Sweet Spots with Diverse Sensing and Actuation

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Abstract— **Diversity within sensors and actuators is visible in many systems. One example is the sensorimotor control system that contains many types of layers that sense and act with different speed and accuracy. Each component design faces the trade-off between speed and accuracy and the diversity of the system then creates sweet spots with optimal performance. The focus of this project was to develop a model with diverse sensing and actuation to explore sweet spots and trade-offs. The aim was to provide examples of diverse systems using LQR and display the difference between only using one certain type of component in comparison with combining components with different properties. For this study two different types of sensors and actuators where considered, one type that was fast but sparse and one that was slow but dense. Distinct sweet spots were achieved that presented cases where the different types failed individually and resulted in excellent performance when combined. The obtained solutions contained a lot of internal feedback which proved necessary to avoid instability. It was concluded that diversity is essential for optimal control which agrees well with the expectations. The results of this paper function as a bridge between familiar control theory and the more novel System Level Synthesis framework.**

I. Introduction

In nature and technology there are endless systems and processes we wish to control. Ideally, we want all the features of the sensors and actuators to be perfect but that is often infeasible to achieve due to fundamental limitations. One trade-off that is present in most systems is the speed and accuracy trade-off (SAT) which states that what we gain in terms of accuracy, we must compensate for by reducing the actuation speed [1]. Therefore, in a system we often end up with multiple layers of components that are differently fast and accurate. For example a fast and unconscious reflex layer and a slow and conscious planning layer as described in [2].

The combined architecture of these diverse layers creates a final complex system that behaves both fast and accurate which ends up in a so called Diversity Enabled Sweet Spot (DeSS) [1]. The DeSS is a sweet spot caused by the diversity of the components which behaves optimally while all layers still obey the SAT law.

Recent development of the System Level Synthesis (SLS) framework enables the ability to model complicated systems where we can introduce constraints in time and space on the sensing, actuation and communication within the system. It is possible to create and combine diversity within the components which results in intricate systems and the framework design accomplishes this without much computational time due to the scalability of the theory [3].

There is a lot left to be discovered about diverse sensing and actuation before introducing the concepts in SLS. This paper aims to provide further understanding of the sweet spots in diverse systems and present examples of how they can be modeled using senior preSLS control theory frameworks. The focus is to explore the impact of diversity for the fundamental features of DeSS using preSLS control theory, as well as to create a bridge between familiar theory and the more novel, powerful features of SLS.

Previous work of Nakahira et al. [2], shows how the sensorimotor control system can be divided into two separate subsystems with a mountain bike example. There is one fast but inaccurate bump-reflex layer and one slow and more accurate trail-planning layer. This example used a scalar formulation with no internal feedback pathways (IFP) where the errors of the system were simply added. IFP refers to internal feedback loops which are an architectural feature in the system where the components utilizes information about previous actions to predict future actions. The work shown in this paper is different and uses a formulation with multiple states in a diverse system where IFP becomes necessary.

A brief introduction of control theory concepts and examples of systems with multiple actuators will be presented in Section II. Section III describes the system setup and the resulting model. Analysis of the results will be presented in Section IV and the extension to the SLS framework is described in Section V. Final conclusions are discussed in Section VI.

II. Background

Control theory provides many helpful tools to model the dynamics of a system. For this study a linear–quadratic regulator (LQR) problem has been used to provide a model of a system with diverse sensing and actuation. Some necessary background theory is presented here together with examples of diverse sensing and actuation in reality which motivates the interests of this study.

A. Diverse Sensing and Actuation

The initial example that sparked the desire to explore diverse sensors and actuators is the observation of delays in the sensorimotor control system. Following the example of [2], the sensorimotor control system can roughly be divided into a fast layer with proprioception, and then another slow layer with the visual cortex. The proprioception is our ability to sense the position and movements of our body where the reflexes have a reaction time of around 95.6 ± 10.6 ms according to experiments by [4].

The visual reaction time is much slower than the proprioception at around 247.6 ± 18.54 ms according to [5].

Thus suggesting that the part of the sensorimotor control system related to vision has larger delays. The reason is due to the large quantity of computations needed to interpret the information from the raw retinal image. The visual cortex needs to translate a lot of raw data into a decision of which actions needs to be taken [6].

Although the reaction time is faster, the proprioception is however less accurate than vision and when performing sensorimotor tasks we rely much more on vision for precision and accuracy in the movement [7]. Thus there exists a SAT obeying system containing diverse components where one layer is fast and inaccurate which together with a slow and accurate layer creates a DeSS.

Within vision there is a lot of internal feedback between the components of the visual network and IFP is commonly found in the visual area [8]–[10]. In general the world is slow changing in relation to the $\sim$ 200 ms it takes for us to see everything in a room. Using IFP, our vision can use predictions to compute error signals which are smaller and can be sent faster than the full raw image. The internal delays of vision also result in stale images by the time they arrive to the motor areas which also motivates the benefits of the predictions from IFP. There are also experiments which point to IFP from motor to visual areas [11].

Another motivating example of diverse sensing and actuation is the immune system response. The complexity of the immune system is undeniable. The components of the immune system protect us from pathogens and can initially be divided into two parts: the innate immunity and the adaptive immunity [12]. The innate immunity provides a fast but not very accurate protection and is able to limit infections to some extent. Here cells like macrophages, neutrophils and basophils respond within minutes to hours with a rapid response. The adaptive immunity is the accurate but slow defense with specialized protection against pathogens. In comparison the adaptive immunity may take weeks to respond but when it does, it works very effectively with T lymphocytes and B lymphocytes that produces an efficient and long lasting protection [12]. Thus the immune system also presents a DeSS system with a fast but inaccurate layer and an accurate but slow layer.

IFP is also visible in the immune system where T cells can be further divided into CD4+ T cells, or T helper cells, and CD8+ T cells, or cytotoxic T cells, which function as the names suggest. The CD8+ T cells kills abnormal or damaged cells, which can be virus infected or cancerous [13]. The CD4+ T cells on the other hand help regulate the responsiveness of the killer cells and promotes interaction in between cells in the immune system [14]. The T cells are activated via negative feedback [15] suggesting a lot of IFP present in the immune system in order to recognize and be sensitive to changes and irregularities.

The ubiquity of SAT, DeSS and IFP in both the sensorimotor system and the immune system motivates us to seek fundamental theory on the optimal usage of diverse components (e.g. diverse sensors and actuators). The current goal is not to model these biological systems in fine detail, but to study simple, human-interpretable examples that give rise to fundamental theory on the existence and characteristics of DeSS and IFP.

