File Fragmentation over an Unreliable Channel

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***Abstract*—When files are transmitted over an unreliable channel, loss of data will occur, typically with the need of resending the whole file. Instead of resending the file until it is successfully transmitted, the file is fragmented, and the fragments are sent in order through the channel. We examine the optimal fragmentation policy which minimizes the expected file transfer time and show that for non-decreasing failure rate distributions, constant, file size dependent fragmentation is optimal. Assuming the failure probability is known, this suggests that the file can be fragmented in advance. Furthermore, we bound the optimal fragment size, and show that the optimal fragment size approaches a file size independent value for large file sizes. We also explore the sensitivity of the average file transfer time to model error, and give an upper bound on the penalty of model error under two typical error models.**

I. Introduction

It has been recently shown that as long as file sizes have infinite support, the total completion times are heavy-tailed, provided files are not fragmented. A key feature for this result is that files have to be resent after a failure [1]. File fragmentation into packets is enforced by many internet protocols, such as the transmission control protocol (TCP) [2]. In this paper we explore the possibility of reducing the retransmission time due to packet losses, by controling the fragment sizes. We minimize the total file transfer time, by minimizing the resending time due to lost packets. While [3] introduces a dynamic fragmentation algorithm for file transfers, we will examine an optimal static fragmentation policy. While a dynamic fragmentation policy may require additional computations to be performed, a static policy can be predetermined with knowledge of the channel statistics and thus does not require real time computations of packet sizes.

II. Model

A file of size L is to be sent over an unreliable channel with unit transmission rate. Every packet cosists of a data section, and a fixed-size overhead, making the fragment size bigger than the actual data. Let $x_i + \phi$ be the packet size of the ith transmission, where x_i represents the data size, and ϕ the fixed overhead size. The ith transmission will be successful if $A_i \geq x_i + \phi$, where $(A_i, i = 1, 2, \dots)$ are i.i.d. random variables with common distribution $F(x)$. This model thus covers randomly introduced packet errors over the channel that are indepentent, but fails to model other packet errors such as errors due to congestion.

The file is fragmented into packets. The objective is to choose a file fragmentation policy that minimizes the expected transfer time of the file. In general, the optimal fragmentation policy will depend on the file size and the failure process (A_i).

A Markov policy is a function $x(l)$ of the remaining file size l with the following interpretation. Given l, a packet of size $x(l) + \phi$ is formed. If the packet is successfully transmitted, the remaining file size will be $l - x(l)$. If the transmission fails, the file size remains unchanged and therefore the next fragment remains $x(l)$, until the packet is successfully transmitted. Define $\bar{F}(x) = 1 - F(x)$. The expected time it takes to successfully transmit a fragment is $(x(l)+\phi)/\bar{F}(x(l)+\phi)$, the cost per trial multiplied by the expected number of trials that is geometrically distributed with parameter $F(x(l) + \phi)$. This implies that if we let $J(l)$ denote the expected completion time when the file size is l under a generic Markov policy $x(l)$, then

$$J(l) = J(l - x(l)) + \frac{x(l) + \phi}{\bar{F}(x(l) + \phi)}$$

Given any Markov policy $x(l)$, consider the sequence of fragments $x_0, x_1, \dots$, generated from an initial file size L, defined recursively as:

$$x_0 := x(L), \; x_1 := x(L - x_0), \dots$$

such that $\sum_i x_i = L$. Define the expected time to successfully transmit a segment of size x as

$$h(x) = \frac{x + \phi}{\bar{F}(x + \phi)} \tag{1}$$

The expected completion time is thus

$$J(L) = \sum_i h(x_i)$$

Since $h(x) \geq h(0) > \phi > 0$ for all $x \geq 0$, an optimal policy must only have finitely many terms in $J(L)$. Let $J^*(L)$ denote the (minimum) expected completion time under an optimal policy x^*. The optimal fragmentation policy must be the minimizer of $J(L)$, and thus

$$J^*(L) = \min_{M\in\mathbb{N}} \min_{x_o,\ldots,x_{M-1}} \sum_{i=0}^{M-1} \frac{x_i+\phi}{\bar{F}(x_i+\phi)} \tag{2}$$

$$\text{where} \quad \sum_{i=0}^{M-1} x_i = L$$

$$\text{and} \quad x_i > 0,\ i = 0,\ldots,M-1,\ M = 1,2,\ldots$$

The following lemma proves the intuition that larger files have strictly longer optimal average transmission times than smaller files.

