.

Definition 2 *The **Hamiltonian** of* (16) *is*

$$H(x,u,\lambda) \triangleq L(x,u) + \lambda' f(x,u). \tag{18}$$

Definition 3 *The **H-minimal control** u^* is defined as*

$$u^*(x,\lambda) \triangleq \arg\min_u \quad H(x,u,\lambda) \tag{19}$$

Definition 4 *The **Hamilton-Jacobi-Bellman equation** is*

$$\frac{\partial V(t,x)}{\partial t} = -H\left(x, u^*\left(x, \frac{\partial V(t,x)}{\partial x}\right), \frac{\partial V(t,x)}{\partial x}\right). \tag{20}$$

The following proposition is proved in [8]:

Proposition 2 *Suppose that*

1. *The control u_T generates a periodic solution x_T of* (15) - (17)*.*

2. *There exists a continuously differentiable solution* $V(t,x)$ *of* (20) *such that*

$$V(0,x) - V(T,x) = C(T), \tag{21}$$

where $C(t)$ *is a real function.*

Then, u_T *is optimal to* (15) - (17) *if*

$$u_T = u^*\left(x_T, \frac{\partial V(t,x_T)}{\partial x}\right). \tag{22}$$

Next, optimality conditions are first stated for a periodic smoothing problem with continuous data and a general expression for the optimal control is given. Finally, the optimal control for Problem 4.1 in the limit $N \to \infty$ is derived.

5.2.2 Optimality Conditions: Continuous time, continuous data

Consider the following problem:

Problem 5.2

$$\min_{u\in L_2[0,T]} J(u,x) = \frac{1}{2}\int_0^T \left[u(t)'Q^{-1}u(t) + (z(t)-Cx(t))'R^{-1}(z(t)-Cx(t))\right] dt \tag{23}$$

$$\begin{aligned} \text{s.t.}\quad \dot{x} &= Ax + Bu && (24)\\ x(0) &= x(T). && (25)\end{aligned}$$

This may be viewed as a smoothing problem with continuous data $z(t)$ or as a problem of tracking a curve given by $z(t)$. Optimality conditions for problems of this type were derived in [8]. A review of the results follows. For brevity, throughout this section we use the notation V_t and V_x to denote the partial derivatives of V. The Hamiltonian corresponding to (23) is

$$H(x,u,\lambda) = \frac{1}{2}u'Q^{-1}u + \frac{1}{2}x'C'R^{-1}Cx - x'C'R^{-1}z + \frac{1}{2}z'R^{-1}z + \lambda'(Ax+Bu). \tag{26}$$

The $H-minimal\ control$ u^* is derived as

$$\frac{\partial H(x,u,\lambda)}{\partial u} = Q^{-1}u + B'\lambda \quad \Rightarrow \quad u^* = -QB'\lambda, \tag{27}$$

so $u^*(x_T,V_x) = -QB'V_x$. Then the Hamilton-Jacobi-Bellman equation is

$$V_t \quad = \quad \frac{1}{2}V_x'BQB'V_x - \frac{1}{2}x'C'R^{-1}Cx + x'C'R^{-1}z - \frac{1}{2}z'R^{-1}z - V_x'Ax. \tag{28}$$

In [8] the following form for $V(t,x)$ is proposed for Problem 5.2:

$$V(t,x) = \frac{1}{2}x'Px + x'\phi + s. \tag{29}$$

Here, ϕ, P and s should be chosen so that (28) is satisfied, and $V(0,x) - V(T,x) = C(T)$ for some real function $C(t)$. Furthermore, P is a symmetric, positive semidefinite matrix. The derivatives V_t and V_x are

$$V_t = \frac{1}{2}\dot{x}'Px + \frac{1}{2}x'P\dot{x} + \frac{1}{2}x'\dot{P}x + \dot{x}'\phi + x'\dot{\phi} + \dot{s} \tag{30}$$

$$V_x = Px + \phi. \tag{31}$$

Then, since

$$u_T = u^*(x, V_x) = -QB'(Px + \phi) \tag{32}$$

we get

$$\dot{x}_T = Ax_T + Bu_T = Ax_T - BQB'(Px_T + \phi) = (A - BQB'P)x_T - BQB'\phi. \tag{33}$$