In reality most of the delays are caused by limitations in communication between components. For this study the origins of the delays are simplified. The idea is to push all delays into the sensors and actuators to fit the preSLS theory. Given the abstractions and simplifications the goal is to investigate if it is possible to find stories that shows DeSS and IFP using preSLS theory in a simple and approachable way.

B. LQR

Linear-quadratic optimization refers to a problem formulation where the objective is to minimize a cost that is described by a quadratic function, as in Equation (1), and the system dynamics are described by linear differential equations, as in Equation (2) [16]. Here x_k denotes the state vector, u_k the input, y_k the output and the disturbance w_k is assumed to be Gaussian distributed.

$$J = \lim_{\tau \to \infty} \sum_{k=1}^{\tau} x_k^T Q x_k + u_k^T R u_k \tag{1}$$

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 u_k + B_2 w_k \\ y_k &= Cx_k \end{aligned} \tag{2}$$

The optimal control is given by $u_k = -Kx_k$ where K is the optimal controller and is given by:

$$K = (R + B_1^T P B_1)^{-1}(B_1^T P A)$$

and P is solved for using the discrete time algebraic Riccati equation (DARE):

$$P = A^T PA - (A^T PB_1)(R + B_1^T PB_1)^{-1}(B_1^T PA) + Q \tag{3}$$

For this study only the dual problems of State Feedback (SF) and Full Control (FC) will be considered where either $C = I$ or $B_1 = I$ respectively in Equation (2). The general Output Feedback (OF) case will not be considered and is left for future studies. Additionally, the sensor noise is assumed to to be zero which is equivalent to $R = 0$ in Equation (1).

One key property of the states is the concept of controllability. A state x is said to be *controllable* if in finite time, an input can yield the desired state x. If all states are controllable then the whole system is controllable [16].

A related theorem states that a system of order n is controllable if and only if the controllability matrix $\mathcal{S}$ has full rank where the controllability matrix is given by:

$$\mathcal{S}(A, B) = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

Here A and B are the matrices of Equation (2) [16].

The properties of this theorem yields a trivial case where actuation density is demanded. If $A = I$ then the stability matrix becomes:

$$\mathcal{S} = \begin{bmatrix} B & B & \dots & B \end{bmatrix}$$

Thus B must be of the same rank as A for the system to be controllable, meaning that for each state we must also have an actuator. This density requirement of the actuators is considered trivial and will not be further discussed. The aim is instead to look at diversity both in time and sparseness to see how diversity can yield stable control.

III. Methodology

The cases of diverse sensing and diverse actuation are managed separately. When modeling diverse sensing, the actuation is assumed to be perfect and vice versa. This is to highlight the impact of modifying the components respectively. For both cases we assume the system setup to be a ring formation with n states where the closest neighbors interact with each other as shown in Figure 1. We assume that all interactions between neighboring states and own interactions with previous time steps are equal to some constant α which yields the following $(n \times n)$ A matrix describing the dynamics:

$$A = a * \begin{bmatrix} \alpha & \alpha & 0 & 0 & \dots & \alpha \\ \alpha & \alpha & \alpha & 0 & \dots & 0 \\ 0 & \alpha & \alpha & \alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \alpha & \alpha & \alpha \\ \alpha & 0 & \dots & 0 & \alpha & \alpha \end{bmatrix} \tag{4}$$

a is a scaling parameter which scales the eigenvalues of the A matrix. α has been chosen to be $1/3$ to make A neutrally stable for $a = 1$, i.e. the maximum eigenvalue is equal to 1. The scaling parameter a can be adjusted to make eigenvalues of the A matrix become > 1 which creates an unstable system.

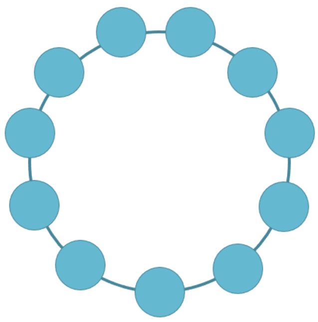


Fig. 1: The ring formation with n states where each state is connected to two neighbors.

For both cases of diverse sensing and actuation we consider two types of components: one that is fast but sparse, and one that is slow but dense, motivated by the examples discussed in Section II. Sparse and dense in this context refers to the quantity of sensors or actuators in the system in comparison to the states. Dense actuation means that we have as many actuators as states which can act on all states at once. Sparse actuation on the other hand means that the control is limited to a smaller section of the system.

A. Diverse Actuation

Suppose that we have a time discrete system with the following dynamics:

$$\begin{aligned} x_{k+1} &= \mathcal{A}x_k + \mathcal{B}_1 u_k + \mathcal{B}_2 w_k \\ y_k &= C x_k \end{aligned} \tag{5}$$

where $C = I$ for the diverse actuation SF case. The first n entries of x_k are the actual, real states and the rest denotes the delayed, internal actuation states. $\mathcal{A}$ and $\mathcal{B}_1$ are the matrices containing the dynamics of the entire system, including the dynamics of the internal, delayed states due to the speed diversity of the actuators.

The inputs in u are divided into two parts, one actuation that is fast and sparse, and another actuation which is slow and dense. The corresponding parts of the $\mathcal{B}_1$ matrix, $\mathcal{B}_1 = [B_f \quad B_s]$, are denoted B_f for fast and B_s for slow. The format of the $\mathcal{B}_1$ matrix depends on two additional parameters besides the number of states n: the relative delay d between the fast and slow actuation, and the dimension m of the fast actuation, which is how many m eigenvectors of A that are included in the fast component.

The fast actuators corresponding matrix B_f is of the size $(n \cdot (d+1) \times m)$ where the top $(n \times m)$ block is a matrix with the m eigenvectors corresponding to the m largest eigenvalues of A. The remaining entries of B_f are zero:

$$B_f = \begin{bmatrix} E & 0 & \dots & 0 \end{bmatrix}^T \tag{6}$$

where we let E be the block matrix of the m eigenvectors of A. Thus B_f represents the instant but sparse actuation which acts in the m "worst directions" of the spread when given an impulse to the system.

The slow actuators corresponding matrix B_s is of the size $(n \cdot (d+1) \times n)$. With no delay $B_s = I$ and with delays we have that B_s contains d stacked $n \times n$ blocks of zeros and the bottom $n \times n$ block is the identity.

$$B_s = \begin{bmatrix} 0 & 0 & \dots & I \end{bmatrix}^T \tag{7}$$

Thus B_s represents dense actuation, which can act on the whole system, that is delayed d time steps. To include the delayed states of the system, the extended matrix $\mathcal{A}$ then becomes:

$$\mathcal{A} = \begin{bmatrix} A & I & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & 0 & & \ddots & \\ \vdots & & & & I \\ 0 & & \dots & & 0 \end{bmatrix} \tag{8}$$

where a delay d yields the dimension of $\mathcal{A}$ as $(n \cdot (d+1) \times n \cdot (d+1))$.