Lemma 1. *$J^*(L)$ is strictly increasing in L.*

Proof: Consider two files of size L_1 and L_2 where $L_2 > L_1$. We know that an optimal fragmentation policy for L_2 exists. Suppose that the last sent package of this policy is of size x^* and that $L_2 - L_1 < x^*$. We have:

$$\begin{aligned} J^*(L_2) &= J^*(L_2 - x^*) + \frac{x^*+\phi}{\bar{F}(x^*+\phi)} \\ &> J^*(L_2 - x^*) + \frac{x^* - (L_2 - L_1) + \phi}{\bar{F}(x^* - (L_2 - L_1) + \phi)} \\ &\geq J^*(L_1) \end{aligned} \tag{3}$$

since $J(L_1)$ is the optimal fragmentation policy. For file sizes where $L_2 - L_1 > x^*$, use $J^*(L_2) = J^*(L_2 - x^*) + \frac{x^*+\phi}{\bar{F}(x^*+\phi)} \geq J^*(L_2 - x^*)$ recursively, and then apply (3). ■

III. Optimal fragmentation

A. Unique constant fragmentation

We first define the function

$$g(x) = \frac{x+\phi}{x\bar{F}\left(x+\phi\right)} \tag{4}$$

Furthermore, let

$$a = \operatorname*{argmin}_x g(x) \qquad x \in \Re^+$$

Knowing that the optimal cost of sending a file of size L is given by equation (2), we can conclude the following theorem:

Theorem 2. *If the density function $f(x) = F'(x)$ exists, and the failure rate $\lambda(x) = f(x)/\bar{F}(x)$ is continuous and non-decreasing, the minimizer (M^*, x_i^*) of (2) is semi-unique, in the sense that:*

1) *x_i^* is unique with equal fragment size, $x_i^* = L/M^*$ for $i = 1,\ldots,M^*$.*
2) *M^* equals $\lfloor L/a \rfloor$ or $\lceil L/a \rceil$ whichever produces a smaller value for $g(L/M^*)$.*

Proof: We will first prove that, given any M, the minimizer x_i^* of the inner minimization exists, is unique, and $x_i^* = x^*$ for all i. We then prove that the optimal M^* is either $\lfloor L/a \rfloor$ or $\lceil L/a \rceil$.

Solving the inner minimization in (2) raises the following necessary KKT condition for the optimum $x* = (x_0^*, \ldots, x_{M-1}^*)$ [4]:

$$\begin{aligned} &\frac{\mathrm{d}h(x_i^*)}{\mathrm{d}x_i^*} = \frac{1}{\bar{F}(x_i^*+\phi)} + (x_i^*+\phi)\frac{f(x_i^*+\phi)}{\left(\bar{F}(x_i^*+\phi)\right)^2} = \lambda \\ &\forall \qquad i = 0,\ldots,M-1 \\ &\sum_{i=0}^{M-1} x_i^* = L \end{aligned} \tag{5}$$

By assumption $\lambda(x)$ is non-decreasing. Moreover $1/\bar{F}(x)$ is non-decreasing and $x/\bar{F}(x)$ is strictly increasing. Therefore $h'(x)$ is strictly increasing, which is equivalent to $h(x)$ being convex. Thus the minimization problem is convex, and the KKT condition is sufficient. A unique solution $x^* = (x_1^*, \ldots, x_M^*)$ exists. Moreover, since all x_i^* are uniquely determined by (5), they are the same and $x_i^* = L/K$ for all i. This reduces the minimization (2) to

$$M\frac{y+\phi}{\bar{F}(y+\phi)} = M\frac{\frac{L}{M}+\phi}{\bar{F}\left(\frac{L}{M}+\phi\right)} = L\frac{\frac{L}{M}+\phi}{\frac{L}{M}\bar{F}\left(\frac{L}{M}+\phi\right)}$$

Since L is constant, this is equivalent to solving

$$x^* = \operatorname*{argmin}_x g(x) \qquad x = \left\{L, \frac{L}{2}, \frac{L}{3}, \ldots\right\} \tag{6}$$