Somewhat tedious calculations yield, after plugging in (30), (31) and (32) into (28), that P, ϕ and s must satisfy

$$\dot{P} = -A'P - PA + PBQB'P - C'R^{-1}C \tag{34}$$

$$\dot{\phi} = -(A - BQBP)'\phi + C'R^{-1}z \tag{35}$$

$$\dot{s} = \frac{1}{2}\phi'BQB'\phi - \frac{1}{2}z'R^{-1}z \tag{36}$$

$$P(T) = P(0) \tag{37}$$

$$\phi(T) = \phi(0) \tag{38}$$

$$s(T) = s(0) - C(T). \tag{39}$$

In [8] the constant, positive definite solution $\overline{P}$ to the *algebraic* Riccati equation $\dot{P} = 0$ is chosen. For the special case (23) - (25), with linear dynamics and continuous data, the trivially periodic $\overline{P}$ satisfies the optimality conditions.

Moreover, for $\overline{P}$ constant and positive definite and with A, B and Q as defined by (5), all eigenvalues of $A - BQB'P$ have negative real part. In fact, it is observed in [8] that this holds for all matrices A, B, C, Q^{-1} and $R^{-1} = D'D$ such that (A,B) is controllable and $(A, D'C)$ is observable, and Q^{-1} positive definite. It follows that no eigenvalues of $(I - e^{A-BQB'PT})$ and $(I - e^{-(A-BQB'P)'T})$ are equal to 1 or -1. Hence (33) and (35) have unique, T-periodic solutions.

In order to generalize to scenarios with discrete data, P must however be time-varying. To the extent of the authors' knowledge, necessary conditions for existence of periodic solutions to Riccati-type equations are generally difficult to find.

5.2.3 Optimality Conditions: Continuous time, discrete data

We restate Problem 4.1 for the reader's convenience:

Problem 5.3

$$\min_{u \in L_2[0,T]} J(u,x) = \frac{1}{2}\int_0^T u(t)'Q^{-1}u(t)dt + \frac{1}{2}\sum_{i=1}^{N}(t_i - t_{i-1})(z(t_i) - Cx(t_i))'R^{-1}(z(t_i) - Cx(t_i)) \tag{40}$$

$$\begin{aligned} \text{s.t.} \quad \dot{x} &= Ax + Bu \quad (41) \\ x(0) &= x(T). \quad (42) \end{aligned}$$

Note that here, we represent the data z_i as samples of a function $z(t)$ at times t_i. Due to the periodicity constraint, it is not trivial to find optimality conditions for this problem. A similar problem is studied in [14]. There, the cost function is of the form

$$\sum_{i=0}^{N-1} \left(w_{i+1} |z(t_{i+1}) - Cx(t_{i+1})|^2 + \int_{t_i}^{t_{i+1}} \sigma_{i+1}(t,x,u)dt \right) \quad (43)$$

where $\sigma_i(t,x,u)$ contains, in addition to the u-quadratic term, cross terms for x and u and linear terms in x and u. [14] assumes dynamic constraints of the form (41) but includes a multiplicative stochastic uncertainty. Further, the initial value $x(0)$ is fix.