We now formulate an LQR problem with $Q = G^T G$ and $R = H^T H$. Here $G = \begin{bmatrix} I & 0 & \dots \end{bmatrix}$ where I is the identity with dimension n and d blocks of zero. Thus only the n original states are minimized in the objective function and the delayed internal states are excluded. $H = \epsilon \cdot I$ where $\epsilon \to 0$ is a small value ensuring numerical feasibility. The resulting objective function is as Equation (1).

The optimal control action is $u_k = -Kx_k$, where $K = (R + \mathcal{B}_1^T P \mathcal{B}_1)^{-1}(\mathcal{B}_1^T P \mathcal{A})$, and P is obtained using DARE:

$$P = \mathcal{A}^T P \mathcal{A} - (\mathcal{A}^T P \mathcal{B}_1)(R + \mathcal{B}_1^T P \mathcal{B}_1)^{-1}(\mathcal{B}_1^T P \mathcal{A}) + Q \tag{9}$$

The state feedback problem described in Equation (5) and the LQR weights $Q = G^T G$ and $R = H^T H$ will be referred to as the *Primal problem* or *(SF) problem*. When $H = 0$, which we are assuming by letting $\epsilon \to 0$, the cost of the objective is calculated as $\text{Tr}(\mathcal{B}_2^T P \mathcal{B}_2)$.

B. Diverse Sensing

The diverse sensing case is the dual problem of the SF formulation. Here we instead have the following dynamics:

$$\begin{aligned} x_{k+1} &= \mathcal{A}^T x_k + C^T u_k + G^T w_k \\ y_k &= \mathcal{B}_1^T x_k + H^T v_k \end{aligned} \tag{10}$$

where the first n entries of x_k are the real states and the rest now denotes the delayed, internal sensed states. This is the FC case which corresponds to $C = I$.

The dual LQR problem has the weights $Q = \mathcal{B}_2 \mathcal{B}_2^T$ and $R = 0$. The optimal controller $u_k = -My_k = -M\mathcal{B}_1^T x_k$, where $M = \mathcal{A}^T F \mathcal{B}_1(\mathcal{B}_1^T F \mathcal{B}_1 + R)^{-1}$ is obtained by solving DARE:

$$F = \mathcal{A}^T F \mathcal{A} - (\mathcal{A}^T F \mathcal{B}_1)(\mathcal{B}_1^T F \mathcal{B}_1 + R)^{-1}(\mathcal{B}_1^T F \mathcal{A}) + Q \tag{11}$$

Note that $F = P$ which gives $M = K^T$. Again we have that the cost when $H = 0$ is calculated as $\text{Tr}(\mathcal{B}_2^T F \mathcal{B}_2) = \text{Tr}(\mathcal{B}_2^T P \mathcal{B}_2)$, i.e. due to the duality of the two problem formulations the cost is equal for both cases.

Diverse sensing assumes perfect actuation and thus the dynamics described by Equation (10) will be referred to as the *Dual problem* or *(FC)* problem.

Later we study the behaviour of the systems with diverse sensing and actuation when exposed to an impulse in the first time step. The aim is to search for trade-offs and DeSS by varying the different parameters and to find interesting examples showing variations of the performance when using the fast and slow components separately and combined.

IV. Results and Analysis

The system setup has four parameters which affect the outcome: the number of states n, the scaling parameter a, the delay of the slow component d and the sparseness of the fast component, which is determined by how many m eigenvectors of A are included. The open loop with no control results in an unstable system for $a \geq 1$ since for these values the eigenvalues of A become larger than or equal to 1. For these occasions an impulse given at any place in the ring will cause the states to grow rapidly.

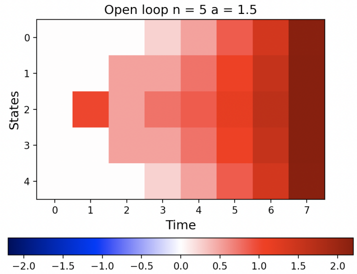


Fig. 2: Open loop with no actuation when given an impulse with $n = 5$, $a = 1.5$ and cost = ∞.

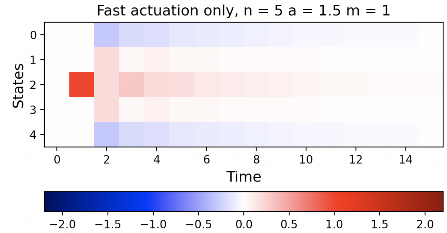


Fig. 3: Fast-only actuation when given an impulse with $n = 5$, $a = 1.5$, $m = 1$ and cost = 1.800.

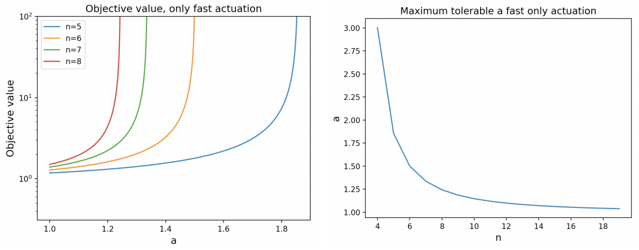
One example of this is shown for $n = 5$, $a = 1.5$ in Figure 2. After 5 time steps the impulse with magnitude 1 has doubled and keeps on growing exponentially to infinity due to the instability of the system. Because of the unstable system, control action is needed and next we look at the results from the diverse actuation and the diverse sensing setup separately.

A. Diverse Actuation

As described above, we consider two types of actuators: fast and slow. First, we study system performance in the absence of diverse actuators. We show that both fast-only actuation and slow-only actuation, on their own, can perform terribly and incur extreme (sometimes infinite) costs. We then show that by allowing the system to use diverse actuators (i.e. both fast and slow), performance is dramatically improved. This demonstrates the existence of a striking DeSS.

1) Fast-only actuation: Extending the example of the open loop with $n = 5$, $a = 1.5$, applying a fast actuator with $m = 1$ yields the following impulse response as shown in Figure 3. It is clear that the actuation is able to stabilize the system and can attenuate the impulse completely. This is due to that the second eigenvalue of the A matrix is smaller than 1 and the fast actuator is able to attenuate the disturbances in the "worst direction". Thus the system decays to zero after the impulse.