Since $f(x)$ by assumption is continuous and $\lim_{x\to 0} g(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, any unconstrained minimum a of $g(x)$ must also be an extremum. Thus, setting $g'(x) = 0$ yields:

$$\frac{f(x+\phi)}{\bar{F}(x+\phi)} \cdot \frac{x(x+\phi)}{\phi} = \xi(x) = 1$$

Since $f(x+\phi)/\bar{F}(x+\phi)$ is non-decreasing, $x(x+\phi)/\phi$ is strictly increasing, $\xi(0) = 0$, $\lim_{x\to\infty} \xi(x) = \infty$, and $f(x)$ is continuous, the equation $\xi(x) = 1$ will have one unique solution, which is the unique minimizer of $g(x)$. Since the minimizer is unique, this will guarantee that L/M^* is the unique solution of (6) for either $M^* = \lfloor L/a \rfloor$ or $M^* = \lceil L/a \rceil$, whichever gives rise to the lowest value of $g(x)$. ■

Remark 1. *[5] provides a useful sufficient condition for Theorem 2 to hold. If $f(x)$ is log-concave, so is $\bar{F}(x)$, implying that $\lambda(x)$ is non-decreasing. This is very useful in cases where the cumulative distribution function is unknown, such as for the Gaussian distribution.*

B. Optimal fragment size

Under the assumption that $\lambda(x)$ is non-decreasing, we can bound the optimal fragment size x^* in terms of a. First note that since $g'(x) < 0$ for $x < a$, $g(L/M) > g(L) > g(a)$ for all $M \in \mathrm{N}$. This implies $x^* = L$. We therefore only consider the case where $L > a$ in this section.

Lemma 3. *Suppose that $\lambda(x)$ is non-decreasing, then $\frac{a}{2} \le \frac{a}{1+a/L} \le x^* \le \min\left(2a, \frac{a}{1-a/L}\right)$ for $L > a$.*

Proof: We know that for some integer M:

$$\frac{L}{M+1} \le a \le \frac{L}{M}$$

We know from theorem 2 that $x_i^* = L/M$ or $x_i^* = L/(M+1)$. Therefore, if $a \le L$;

$$x^* \le a \le \frac{M+1}{M} x^* \le 2x^* \qquad \text{or}$$
$$\frac{x^*}{2} \le \frac{M}{M+1} x^* \le a \le x^*$$

so it must be that

$$\frac{a}{2} \le x^* \le 2a$$

We can also conclude that

$$\frac{L}{a} - 1 \le M \le \frac{L}{a}$$

implying

$$a \le \frac{L}{M} \le \frac{a}{1-a/L} \quad \text{and} \quad \frac{a}{1+a/L} \le \frac{L}{M+1} \le a$$

and since either $L/M = x^*$ or $L/(M+1) = x^*$:

$$\frac{a}{1+a/L} \le x^* \le \frac{a}{1-a/L}$$

Combining the two results completes the proof. ∎

Corollary 4.

$$\lim_{L\to\infty} x^*(L) = a$$

One might think that the above results hold generally, or that there is a more general bound on x^*. This is however not true. Consider:

$$\bar{F}(x) = \begin{cases} 1 & 0 \le x < 1 \\ 1/x & 1 \le x < a \\ 1/a & a \le x < b \\ b/(ax) & b \le x < \infty \end{cases} \tag{7}$$

where

$$a = (2-\epsilon)\frac{b}{b+1}, \qquad \epsilon > 0 \qquad b > 3$$

We see directly that

$$g(1) = 2$$
$$g(b) = 2 - \epsilon b \le g(x) \quad \forall x \ge 0$$

We conclude that $a = g(b)$ is the unique minimizer of $g(x)$. Consider a file of size $L = b+1$. The cost of sending the file in two packets of size b and 1 is:

$$J_{b+1}(L) = g(1) + bg(b) = 2 + (2-\epsilon)b = 2L - \epsilon b \tag{8}$$

If we send 2 packets, where one packet is smaller than 1, it is clear that the cost $J(L)$ will increase. We thus consider the case where both packages are bigger than 1. Due to symmetry, we only consider the case where we decrease the packet of size b to a size greater than $L/2$. By doing this, we strictly increase the unit cost $g(x)$ of sending the packet b. If the unit cost of sending the packet of size 1 is also increased, this fragmentation policy cannot be better. Thus, if the unit cost of sending a packet of size $\frac{L}{2}$ is higher than sending a packet of size 1, this policy gives a strictly higher cost for sending the file. Thus, we require