(40) may be viewed as a special case of (43), where some terms are removed. Removing the stochastic terms from the dynamic constraints, the control problem in [14] is the same as Problem 5.3 except for the boundary constraints. In [14], using functions V_i of the form (29), the following relations are obtained (where we have substituted for the notation in the current paper):

$$\begin{aligned} \dot{P}_i &= -A'P_i - P_iA + P_iBQB'P_i \quad (44) \\ P_i(t_{i+1}) &= P_{i+1}(t_{i+1}) + (t_{i+1} - t_i)C'R^{-1}C \quad (45) \\ \dot{\phi}_i &= -(A - BQB'P_i)'\phi_i \quad (46) \\ \phi_i(t_{i+1}) &= \phi_{i+1}(t_{i+1}) - (t_{i+1} - t_i)C'R^{-1}z(t_{i+1}) \quad (47) \\ \dot{s}_i &= \frac{1}{2}\phi_i'BQB'\phi_i \quad (48) \\ s_i(t_{i+1}) &= s_{i+1}(t_{i+1}) + \frac{1}{2}(t_{i+1} - t_i)z(t_{i+1})'R^{-1}z(t_{i+1}). \quad (49) \end{aligned}$$

On the interval $[t_i, t_{i+1}]$ the resulting optimal control is

$$u_i = -QB'(P_ix + \phi_i). \quad (50)$$

The authors believe that Proposition 2 can be generalized to allow for a piecewise continuous function of the form

$$V(x,t) = \sum_{i=0}^{N-1} V_i(x,t)[H(t - t_i) - H(t - t_{i+1})] \quad (51)$$

where $H(t)$ is the Heaviside step function and $V(x,t)$ has the property

$$V(x,0) - V(x,T) = C(T). \quad (52)$$

This would imply that the piecewise continuous control

$$u_{T,N} = u_i(t) \quad t \in [t_i, t_{i+1}], \quad i = 0, \dots, N-1 \quad (53)$$

is optimal to Problem 5.3 if there exist solutions to (44) - (49) such that

$$
\begin{aligned}
P(T) &= P(0) && (54)\\
\phi(T) &= \phi(0) && (55)\\
s(T) &= s(0) - C(T), && (56)
\end{aligned}
$$

with $P(t), \phi(t), s(t)$ defined analogously with $V(x,t)$ in (51). Although simulation results support this claim, a proof is yet to be constructed. A weaker result is stated in the following proposition.

Proposition 3 *As $N \to \infty$, and under the assumption that* (54) - (56) *hold, $u_{T,N}$ defined by* (53) *converges to u_T defined by* (32)*.*

Proof The proof follows from the definition of the derivative. See Appendix B for details.

■

Remark 2 *With a clever choice of discretization of Problem 5.3, $u_{T,N}$ can be computed as the solution of an unconstrained, quadratic programming problem, avoiding the difficulty of finding solutions to* (44) - (49)*. This is discussed in further detail in [17], [18], [19] and [20].*

6 Simulations

In this section, we show an example of curve estimation with Problem 4.1 and Problem 4.2. We let $y_{true}(t)$ denote the underlying curve while $(y^k(t), u^k_{T,N}(t))$ denotes the optimal smoothing solution at iteration k. For $k = 1$ the closed form (Problem 4.1) was used, and for $k = 2, 3, \ldots$ the recursive form (Problem 4.2), yielding the control $\tilde{u}^k_{T,N}(t)$ (9). Throughout the simulations, a noise $\xi_i \in \mathbf{N}(0, 0.05)$ was added to the samples z_i of $y_{true}(t_i)$ to simulate measurement noise. An example is shown in Figure 1 for $k = 1, 5, 10$. As expected, the resulting control $u_{T,N}$ (53) is periodic with breaks at the interpolation points t_i. It is also worth noting that as k grows, the magnitude of $\tilde{u}^k(t)$ decreases as a consequence of the error convergence.

To further evaluate the recursive problem, a study of error convergence was performed for Problem 4.2. For reference, Problem 4.1 was solved for an increasing number of data points N. Denote by $y^k(t)$ and R^k the solution and error of Problem 4.2 at iteration k and let y^N and R^N denote the solution and error of Problem 4.1 for N data points. The errors are computed as

$$
R^{(\cdot)} = \int_0^T (y^{(\cdot)}(t) - y_{true}(t))^2 dt. \tag{57}
$$

Results are shown in Figure 2. Mean values of R^k and R^N are plotted for 25 test cases. For $k = 1, \ldots, 100$ Problem 4.2 was solved with $N(k) = 15$ data points at each iteration, while Problem 4.1 was solved with $N(k) = 15k$ data points. Figure 2 shows that the performance of the recursive problem is almost as good as that of the closed form, without the drawbacks discussed in Remark 1.