For $m = 1$ the relationship between the objective value and the value of a is shown in Figure 4a. Here it is clear how the function of the objective value becomes almost vertical for certain values of a which shows the breaking points of where the fast-only actuation fails and results in infinite cost. It is for these breaking points that more than m eigenvalues become larger than or equal to 1 and for which



(a) (b)

Fig. 4: Relations of a and n where fast actuation results in infinite costs for $m = 1$: (a) Objective value as a function of a for different n. (b) a values where fast actuation fails as a function of n.

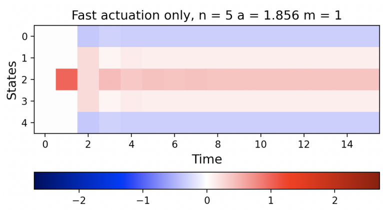


Fig. 5: Fast-only actuation when given an impulse with $n = 5$, $a = 1.856$, $m = 1$ and cost = ∞.

the eigenvectors in the fast-only actuation are no longer sufficient to cover all the impulse propagation directions. Keeping $m = 1$, the breaking points where the fast-only actuation fails for the different n values are described in Table I.

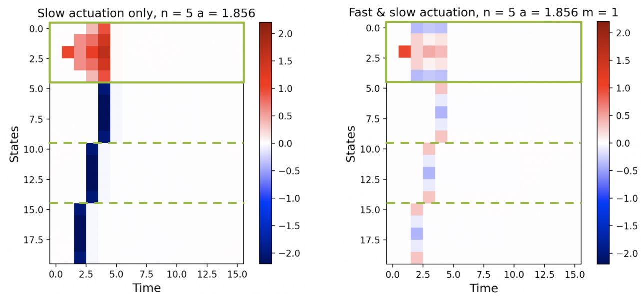
Figure 4b shows the breaking point values of a for which the fast-only actuation with $m = 1$ results in infinite costs. Here it is evident that as the n number of states increase, the system is less able to control instability.

n	5	6	7	8
a	1.856	1.503	1.336	1.244

TABLE I: Breaking point value for a where fast-only gives infinite cost with $m = 1$.

If we examine the case of the blue line in Figure 4a we get that for $n = 5$, $m = 1$ the fast-only actuation fails for $a = 1.856$. We now fix the parameters to be $n = 5$, $m = 1$ and $a = 1.856$. The corresponding impulse response for fast-only actuation is show in Figure 5, and comparing with Figure 3, we note how the impulse response is different in this case and does not get attenuated to zero.

The breaking point of where the fast-only actuation fails occurs when the cost becomes infinite. The fast-only component can only act in the "worst case" direction and for this setup the second and third eigenvalue has become 1 which yields this step response where the states become constant. Hence, the fast-only component manages to attenuate the exponentially growing part of the system but can not act on the constant steps which yields the infinite cost.



(a) Slow only actuation (b) Fast & slow actuation

Fig. 6: Impulse responses for $n = 5$, $a = 1.856$, $m = 1$ and $d = 3$ with cost = a) 13.726, b) 2.279.

2) Slow-only actuation: We now consider the case of slow-only actuators which are delayed $d = 3$ time steps. If the setup with $n = 5$, $m = 1$ $a = 1.856$ is repeated when only allowing slow actuation with $d = 3$, the system behaves as shown in Figure 6a.

Note that the top 5 states of Figure 6a display the actual states where as the bottom states are the internal, delayed states. For all following impulse responses, the top 5 states are the actual ring states of the system and are encircled with green for clarification. The dashed lines display the distinction between the delayed states.

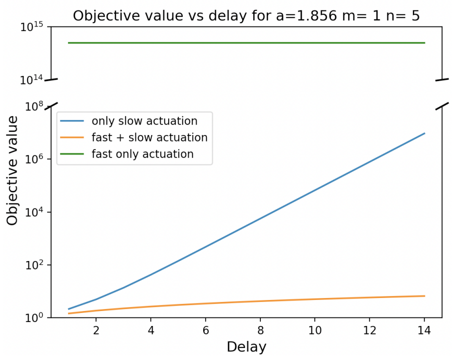
In comparison with the impulse response of the open loop in Figure 2 it is clear that the behaviors of the first three time steps are the same, and the slow-only actuation behaves like there is not any control action present. This is due to the delay of the slow actuators. Once the slow actuators are caught up, the impulse propagation is killed off immediately, showing the effects of the slow actuator being dense.

3) Fast & Slow actuation: The combination of the fast and slow actuation is what is of most interest since this setup is what creates the diversity of the system. The impulse response when utilizing both types of actuators is shown in Figure 6b.

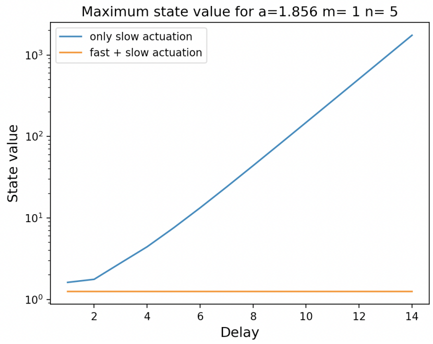
This case shows a combination of the behaviors from both previous setups where the initial impulse response in the first three time steps are equal to the fast-only response and then all impulse propagation is attenuated completely when the slow actuator has responded.

Since the fast-only actuation yields infinite cost for $a = 1.856$ we fix the scaling parameter a as this breaking point and study the objective value of the slow-only compared with the fast & slow actuation together while varying the delay d. The corresponding result is shown in Figure 7a.

4) DeSS: A dramatic sweet spot has been found. The results are clearly showing how much better the performance is when combining the two different actuators in comparison with using only fast or slow actuation separately. This striking sweet spot shows how the mixture of fast and slow actuation results in a low cost at the same time as the fast-only actuation results in infinite costs and the costs of the slow-only actuation is increasing exponentially when the delay increases. Moving forward, the green line showing



(a) Objective value



(b) Maximum amplitude

Fig. 7: Objective value and maximum amplitude of slow-only and fast & slow actuation for $n = 5$, $a = 1.856$ and $m = 1$.

