Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan, July 24-28, 2006

Transfer Function Approximation and Identification Using Magnitude and Phase Criteria

Henrik Sandberg, Alexander Lanzon, and Brian D.O. Anderson

***Abstract*— In this paper, we show how convex optimization can be used for model reduction and identification of transfer functions. Two different methods are presented. In the first method magnitude functions are matched, and in the second method phase functions are matched. The weighted error bounds have direct interpretation in a Bode diagram. Both methods are suitable to engineers working with Bode diagrams. Furthermore, we see that transfer functions that have similar magnitude or phase functions also have a small relative H-infinity error under some minimum phase assumptions. The error bounds come from bounds associated with the Hilbert transform operator restricted in its application to rational transfer functions. Two examples are included to illustrate the results.**

***Keywords*— Model approximation, model reduction, system identification, magnitude, phase, Hilbert transform, semidefinite programs, KYP lemma, sum of squares decomposition, LMI**

I. Introduction

It has become increasingly popular in control and systems theory to use convex optimization to solve problems of various nature, see, for example, [3], [4]. One reason is the efficient software available for solving semidefinite programs in polynomial time. Another reason is that many well-known problems, such as H_∞ optimization, can be cast as semidefinite programs using Linear Matrix Inequalities (LMIs). It is, however, not known how to use these tools to obtain optimal transfer function approximations in the H_∞ norm. The reason is a nonconvex rank constraint that is needed to enforce the degree of the approximation [4, Corollary 7.11]. For this reason, various suboptimal methods, such as balanced truncation [4], [9], [10], are often being used for model reduction.

One possibility for obtaining an optimal model reduction problem that can be solved in polynomial time is to change the norm. The Hankel norm model reduction problem can be solved in polynomial time, and often yields good approximations in H_∞ norm as well, see [5]. Other new interesting approaches are to use the real part norm, see [8], or a special parametrization of unstable transfer functions together with the L_∞ norm, see [17]. In particular, the methods in [8], [17] give tighter lower bounds on the H_∞ approximation error than Hankel approximation, easily incorporate weights, and can be used with samples of frequency data or the impulse response. This opens up the possibility of using the methods in [8], [17] for system identification as well.

In this paper, we show that transfer functions of given degree can be matched optimally to magnitude and phase data in polynomial time, using standard LMI solvers. The proposed methods share some advantages with the methods in [8], [17], such as the possibility of using weights and samples of data, but also have direct interpretations in Bode diagrams. Two examples of the problems we solve in the SISO case (for ease of illustration) are

$$\|\log|G_a| - \log|G|\|_\infty \to \min \tag{1}$$

$$\|\arg G_a - \arg G\|_\infty \to \min \tag{2}$$

where G is a given transfer function, and G_a is a stable approximation of user-specified degree. Since we match either magnitude or phase, we must put additional assumptions on G_a to obtain unique solutions. Generally, we will require G_a to be minimum phase and stable. This should be the situation of most interest. However, G_a can easily be modified to include unstable poles or zeros. Two other examples of the problems we solve in the SISO case (for ease of illustration) are to find G_a such that

$$\log|w_2(j\omega)| \le \log|G_a(j\omega)| - \log|G(j\omega)| \le -\log|w_1(j\omega)| \tag{3}$$

$$\arg w_2(j\omega) \le \arg G_a(j\omega) - \arg G(j\omega) \le -\arg w_1(j\omega) \tag{4}$$

for all ω, where w_1, w_2 are user-specified weights. In (3) it is assumed that $\|w_1\|_\infty \le 1$ and $\|w_2\|_\infty \le 1$, and in (4) it is assumed that $\arg w_1 \le 0$ and $\arg w_2 \le 0$. For the magnitude approximation case, we will be able to deal with MIMO systems as well. For the phase approximation case, we will not deal with MIMO systems. One reason for this is the splintering of definition of what phase is in the MIMO case [1], [7], [11]. Furthermore, we obtain a priori bounds on the relative error in the H_∞ norm, $\|G^{-1}(G - G_a)\|_\infty$. Many other methods, such as optimal Hankel norm and real part norm approximation, give a priori bounds on the additive error $\|G - G_a\|_\infty$. To

This work was supported by the Australian Research Council under Discovery-Projects Grant DP0342683, and by National ICT Australia Ltd., which is funded through the Australian Government's *Backing Australia's Ability Initiative*. Parts of this work were performed during a month's visit of the first author at the Australian National University.