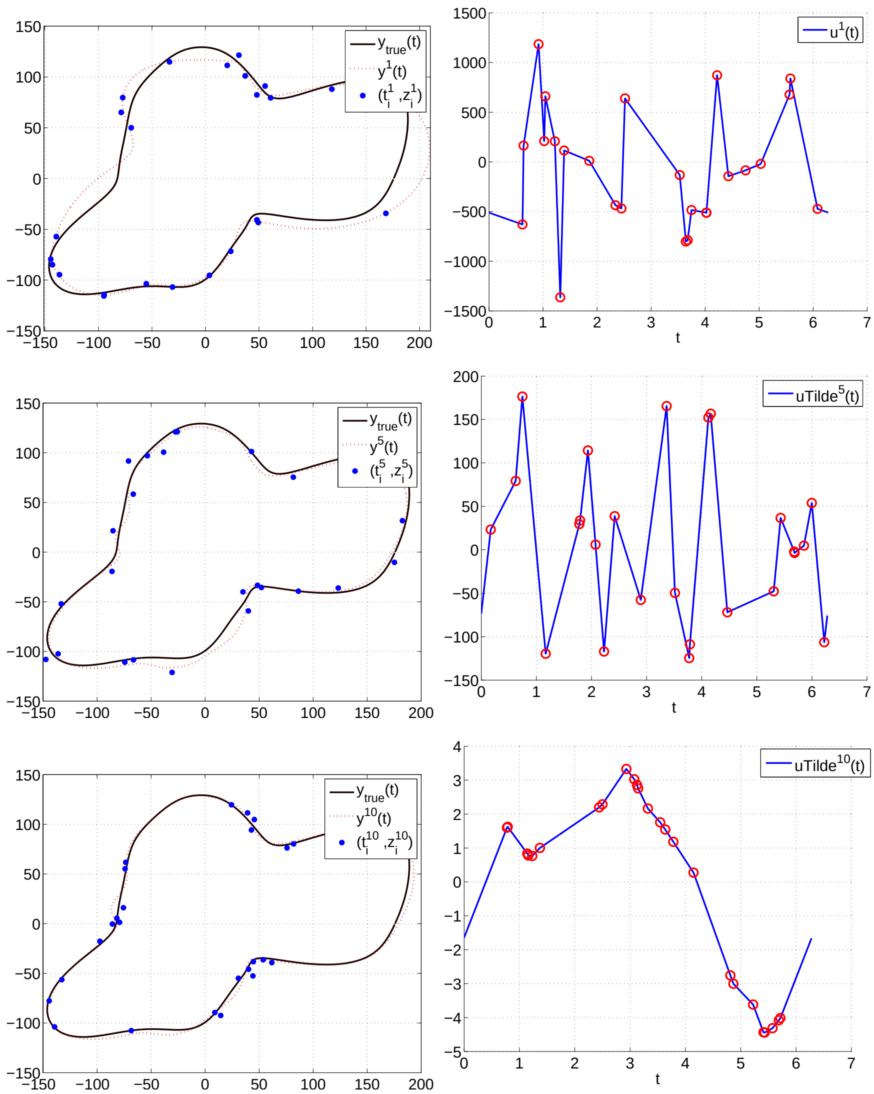

Figure 1: An example. **Left:** Estimates $y(t)$ (dashed) are shown at iterations $k = 1, 5, 10$ and compared with the underlying curve y_{true} and the sampled input data (t_i, z_i) (stars). **Right:** The corresponding controls $u^1_{T,N}$, $\tilde{u}^5_{T,N}$ and $\tilde{u}^{10}_{T,N}$. The interpolation points at $t = t_i$ are marked with circles.