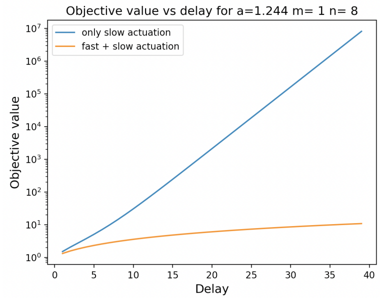
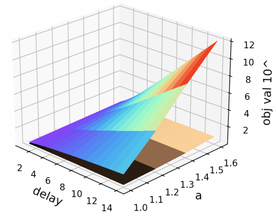
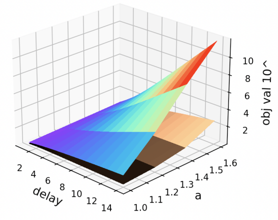


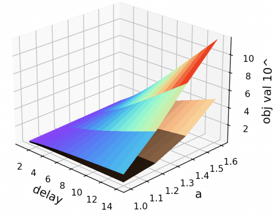
Fig. 8: Maximum objective value of slow-only and fast & slow actuation for $n = 8$, $a = 1.244$ and $m = 1$



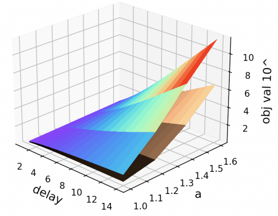
(a) $n = 5$



(b) $n = 6$



(c) $n = 7$



(d) $n = 8$

Fig. 9: Objective value as a function of a and d for $m = 1$ and $n = (5, 6, 7, 8)$.

the fast-only result will be left out since it is infinite when the fast-only actuation fails, which is where the sweet spots occur.

Another really important result is the difference in the worst case maximum amplitude for the different cases. The slow-only case yields in an infinite maximum amplitude as the delay increases. This is due to that the slow-only impulse acts as the open loop until the delayed slow actuation may operate. When using the combination of fast & slow, the maximum amplitude instead remains to be close to the impulse value. Thus the "worst-case" states are significantly better when using the combination of the actuators which is also displayed in Figure 7b. Thus, we can conclude that delayed actuation is acceptable if combined with a fast but sparse actuator. Hence, the theoretical sweet spot found here is consistent with what real examples, such as the sensorimotor control system, suggest.

Building onto the information obtained from Figure 4a we search for similar sweet spots for higher values of n. From Table I the corresponding a value that causes the fast-only actuation to result in infinite cost for $n = 8$ is $a = 1.244$. Studying the objective value for the slow-only and the fast & slow actuation, with $n = 8$, $a = 1.244$ and $m = 1$, when varying the delay of the slow actuation yields the following results shown in Figure 8. Again there is a sweet spot present where it is clear how the combination of two actuators, who separately results in infinite or large costs, creates a low-cost system together. The increase in n causes the vertical breaking point in a to be pushed to the left in Figure 4a. However, the decrease of the a value is generating a less dramatic sweet spot where the delay d needs to increase a lot in order for the fast & slow actuation to greatly outperform the slow-only actuation. Decreasing the scaling parameter a is generating a system with slower dynamics which in turn requires the delay to increase in order to demonstrate a visible effect.

The fast-only actuation does not depend on the delay d and in order to display how the cost depends on a and d for the slow-only and fast & slow actuation, 3D surface plots are shown in Figure 9 for different n while keeping the fast actuator with dimension $m = 1$.

Here the rainbow colored surface is the objective value with slow actuation only, and the copper colored surface shows the objective value of the fast & slow actuation. What is noticeable is that as the number of states increases, the performance of the fast & slow actuation gets worse, signaling that the fast actuation becomes less useful. This is reasonable since we are keeping $m = 1$ constant and thus as the number of states increases the fast actuation becomes

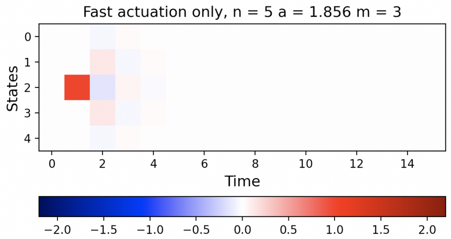


Fig. 10: Fast-only actuation when given an impulse $n = 5$, $a = 1.856$, $m = 3$ and cost = 1.079 .

less comprehensive.

5) The (n, m) relation: Since the fast-only actuation becomes less effective as n increases, we study the impact of increasing the m value. For $m = 1$ we picked the fast actuator to be the eigenvector corresponding to the largest eigenvalue. When increasing m, more eigenvectors are included corresponding to the largest eigenvalues in descending order. If there are multiple eigenvectors corresponding to the same eigenvalue, m is increased to include both in the fast actuator. For example with $n = 5$, the second and the third eigenvalues are equal. If m is chosen to 2 in this case, it will automatically be increased to 3.

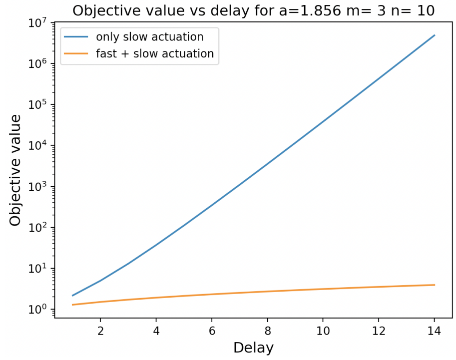
With $n = 5$ and $m = 1$ it was previously shown that $a = 1.856$ with fast-only actuation yielded an infinite cost. Increasing to $m = 3$ instead gives a finite cost with the corresponding impulse propagation as shown in Figure 10.

In this case, the fast actuation alone is able to attenuate the impulse which is a big difference from the previous case of Figure 5. In fact, the fast-only actuation can achieve a stable system as long as m is larger than or equal to the number of eigenvalues of A that are ≥ 1. Increasing m has an impact on the fast actuation that pushes the vertical lines in Figure 4a, which shows the breaking points of fast-only actuation, to the right. Thus, intuitively, decreasing the sparseness of the fast actuation improves the performance in terms of stability.

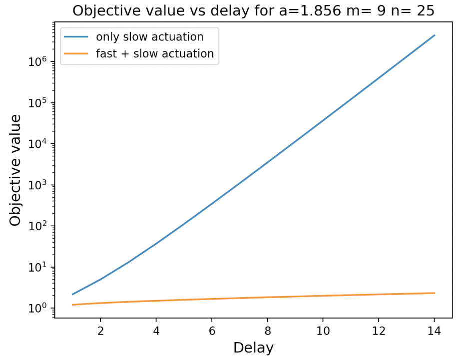
We now search for similar sweet spots as the one obtained for $n = 5$, $m = 1$, $a = 1.856$ where the increase in delay of the slow actuation showed the benefits of using both actuators as displayed in Figure 7a. a is kept at $a = 1.856$ and pairs of (n, m) that shows the same sweet spots are searched for.

We search for two things, first we want that the pair (n, m) is so that the fast actuation alone leads to infinite cost for $a = 1.856$, secondly we want the impact of the fast actuation to be great enough that the performance of the fast & slow actuation together outperforms the result of the slow only actuation. Although the computations are very sensitive to small changes in the parameters, there seems to be a pattern of (n, m) pairs that yields what we are searching for. With $a = 1.856$ the fast only actuation fails for the pairs $(n, m) = (5, 1), (10, 3), (15, 5), (20, 7), (25, 9), \ldots$.