H. Sandberg is with California Institute of Technology, Control and Dynamical Systems, M/C 107-81, Pasadena, CA 91125, USA. Email: henriks@cds.caltech.edu

A. Lanzon and B.D.O. Anderson are with Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia, and with National ICT Australia Ltd., Locked Bag 8001, Canberra ACT 2601, Australia. Email: Alexander.Lanzon@anu.edu.au and Brian.Anderson@nicta.com.au

go from an additive bound to a relative bound using just the submultiplicative property of the H_∞ norm is generally too conservative. When reducing plant models that should be used in feedback control design, the relative error is often a better reduction criterion, see [10, p.101].

To illustrate the methods developed here, two examples are included. They show how magnitude and phase approximation can be used for model reduction, system identification, and filter design in Bode diagrams. Hence, the results should be of interest to both control and signal processing engineers.

The organization of the paper is as follows. In Section II, we formulate and solve the magnitude approximation problem. In particular, how the problem is formulated as a semidefinite program is explained in Section II-A. In Section III, we formulate and solve the phase approximation problem. The details of the semidefinite program are given in Section III-A. The methods are illustrated in Section IV, where two different examples are given. Finally, in Section V we have conclusions and suggestions for future work. Proof of all results and further material and examples are included in [16].

Preliminaries: The modulus on $\mathbb{C}$ is denoted by $|\cdot|$. The p singular values of a $p\times p$ complex matrix M are denoted by $\sigma_i(M)$, $i = 1\dots p$, in decreasing order. The maximum and minimum singular values are also denoted by $\overline{\sigma}(M)$ and $\underline{\sigma}(M)$, respectively. M^* is the complex conjugate transpose of M. The L_∞ norm $\|\cdot\|_\infty$ of a $p\times p$ matrix function $G(s)$ of a complex variable s is defined by $\|G\|_\infty := \operatorname{ess\,sup}_\omega \overline{\sigma}(G(j\omega))$, and $G\in L_\infty$ if $\|G\|_\infty$ is finite. A matrix function $G\in H_\infty$ if it is bounded and analytic in the open complex right half plane ($\mathbb{C}_+$), and $G\in H_\infty^-$ if it is bounded and analytic in the open left half plane ($\mathbb{C}_-$). $G\in RH_\infty(RL_\infty)$ if it is in $H_\infty(L_\infty)$ and is rational in s with real coefficients. Furthermore, $G\in R_nH_\infty$ if $G\in RH_\infty$ and has McMillan degree less than or equal to n. The conjugate system $G^\sim$ is defined by $G^\sim(s) := G(-s)^T$ (and $G(j\omega)^* = G^\sim(j\omega)$ if G has real coefficients). A square system G is all-pass if $G(j\omega)^*G(j\omega) = I$. We call a matrix function G minimum phase if it has full rank in $\mathbb{C}_+$, and stable if it belongs to H_∞.

II. Magnitude Approximation

In this section, we formulate an optimization problem that allow us to approximate a transfer function G with another transfer function G_a (of smaller degree) such that the gains $\sigma_i(G(j\omega))$ and $\sigma_i(G_a(j\omega))$ are close. The following assumptions on the model G and the weights w_1, w_2 will be used for magnitude approximation:

Assumption 1: $G, G^{-1}, w_1, w_1^{-1}, w_2, w_2^{-1} \in RL_\infty$, w_1, w_2 are scalar, and $\|w_1\|_\infty \le 1, \|w_2\|_\infty \le 1$.

Remark 1 (Strictly proper models): The condition $G^{-1}\in RL_\infty$ in Assumption 1 excludes strictly proper models G and G with zeros on the imaginary axis. The condition $G^{-1}\in RL_\infty$ can be relaxed, however. For example, one can instead require that there is a positive integer n such that $\lim_{s\to\infty}\det(s^nG(s))\neq 0$. The reason for the conditions on G is to easily guarantee that there exists a minimum phase spectral factor in $R_{mp}H_\infty$.