7 Conclusions

In this paper we introduced a recursive smoothing spline approach to estimation of closed curves in $\mathbb{R}^2$. We derived periodic smoothing splines recursively from noisy data by solving an optimal control problem for a linear system. It was shown that

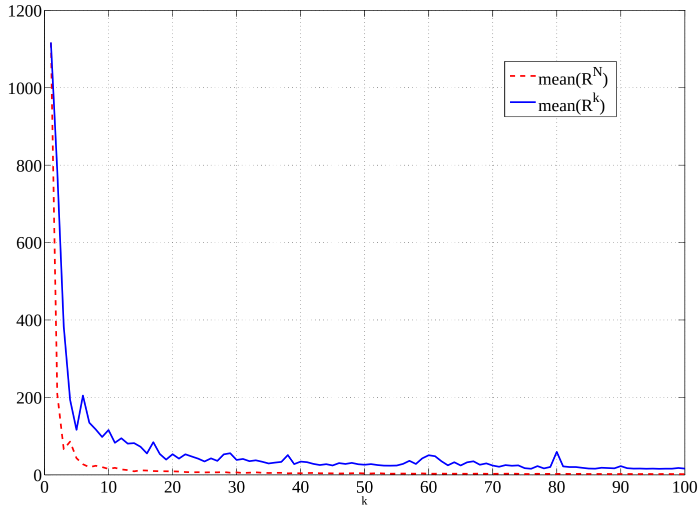

Figure 2: Convergence result for Problem 4.2 for increasing k (solid) compared to Problem 4.1 for increasing N (dashed).

a simple, linear transform makes the recursive problem identical to the closed form problem and that the problem is proper generically. Optimality conditions were examined using Hamilton-Jacobi-Bellman theory and simulations demonstrate satisfying error convergence for the recursive method.

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A Proof of Proposition 1

Proof

From the dynamical constraints (2) it is obtained that

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds, \tag{58}$$

which, inserting the matrices and vectors of (5), is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1(0) + tx_2(0) + \int_0^t (t-s)u(s)ds \\ x_2(0) + \int_0^t u(s)ds \end{pmatrix}. \tag{59}$$

At the terminal time, we obtain

$$\begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix} = \begin{pmatrix} x_1(0) + Tx_2(0) + \int_0^T (T-s)u(s)ds \\ x_2(0) + \int_0^T u(s)ds \end{pmatrix}, \tag{60}$$

and since $x(0) = x(T)$ this yields

$$x_2(0) = -\frac{1}{T}\int_0^T (T-s)u(s)ds \tag{61}$$

$$\int_0^T u(s)ds = 0. \tag{62}$$

It follows from (62) that the only feasible constant control is $\overline{u}^* \equiv 0$. For $\overline{u}^* \equiv 0$, the spline is $\overline{x}_1^* \equiv \overline{z} = \frac{1}{N}\sum_{i=1}^N z_i$. Therefore $\overline{J}^* = J(0,\overline{z})$

Denote by $\hat{x}_1(t)$ the interpolating, periodic, cubic spline that interpolates the points $(z_i - \overline{z})$. This spline exists and is unique for distinct sampling angles t_i, and if there is at least one $z_i \neq \overline{z}$ in the set, $\hat{x}_1(t) \not\equiv \overline{z}$.

It follows that $\hat{u}(t) = \ddot{\hat{x}}_1(t)$ is well defined, non-zero and lies in the feasible region of Problem 4.1. Now let $\alpha \in \mathbb{R}$ and consider

$$\Gamma(\alpha) = J(\alpha\hat{u}(t), \alpha\hat{x}(t) + \overline{z}) = \frac{\alpha^2}{2}\int_0^T \hat{u}(t)^2 dt + \frac{\varepsilon^2}{2}\sum_{i=1}^N (t_i - t_{i-1})(z_i - \alpha\hat{x}_1(t_i) - \overline{z})^2 =$$

$$\frac{\alpha^2}{2}\int_0^T \hat{u}(t)^2 dt + \frac{\varepsilon^2}{2}\sum_{i=1}^N (t_i - t_{i-1})(1-\alpha)^2(z_i - \overline{z})^2. \tag{63}$$