With these pairs of (n, m) where the fast actuation fails we now repeat the case study of the performance of the slow only actuation in comparison with the fast & slow actuation when varying the delay d. The corresponding result



(a) $n = 10$ and $m = 3$



(b) $n = 25$ and $m = 9$

Fig. 11: Objective value of slow-only and fast & slow actuation as a function of d with $a = 1.856$ for different n and m.

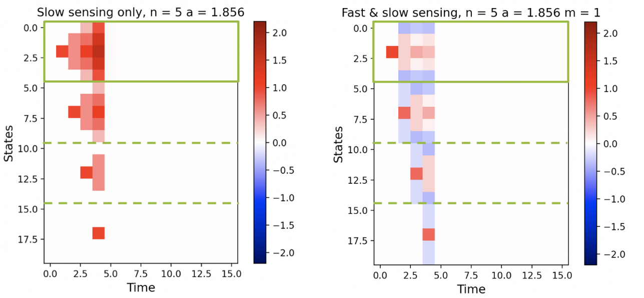
is shown in Figure 11a for $(10, 3)$ and in Figure 11b for $(25, 9)$. The results for all of the (n, m) pairs $(n, m) = (5, 1), (10, 3), (15, 5), (20, 7), (25, 9)$ have the same appearance, signaling a pattern for sweet spots.

This result is displaying dramatic sweet spots where the combination of both fast and slow actuation is essential for the system to function properly. It is showing that delays in actuation are acceptable if they are combined with a fast and sparse response. Even in a very simple system with only two types of actuators, dramatic DeSS are observed, suggesting that DeSS is a fundamental feature in the presence of diverse actuation. These results are however only scratching the surface of the sweet spots that occurs in a system with diverse actuators and more complex setups are needed to describe structures like the sensorimotor control system or the immune system.

B. Diverse Sensing

We previously considered the case of perfect sensing and diverse actuation. Mathematically, this is the dual problem to that of perfect actuation and diverse sensing; thus, our observations about diverse actuation and DeSS also apply to diverse sensing.

1) Fast-only sensing: The duality between the SF and FC cases causes the impulse response of fast-only sensing to be identical to the impulse response of fast-only actuation.



(a) Slow-only sensing (b) Fast & slow sensing

Fig. 12: Impulse responses for $n = 5$, $a = 1.856$, $m = 1$ and $d = 3$ with cost = a) 13.726, b) 2.279.

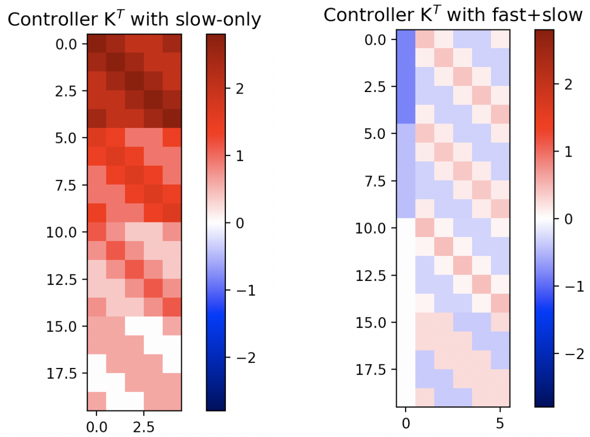
The lack of the delay elements results in symmetric matrices and thus the primal problem becomes the same as the dual problem for the fast-only case.

The physical interpretation of this is that with one component that is fast-only and the other one is assumed to perform perfectly, the sparseness of the fast component plays the deciding role of the outcome. Meaning that if the actuation is perfect but the sensing is sparse, we can not sense the states adequately to attenuate the impulse completely when introduced to an unstable system. On the other hand if the sensing is perfect but the actuation is sparse we can not adequately act on all the states, resulting in the impulse not being perfectly attenuated.

Given the previous information of Figure 4a, we know that the fast-only sensing will fail to control the system for $n = 5$, $m = 1$ and $a = 1.856$. Since the fast-only cases for both sensing and actuation are the same, the obtained impulse response from fast-only sensing is the same as in Figure 5. This is the setup where the system needs a slow sensor in order to function properly, hence this setup will be used when studying the system response from an impulse using slow-only and fast & slow sensing.

2) Slow-only sensing: Using slow-only sensing where the dense but slow sensors are delayed $d = 3$ time steps the impulse response is shown in Figure 12a. Again note that the top 5 states of Figure 12a display the actual states where as the bottom states are the internal, delayed, sensed states. The open loop behavior is clearly visible in the first three time steps. This is due to the delay of the slow sensing. In the impulse response it is clear how the internal, delayed states "grow" for each step while approaching the real states. Once the slow sensors are caught up, the impulse propagation is killed off immediately, showing the effects of the slow sensor being dense and the actuation being perfect.

3) Fast & Slow sensing: Combining the fast and slow sensors we again obtain a diverse system where the impulse response, shown in Figure 12b, emulates a merge between the responses of fast-only and slow-only sensing. The first $d+1$ time steps are identical to the fast-only step response and the following time steps are equal to the slow-only response, i.e. equal to 0. Thus the combination of fast & slow results in a response that looks like the "best of both worlds".



(a) Slow-only (b) Fast & slow

Fig. 13: The optimal controller K^T for the diverse sensing setup with $n = 5$, $a = 1.856$, $m = 1$ and $d = 3$

Due to the duality of the actuation and sensing problems, the top 5 actual states behave the same, i.e. the ring formation for both cases looks the same when given an impulse, which is evident when comparing the corresponding cases in Figure 12 with Figure 6. The duality also yields that the cost of the different setups are the same, as mentioned in Section III. Therefore, the figures displaying the costs as function of the delay d and scaling parameter a still applies to the sensing problem. The same goes for the dramatic sweet spots where the diverse system outperforms the separate types of components substantially. Diverse sensing, like diverse actuation, produces dramatic DeSS even in a simple system. This again suggests that DeSS is a fundamental feature in systems with diversity.

C. IFP

We now observe the structure of the controllers that give the dramatic DeSS found above; we focus on the case of diverse sensing. We show that IFP is an essential architectural feature to enable DeSS and that the removal of it worsens the performance of the system.