The optimization problem we shall try to solve is: Given a $p\times p$ MIMO transfer function G and weights w_1, w_2 that satisfy Assumption 1, and a positive integer m, find γ and an approximation G_a such that

$$\min_{\gamma, G_a} \gamma, \quad \text{subject to} \tag{5}$$

$$\frac{|w_2(j\omega)|}{\sqrt{1+\gamma}} \le \sigma_i((G^{-1}G_a)(j\omega)) \le \frac{\sqrt{1+\gamma}}{|w_1(j\omega)|}, \tag{6}$$

$$\gamma \ge 0, \quad \text{and} \quad G_a, G_a^{-1} \in R_mH_\infty, \tag{7}$$

for all ω and $i = 1\dots p$. As we shall see in this section, in the SISO case ($p = 1$), the problem (5)–(7) is a quasiconvex minimization problem (see [2, p.145]) and we will find the optimal approximation G_a in R_mH_∞. In the MIMO case ($p > 1$), we will find a solution if we allow G_a to belong to a set larger than R_mH_∞. This set contains R_mH_∞ and is a subset of $R_{mp}H_\infty$. Next, we motivate why it is interesting to find solutions to (5)–(7).

The constraints (6) can be reformulated in a number of ways, as is seen in the following theorem.

Theorem 1: Assume that w_1, w_2 have no zeros on the imaginary axis. Then (6) is equivalent to each of

$$\frac{|w_1(j\omega)|}{\sqrt{1+\gamma}} \le \sigma_i((G_a^{-1}G)(j\omega)) \le \frac{\sqrt{1+\gamma}}{|w_2(j\omega)|} \tag{8}$$

$$\begin{aligned} &\|w_1G^{-1}G_a\|_\infty \le \sqrt{1+\gamma} \\ \text{and} \quad &\|w_2\, G_a^{-1}G\|_\infty \le \sqrt{1+\gamma}, \end{aligned} \tag{9}$$

and implies

$$\frac{|w_2(j\omega)|}{\sqrt{1+\gamma}} \le \frac{\sigma_i(G_a(j\omega))}{\sigma_i(G(j\omega))} \le \frac{\sqrt{1+\gamma}}{|w_1(j\omega)|} \tag{10}$$

for all ω, and $i = 1\dots p$.

The following corollary to Theorem 1 shows that if an approximation G_a satisfies (6), then its gains $\sigma_i(G_a(j\omega))$ are close to $\sigma_i(G(j\omega))$. In particular, it shows that the difference between the logarithm of the singular values of G and G_a is bounded. That is, the distance between the curves typically plotted for MIMO systems in Bode magnitude plots, is bounded.

Corollary 2: The inequalities (6) imply

$$\begin{aligned} &-\frac{1}{2}\log(1+\gamma) + \log|w_2(j\omega)| \\ &\qquad \le \log\sigma_i(G_a(j\omega)) - \log\sigma_i(G(j\omega)) \\ &\qquad\qquad \le \frac{1}{2}\log(1+\gamma) - \log|w_1(j\omega)| \end{aligned} \tag{11}$$

for all ω, and $i = 1\dots p$.

Remark 2 (Constraints on γ and w_1, w_2): From the previous inequalities, it should be clear why we require $\gamma \ge 0$ and $\|w_1\|_\infty \le 1, \|w_2\|_\infty \le 1$. If these conditions are violated, it is possible that a solution where $G_a(j\omega) = G(j\omega)$, for some or all ω, does not satisfy

$$B_m(\omega) := \begin{pmatrix} b^{11}_{2m}\omega^{2m} + b^{11}_{2m-2}\omega^{2m-2} + \ldots + b^{11}_0 & \ldots & \begin{matrix}[(b^{p1}_{2m}\omega^{2m} + b^{p1}_{2m-2}\omega^{2m-2} + \cdots + b^{p1}_0) \\ -j\omega(b^{p1}_{2m-1}\omega^{2m-2} + \cdots + b^{p1}_1)]\end{matrix} \\ \vdots & \ddots & \vdots \\ \begin{matrix}[(b^{p1}_{2m}\omega^{2m} + b^{p1}_{2m-2}\omega^{2m-2} + \cdots + b^{p1}_0) \\ +j\omega(b^{p1}_{2m-1}\omega^{2m-2} + \cdots + b^{p1}_1)]\end{matrix} & \ldots & b^{pp}_{2m}\omega^{2m} + b^{pp}_{2m-2}\omega^{2m-2} + \ldots + b^{pp}_0 \end{pmatrix} \quad (12)$$