$$J(L/2) > J(1) \Leftrightarrow$$
$$(1+L/2)(2-\epsilon)\frac{b}{b+1} > 2 \Leftrightarrow$$
$$(1+L/2)(2-\epsilon)(L-1) > 2L \Leftrightarrow$$
$$\epsilon < 2 - \frac{2L^2}{L^2+L-2} = \frac{2(L-2)}{(L-1)(L+2)}$$

If we thus choose ϵ so that the above inequality is fulfilled, the fragmentation policy $b+1$ is always optimal in the class of two-fragment-policies. The policy is also optimal among all fragmentation policies, since all new introduced packages will have a strictly higher unit cost than the cost of the packet of size b, and due to symmetry, the packet of size 1 can be increased at most to the size $\frac{L}{2}$. Thus, the fragmentation policy $b+1$ is optimal. Thus $1 = x_1^*$ and $1 \ne a = b$. Since b can be arbitrary large, we cannot bound the optimal fragment size x^* in terms of a in general.

IV. Model error

A. The cost of the greedy policy under model error

Suppose we have insufficient knowledge of the statistics of the failure process. In this section, we derive bounds on the penalty for applying the optimal policy x^* designed for a failure distribution $\hat{F}$, when the actual distribution is F. Variables with a hat will be used to denote quantities defined with respect to $\hat{F}$, e.g., $\hat{a}$ and $\hat{x}^*$, while a and x^* are those for the true distribution F. Further, let $g^* := g(a) = \min_x g(x)$ where g is defined in (4).

We will compare the expected cost $J^{\hat{a}}$ under F of the policy $\hat{x}^*$ optimal for $\hat{F}$, with the true minimum cost $J^*(L)$. The following result specifies the cost increment in terms of the per-bit cost function g.

Lemma 5.

$$\lim_{L\to\infty} \frac{J^{\hat{x}^*}(L) - J^*(L)}{L} = g(\hat{a}) - g^*$$

Proof: First note that, for any constant fragment size, x,

$$J^x(L) = \left\lfloor \frac{L}{a} \right\rfloor x\, g(x) + x' g(x'),$$

where $x' = L - \lfloor L/x \rfloor x \in [0, x)$. Since $x'g(x') = h(x')$, and $h(\cdot)$ is non-decreasing, this implies

$$|J^x(L) - Lg(x)| \le h(x).$$

Setting $x = \hat{x}^*$, and applying $Lg^* \leq J^*(L) \leq (L+a)g^*$ gives

$$J_G^{\hat{a}}(L) - J^*(L) =$$
$$L\left(g(\hat{a}) - g^*\right) + \alpha(L)h(\max(a, \hat{x}^*))$$

for some $\alpha : \mathbb{R}_+ \to (-1, 1)$. Dividing by L and taking the limit as $L \to \infty$, combined with Corollary 4 and the continuity of g gives the first euqality. ■

We will now bound the modeling errors by finding upper bounds on $g(\hat{a}) - g^*$.

B. General distribution with relative uncertainty in ccdf

Consider the case where the distribution is believed to be $\hat{\bar{F}}(x)$, but the real distribution $\bar{F}(x)$ is

$$\bar{F}(x) = \hat{\bar{F}}(x)(1 + \Delta(x)) \tag{9a}$$

where

$$-1 < -\Delta_{\min} \leq \Delta(x) \leq \Delta_{\max} \tag{9b}$$

Lemma 6.

$$g(\hat{a}) - g^* \leq \frac{\Delta_{\max} + \Delta_{\min}}{(1 + \Delta_{\max})(1 - \Delta_{\min})}\, \hat{g}^*$$

Proof: By insertion of equation (9) into equation (4) we see that

$$\frac{\hat{g}(x)}{1 + \Delta_{\max}} \leq g(x) \leq \frac{\hat{g}(x)}{1 - \Delta_{\min}} \tag{10}$$