Looking at the impulse responses of Figure 12a and Figure 12b there seems to be a lot of IFP present, meaning that there is information flow traveling backwards to the sensors within the system. This is displayed clearly in Figure 13 which shows the optimal controllers K^T of the diverse sensing problem. The second, third and fourth blocks of K^T shows the IFP of the system. Here the entries of the matrix is shown using a color map to more easily visualize patterns rather than showing large matrices full of numbers.

Figure 13a displays the IFP of the case where only the slow but dense sensing is used. The shape of the A matrix is clearly visible, which is not a coincidence since this is what the internal feedback can utilize in terms of predicting the behaviour of the states in the system. Comparing with the case where both fast and slow sensors are used in Figure

13b, the shape is still similar although including the fast sensor changes the magnitudes of the entries in the controller corresponding to the slow and dense sensor.

The magnitude of the entires in K^T in Figure 13a are much grater than in Figure 13b suggesting that there is much more IFP in the slow-only case. Introducing the fast component in Figure 13 greatly reduces the need for IFP in the delayed states since the fast action is taken and this seems to dominate over the now smaller slow-only based action. It is also clear that there is less IFP present the more delayed the internal state is, i.e. the magnitudes in the blocks of K^T are decreasing for each delayed step. However, for both cases it is clear that the system uses IFP to create optimal control and it is necessary with internal feedback to know which action has been taken previously such that the following delayed states does not act poorly.

The presence of IFP is also studied for the diverse actuation case with the sweet spot setup $n = 5$, $a = 1.856$, $m = 1$ and $d = 3$. The optimal controllers of the slow-only and fast & slow actuation are identical to the transpose of the optimal controllers in the corresponding sensing case shown in Figure 13.

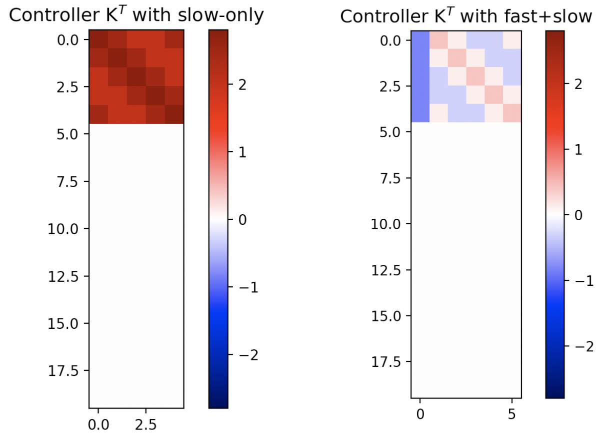
Again the presence of IFP is needed to know what previous action has been taken. The differences between the sensing and actuation cases are that in the SF case IFP is needed because of sensing changes, where as in the FC case IFP is needed because of actuation changes. It is however important to note that the ring states, i.e. the top $n = 5$ states that are the actual states, are identical for both the diverse sensing and actuation cases in Figure 6a, 6b and Figure 12a, 12b. Thus, delays in either component results in the same system behaviour, it is only the internal states that are different.

1) Removing IFP: To further study the impact of IFP we study the case where it is removed in the controllers K^T for both the slow-only and the fast & slow setup. This is done by removing the corresponding IFP rows of K^T which gives the resulting controllers that are shown in Figure 14.

The removal of IFP causes dramatic change. Figure 15 shows the impulse response when using slow-only sensing or actuation with removed IFP, the setup is still $n = 5$, $a = 1.856$, $m = 1$ and $d = 3$. Again it is clear how in both cases of diverse sensing or actuation, the ring states behave equally where as the differences lay in the delayed, internal states.

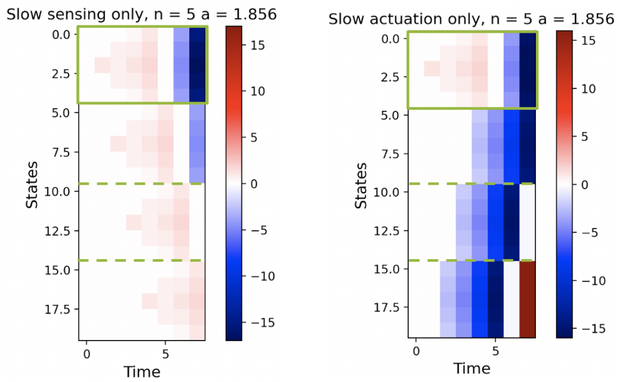
The impulse causes major fluctuation in the states due to the delayed sensing and actuation. Removing IFP causes a gap in the system information about the current state and removes the knowledge of previous actions, which before could be determined perfectly using the internal feedback. Without the IFP, the system is no longer are able to calculate the correct action since the internal feedback of what has previously been done is removed. This generates a system which only acts based on what has been sensed d time steps ago without taking previous, more recent actions into account.

The behavior is clearly visible in the impulse response



(a) Slow-only (b) Fast & slow

Fig. 14: The optimal controller K^T for the diverse sensing setup with $n = 5$, $a = 1.856$, $m = 1$ and $d = 3$ with removed IFP.



(a) Slow only sensing (b) Slow only actuation

Fig. 15: Slow only sensing and actuation with removed IFP when given an impulse with $n = 5$, $a = 1.856$, $m = 1$, $d = 3$ and cost = a) ∞, b) ∞.

where the impulse is attenuated perfectly after $d + 1$ time steps when the delayed sensing has caught up with the delay. But immediately in the next time step the states are fed with more attenuating control action which causes the 0 states to become negative. This is due to that the sensed state d time steps ago was still positive and thus negative actuation was needed and without the internal feedback there is no way of telling that this has already been taken care of in the previous time step.

Thus, more negative actuation is fed into the system causing the states to become negative. Later, these negative states will be sensed and positive action will be taken, but due to the delay and lack of internal feedback the entire system will begin to fluctuate between positive and negative states, each time growing in magnitude. Hence without IFP the system becomes unstable.

This fluctuating behavior for slow-only is shown in Figure

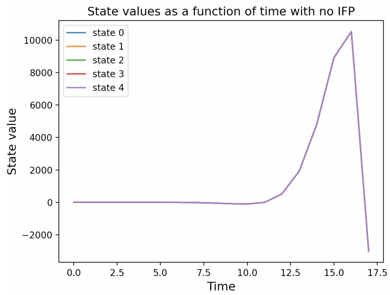
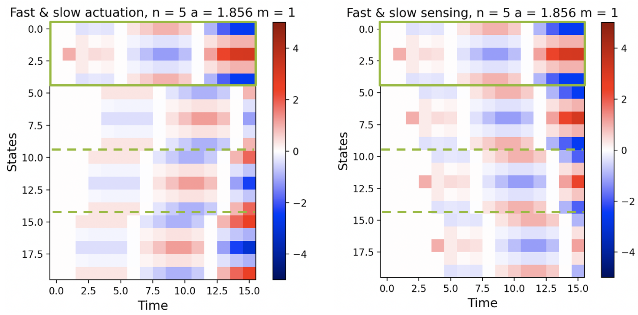


Fig. 16: Slow-only sensing or actuation with removed IFP when given an impulse.