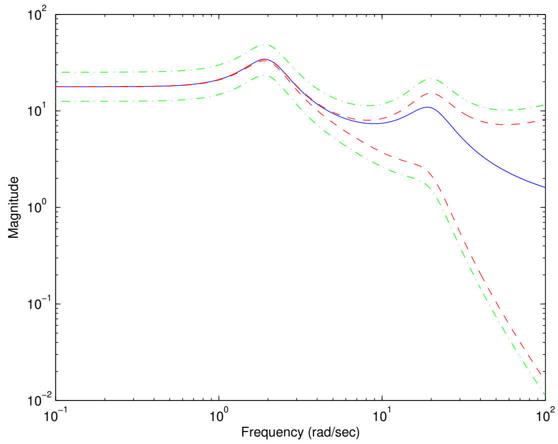

Fig. 1. The interpretation of the weights w_1, w_2, and the constant γ. The full line represents the model G, the dashed lines represent the bounds with $\gamma = 0$, and the dash-dotted lines the relaxed bounds with $\gamma = 1$. The magnitude of the sought approximation G_a should be within the bounds with the smallest possible γ.

the constraints (6). This is obviously undesirable in an approximation problem.

The bounds (11) and the parameter γ have a clear interpretation in a Bode diagram, see Fig. 1. When $\gamma = 0$ we are looking for G_a with magnitude function $|G_a(j\omega)|$ that satisfies the tightest bounds in Fig. 1. These bounds are given by w_1 and w_2. There may be such an approximation or not, depending on the weights w_1, w_2 and model G. When $\gamma > 0$ the bounds are relaxed uniformly, as is seen in Fig. 1. Hence, when γ is minimized in (5), we are looking for an approximation that satisfies the tightest possible bounds. A similar situation holds in the MIMO case, but then for each singular value, as seen in (6) and (8)–(11).

Remark 3 (Equivalence in SISO case): When G is SISO ($p = 1$) the inequalities (6) and (8)–(11) are stating the same thing, since $\sigma_1(G) = |G|$, and (11) is equivalent to (6). Hence, in the scalar case, if we solve (5)–(7) with $w_1 = w_2 = 1$, a solution to the problem (1) is obtained.

Remark 4 (MIMO weights): It is possible to use MIMO weights instead of scalar weights w_1, w_2. Then one should group the weights together with G, i.e., $G \to W_o G W_i$ and all the analysis in this section carries through. In general, it seems enough to use scalar weights, however. MIMO weights could be useful if one would like to have different approximation accuracy for different σ_i in (6) and (8)–(10). Then one needs to carefully design the directional amplification of the weights.

A. Parametrization and the Semidefinite Program

In this section, we see how (5)–(7) is translated into a semidefinite program. First we introduce the parametrization of G_a. The parametrization is critical to be able to formulate the problem as a semidefinite program. Furthermore, it is also important that the parametrization contains all G_a that are interesting. It is shown in Theorem 3 that the following parametrization is a good choice and satisfies these conditions. Let m be a positive integer (which will turn out to be related to the McMillan degree of the approximant G_a) and define an even scalar polynomial

$$a_m(\omega^2) := \omega^{2m} + a_{2m-2}\omega^{2m-2} + \ldots + a_2\omega^2 + a_0,$$

and a hermitian polynomial matrix $B_m(\omega)$ given by (12) where $\{a_i\}$ and $\{b^{jk}_i\}$, $0 \le i \le 2m$, $1 \le k \le j \le p$, are the decision variables. Notice that $B_m(\omega) = B_m(-\omega)^T = B_m(\omega)^* \in \mathbb{C}^{p\times p}$. The polynomial $a_m(\omega^2)$ has m decision variables, and $B_m(\omega)$ has $(p/2)(2pm + p + 1)$ decision variables.