Since equation (10) holds for a, we get

$$\frac{\hat{g}^*}{1 + \Delta_{\max}} \leq \frac{\hat{g}(a)}{1 + \Delta_{\max}} \leq g^* \tag{11}$$

as for $\hat{a}$, from which we get

$$g(\hat{a}) \leq \frac{\hat{g}^*}{1 - \Delta_{\min}} \tag{12}$$

Combining inequalities (11) and (12), we get

$$g(\hat{a}) - g^* \leq \frac{\hat{g}^*}{1 - \Delta_{\min}} - \frac{\hat{g}^*}{1 + \Delta_{\max}}$$
$$= \frac{\Delta_{\max} + \Delta_{\min}}{(1 + \Delta_{\max})(1 - \Delta_{\min})}\, \hat{g}^*$$

■

Remark: If $\Delta_{\min} = \Delta_{\max}$, Lemma 6 implies

$$g(\hat{a}) - g^* \leq \frac{2\Delta_{\max}}{1 - (\Delta_{\max})^2}\, \hat{g}^*.$$

C. Exponential distribution with uncertain parameter

For some simple network structures, such as satellite links, bit-errors occur indepentently of the data that has been sent, resulting in exponential errors [6]. Therefore, consider the case where the distribution function is assumed to be

$$\hat{\bar{F}}(x, \lambda) = e^{-\lambda x}$$

whereas the real distribution $\bar{F}(x)$ is

$$\bar{F}(x, \lambda) = e^{-(\lambda + \Delta(x))x} = \hat{\bar{F}}(x, \lambda)e^{-\Delta(x)x}$$

where $-\Delta_{\min} \leq \Delta(x) \leq \Delta_{\max}$. Let $a(\lambda) := \operatorname{argmin}_x g(x, \lambda)$, and $g^*(\lambda) := \min_x g(x, \lambda) = g(a(\lambda), \lambda)$.

It is obvious that this error model is not covered by the theory in section IV-B, because of the exponentially increasing error term. We can however still bound the penalty of this error, although the relative error may be unbounded.

Lemma 7.

$$g(\hat{a}(\lambda), \lambda) - g^*(\lambda) \leq \hat{g}(\hat{a}(\lambda), \lambda + \Delta_{\max}) - \hat{g}^*(\lambda - \Delta_{\min})$$

Proof: First note that

$$\hat{g}(x, \lambda) = \frac{x + \phi}{x} e^{\lambda x}$$

Since $-\Delta_{\min} \leq \Delta(x) \leq \Delta_{\max}$, and $\hat{g}(x, \lambda)$ is obviously increasing in λ

$$\hat{g}(x, \lambda - \Delta_{\min}) \leq g(x, \lambda) \leq \hat{g}(x, \lambda + \Delta_{\max})$$

which implies

$$\hat{g}^*(\lambda - \Delta_{\min}) \leq g^*(\lambda) \leq \hat{g}^*(\lambda + \Delta_{\max})$$

Thus

$$g(\hat{a}(\lambda), \lambda) - g^*(\lambda) \leq \hat{g}(\hat{a}(\lambda), \lambda + \Delta_{\max}) - \hat{g}^*(\lambda - \Delta_{\min})$$

■

Corollary 8. *If the model error is simply $\bar{F}(x) = e^{-(\lambda+\Delta)x}$, i.e. the parameter in the exponential distribution is misestimated, the corresponding bound on g is:*

$$g(\hat{a}(\lambda), \lambda) - g^*(\lambda) \leq \hat{g}(\hat{a}(\lambda), \lambda + \Delta) - \hat{g}^*(\lambda + \Delta)$$

V. Conclusion and further work

Under the assumption that the failure rate is increasing in file size, optimal file fragmentation is constant and unique, as stated in theorem 2. This implies that with sufficient knowledge of the failure statistics of the channel, files can be pre-fragmented. The fragment sizes are also file size depentend, but for large file sizes the fragment sizes are approximately equal, due to corollary 4.

Further work may focus on characterizing tighter criteria where optimal fragmentation is constant and unique, than the one presented here. Also, implemention issues may be discussed in detail, and how this implementation compares to fragmentation implemented in standard protocols. The current model may also be extended to cover the use of error correcting codes, where k or more bit errors require the packet to be resent.

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