(a) Fast & slow actuation (b) Fast & slow sensing

Fig. 17: Fast & slow sensing and actuation with removed IFP when given an impulse with $n = 5$, $a = 1.856$, $m = 1$, $d = 3$ and cost = a) ∞, b) ∞.

16. Due to the increase in magnitude, the fluctuation appears as a flat line when in fact the states are fluctuating with exponential increase in magnitude. Since the ring states are equal for the dual cases, the figure applies to both slow-only cases of sensing and actuation.

Removing IFP in the fast & slow sensing or actuation case, using the controller of Figure 14b, the impulse responses are shown in Figure 17. Again, the ring states are identical for both cases but the internal, delayed states are different. The impulse response again fluctuates such that the amplitude increases to infinity as shown in Figure 18.

Comparing the results of Figure 16 and Figure 18 it is

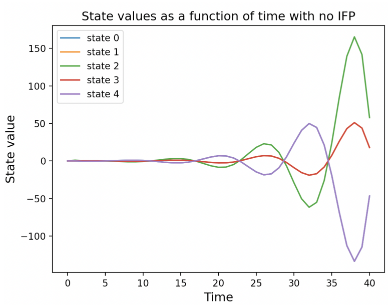


Fig. 18: Fast & slow sensing or actuation with removed IFP when given an impulse with $n = 5$, $a = 1.856$, $m = 1$ and $d = 3$.

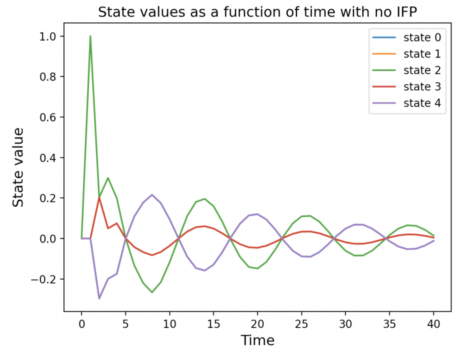


Fig. 19: Fast & slow sensing or actuation with removed IFP when given an impulse with $n = 5$, $a = 1.5$, $m = 1$ and $d = 3$.

evident that the system using both kinds of components behaves better since the magnitude of the state values increase less rapidly. However, for long time intervals, the amplitude of the states will increase to infinity suggesting that IFP is a crucial attribute of the system in order to achieve stability.

What is interesting to note is the result of the fast & slow without IFP for cases where the fast-only component alone is sufficient for the system to achieve stability. A case when this occurs, as previously seen in Figure 3, is the setup $n = 5$, $a = 1.5$, $m = 1$. The impulse response of this setup, without enabling IFP for fast & slow in the controller, gives the following result as shown in Figure 19.

The fluctuations are decreasing in amplitude and for long time intervals the impulse is attenuated to 0. However, the impulse attenuation for fast-only is much faster as we saw in Figure 3. Thus, including the slow components without IFP makes the system performance worse because of the delayed sensing without feedback, suggesting the importance of IFP.

Removing IFP from either the slow-only or fast & slow case results in worse performances of the system. The setup that resulted in sweet spots in previous figures, yielded an unstable system with infinite magnitudes without the internal feedback pathways thus highlighting the importance of its presence.

Without IFP, the slow-only setup will always result in instability because of the delayed reaction. But even with both fast and slow components the system dynamics are more limited and the scaling parameter a needs to be small in order for the system to remain stable. Thus signaling the importance of IFP when delays are present in order to reduce control limits. It is also evident that IFP is a necessary architectural feature to enable DeSS in the presence of delayed layers.

Given the experience of these results that are presented in this paper, it is clear that the preSLS control theory is sufficient to describe a simplified model of diverse systems, and which functions as a bridge between familiar theory and future extensions. The key takeaways are that DeSS is a fundamental feature generated by the diversity of the systems and that IFP is a fundamental feature necessary to enable

DeSS when delays are present.

V. Extension to SLS

Moving forwards there are two directions of further studies. One direction is to extend all of the experiences from the senior, preSLS control theory to the SLS framework. Using the understanding from the fundamental examples, more complex applications to the sensorimotor control system or immune system can be made using SLS. There we still assume full state feedback with full control but the delays are moved from the sensors and actuators. Instead, internal communication and computation delays are imposed which causes an explosion in IFP which demands a switch to the SLS framework.

The first key result of this research is that large delays in sensors or actuators, which exists in the visual cortex, are acceptable if the components are dense and accurate and if these delayed components are complemented by sparse and fast components, like the vestibular system and proprioception. Extending the implementation of diverse actuation in SLS is easily done by first creating a concatenated $\mathcal{B} = \begin{bmatrix} B_1 & B_2 & \ldots \end{bmatrix}$ matrix with all the $B : i$ matrices describing the separate dynamics of each type of actuator. Secondly the delayed parts of the diverse actuation is implemented by constraining the corresponding parts of the delayed actuator in each block matrix of the Φ_u matrix from [3]. The constraint of diverse, delayed actuation and sensing can then be further analyzed in combination with implementing locality and communication constraints using the SLS framework where the scalability property is essential to be able to describe further complex systems.

The second key result is regarding the presence of IFP in the optimal controllers for the delayed components. The necessity of IFP for a well-functioning system is a good motivator for continued studies of these kinds of systems using SLS. As the size and complexity grows, the more need for massive IFP, which will affect the computational time. Hence when trying to model systems as the sensorimotor control system or the immune system, that are exhibiting functions with a lot of IFP, the scalability and locality of the SLS theory is crucial.

The other direction of further studies would be to combine the two separate cases of diverse sensing and actuation by switching to OF dynamics, introduce sensor noise and then use a Kalman filter [16] to estimate the state of the system. This could be done with preSLS theory or with SLS to extend the previously mentioned direction of future studies even further.

VI. Discussions and Conclusions

The results of this paper show that there exists dramatic sweet spots in systems with diverse sensing and actuation where the components fail to control the system separately but create a well-functioning setup when combined. The results also show the importance of internal feedback within delayed components where the system performance becomes unstable without the correct feedback information. Examples of applications where the theory of diverse sensing and actuation can be implemented include the sensorimotor control and the immune system. These are all complex structures which when modeled requires large computations. This yields SLS as a natural future direction of study.

VII. Acknowledgements

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