The derivative $\Gamma(\alpha)$ at $\alpha = 0$ is

$$\frac{\partial\Gamma(\alpha)}{\partial\alpha} = \alpha\int_0^T \hat{u}(t)^2 dt - \varepsilon^2(1-\alpha)\sum_{i=1}^N (t_i - t_{i-1})(z_i - \overline{z})^2 =$$

$$-\varepsilon^2\sum_{i=1}^N (t_i - t_{i-1})(z_i - \overline{z})^2 < 0. \tag{64}$$

It follows that there exists an α^* such that $\Gamma(\alpha^*) < \Gamma(0)$.

■

B Proof of Proposition 3

Proof

Let $t = t_{i+1}$, $\Delta t = (t_{i+1} - t_i)$. Note that if (54) - (55) hold, it follows that

$$u_{T,N}(T) = u_{T,N}(0), \tag{65}$$

Therefore, if we can show that (44) - (47) correspond to (34) - (35) in the limit $\Delta t \to 0$, the proposition is proved.

Convergence for P

From (44) and (45) we get

$$\dot{P}_i(\tau) = -A'P_i(\tau) - P_i(\tau)A + P_i(\tau)BQB'P_i(\tau) \quad \tau \in [t, t+\Delta t] \tag{66}$$

$$P_i(t+\Delta t) = P_{i+1}(t+\Delta t) + \Delta t C'R^{-1}C. \tag{67}$$

From the definition of the derivative, we obtain

$$\begin{aligned}\dot{P}_i(t) &= \lim_{\Delta t\to 0}\frac{P_i(t+\Delta t)-P_i(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{P_{i+1}(t+\Delta t)+\Delta tC'R^{-1}C-P_i(t)}{\Delta t} \\ &= \lim_{\Delta t\to 0}\frac{P_{i+1}(t+\Delta t)-P_i(t)}{\Delta t}+C'R^{-1}C. \end{aligned} \tag{68}$$

Using (66) we arrive at the equality

$$\lim_{\Delta t\to 0}\frac{P_{i+1}(t+\Delta t)-P_i(t)}{\Delta t} = -A'P_i-P_iA+P_iBQB'P_i-C'R^{-1}C. \tag{69}$$

We manipulate the left hand side of (69):

$$\begin{aligned}&\lim_{\Delta t\to 0}\frac{P_{i+1}(t+\Delta t)-P_i(t)}{\Delta t} = \\ &\quad\lim_{\Delta t\to 0}\frac{P_{i+1}(t+\Delta t)-P_{i+1}(t)+P_i(t+\Delta t)-P_i(t)+P_{i+1}(t)-P_i(t+\Delta t)}{\Delta t} = \\ &\quad\quad\dot{P}_{i+1}+\dot{P}_i+\lim_{\Delta t\to 0}\frac{P_{i+1}(t)-P_i(t+\Delta t)}{\Delta t} = \{\quad \text{using (67)}\quad\} \\ &\quad\quad\dot{P}_{i+1}+\dot{P}_i+\lim_{\Delta t\to 0}\frac{P_{i+1}(t)-P_{i+1}(t+\Delta t)-\Delta tC'R^{-1}C}{\Delta t} = \\ &\quad\quad\dot{P}_{i+1}+\dot{P}_i+\lim_{\Delta t\to 0}\frac{P_{i+1}(t)-P_{i+1}(t+\Delta t)}{\Delta t} = \dot{P}_{i+1}+\dot{P}_i-\dot{P}_{i+1}=\dot{P}_i. \end{aligned} \tag{70}$$

$$\Rightarrow \lim_{\Delta t\to 0}\dot{P}_i = -A'P_i-P_iA+P_iBQB'P_i-C'R^{-1}C. \tag{71}$$

Also, from (67) it follows that

$$\lim_{\Delta t\to 0}P_i(t+\Delta t)=P_{i+1}(t). \tag{72}$$

Convergence for ϕ
From (46) and (47) we get

$$\dot{\phi}_i(\tau) = -(A-BQB'P_i)'\phi_i(\tau) \quad \tau\in[t,t+\Delta t] \tag{73}$$
$$\phi_i(t+\Delta t) = \phi_{i+1}(t+\Delta t)-\Delta tC'R^{-1}z(t+\Delta t). \tag{74}$$

From the definition of the derivative, we obtain

$$\begin{aligned}\dot{\phi}_i(t) &= \lim_{\Delta t\to 0}\frac{\phi_i(t+\Delta t)-\phi_i(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\phi_{i+1}(t+\Delta t)-\Delta tC'R^{-1}z(t+\Delta t)-\phi_i(t)}{\Delta t} \\ &= \lim_{\Delta t\to 0}\frac{\phi_{i+1}(t+\Delta t)-\phi_i(t)}{\Delta t}-C'R^{-1}z(t). \end{aligned} \tag{75}$$

Using (73) we arrive at the equality

$$\lim_{\Delta t\to 0}\frac{\phi_{i+1}(t+\Delta t)-\phi_i(t)}{\Delta t} = -(A-BQB'P_i(t))'\phi_i(t)+C'R^{-1}z(t). \tag{76}$$

With the same reasoning as in (70)

$$\lim_{\Delta t\to 0}\frac{\phi_{i+1}(t+\Delta t)-\phi_i(t)}{\Delta t}=\dot{\phi}_i. \tag{77}$$

$$\Rightarrow \lim_{\Delta t\to 0}\dot{\phi}_i=-(A-BQB'P_i)'\phi_i+C'R^{-1}z(t). \tag{78}$$

Also, from (74) it follows that

$$\lim_{\Delta t\to 0}\phi_i(t+\Delta t)=\phi_{i+1}(t). \tag{79}$$

Convergence for s

From (48) and (49) we get

$$\begin{aligned}\dot{s}_i(\tau) &= \frac{1}{2}\phi_i'BQB'\phi_i \quad \tau\in[t,t+\Delta t] && (80)\\ s_i(t+\Delta t) &= s_{i+1}(t+\Delta t)+\frac{1}{2}\Delta t z(t+\Delta t)'R^{-1}z(t+\Delta t). && (81)\end{aligned}$$

From the definition of the derivative, we obtain

$$\begin{aligned}\dot{s}_i(t) &= \lim_{\Delta t\to 0}\frac{s_i(t+\Delta t)-s_i(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{s_{i+1}(t+\Delta t)+\frac{1}{2}\Delta t z(t+\Delta t)'R^{-1}z(t+\Delta t)-s_i(t)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{s_{i+1}(t+\Delta t)-s_i(t)}{\Delta t}+\frac{1}{2}z(t)'R^{-1}z(t). && (82)\end{aligned}$$

Using (80) we arrive at the equality

$$\lim_{\Delta t\to 0}\frac{\phi_{i+1}(t+\Delta t)-\phi_i(t)}{\Delta t}=\frac{1}{2}\phi_i'BQB'\phi_i-\frac{1}{2}z(t)'R^{-1}z(t). \tag{83}$$

Again, with the same reasoning as in (70)

$$\lim_{\Delta t\to 0}\frac{s_{i+1}(t+\Delta t)-s_i(t)}{\Delta t}=\dot{s}_i. \tag{84}$$

$$\Rightarrow \lim_{\Delta t\to 0}\dot{s}_i=\frac{1}{2}\phi_i'BQB'\phi_i-\frac{1}{2}z(t)'R^{-1}z(t). \tag{85}$$

Also, from (81) it follows that

$$\lim_{\Delta t\to 0}s_i(t+\Delta t)=s_{i+1}(t). \tag{86}$$

■