Theorem 3: Assume that G, w_1, w_2 satisfy Assumption 1, and that the integer m and $\gamma \ge 0$ are fixed. If there is a $G_a \in R_m H_\infty$ that satisfies (6), then there exist a_m, B_m that satisfy all of

$$|w_1(j\omega)|^2 B_m(\omega) - (1+\gamma)a_m(\omega^2)(GG^*)(j\omega) \le 0 \quad (13)$$
$$|w_2(j\omega)|^2 a_m(\omega^2)(GG^*)(j\omega) - (1+\gamma)B_m(\omega) \le 0 \quad (14)$$
$$a_m(\omega^2) > 0 \quad (15)$$

for all ω. Conversely, if there are a_m, B_m that satisfy (13)–(15) for all ω, then there is a $G_a \in R_{mp}H_\infty$ that satisfies (6) and $G_a^{-1} \in R_{mp}H_\infty$.

Remark 5 (Equivalence in SISO case): Theorem 3 is stronger in the SISO case when $p = 1$. Then there is an approximation G_a of McMillan degree m that satisfies (6) if and only if (13)–(15) hold. In the MIMO case, a G_a of McMillan degree m can only result when the decision variables in $a_m(\omega^2)$ and $B_m(\omega)$ lie in a proper algebraic variety.

For fixed γ, (13)–(15) is a convex feasibility problem in the decision variables $\{a_i\}$ and $\{b^{jk}_i\}$. This is because a_m and B_m are linear in the decision variables. Using Theorem 3 in the SISO case, we can thus replace (6)–(7) with the convex feasibility problem (13)–(15). This implies that the minimization problem (5)–(7) is *quasiconvex* [2, p.145], and that it can be solved by means of a bisection algorithm.

In the MIMO case, we can still replace (6)–(7) with (13)–(15), and we again obtain a quasiconvex minimization problem. However, we can now only guarantee that the solutions belong to $R_{mp}H_\infty$. More precisely, we can obtain those approximations $G_a \in R_{mp}H_\infty$ that can be written as $G_aG_a^* = B_m/a_m$. This includes all $G_a \in R_mH_\infty$.

The inequalities (13)–(15) consist of an infinite number of constraints because of the dependence on ω. If a state-space realization of $G(j\omega)$ is known, such as in a typical model reduction situation, this dependence can be removed using either the Kalman-Yakubovich-Popov (KYP) lemma [15] or a sum of squares decomposition (in SISO case) [13], [14]. If $G(j\omega)$ is only known on a grid $\{\omega_k\}$, such as in the typical system identification situation, (13)–(14) can be solved directly for the unknowns $\{a_i\}$ and $\{b_i^{jk}\}$ by an LMI solver. Notice, however, that we must also make sure that $a_m(\omega^2) > 0$ and $B_m(\omega) > 0$ for all ω to obtain $G_a, G_a^{-1} \in R_{mp}H_\infty$. This can be ensured by using the KYP lemma.

B. Relative H_∞ Error Bounds

In this section, we see that not only does (6) give simple bounds on the singular values, as shown in Theorem 1 and Corollary 2, but we can also obtain a priori bounds on the relative H_∞ norm error. As is discussed in [10], relative error bounds on plant models are more useful than additive bounds if the plant is used in closed-loop systems.

Theorem 4: Assume that $G, G_a \in RL_\infty$. The inequalities

$$1/\sqrt{1+\gamma} \le \sigma_i((G^{-1}G_a)(j\omega)) \le \sqrt{1+\gamma} \tag{16}$$

for all ω and $i = 1 \dots p$, are equivalent to

$$\begin{aligned} &\|(G^{-1}G_a)(G^{-1}G_a)^* - I\|_\infty \le \gamma \\ \text{and} \quad &\|(G_a^{-1}G)(G_a^{-1}G)^* - I\|_\infty \le \gamma. \end{aligned} \tag{17}$$

If furthermore $(G^{-1}G_a), (G_a^{-1}G) \in R_nH_\infty$ and $(G^{-1}G_a)(\infty) = I$, then (16) implies

$$\begin{aligned} &\|G^{-1}(G - G_a)\|_\infty \le 2n\gamma\sqrt{1+\gamma} \\ \text{and} \quad &\|G_a^{-1}(G - G_a)\|_\infty \le 2n\gamma\sqrt{1+\gamma}. \end{aligned} \tag{18}$$

The second part of Theorem 4 shows that if $G^{-1}G_a$ is rational, biproper, stable, and minimum phase, G_a can be proven to be a fine approximation of G even if we have only matched magnitude data in (16) and (17). This is because the phase can be reconstructed from the magnitude with the Hilbert transform, and the error propagation can be bounded for rational functions. Notice in particular that the bound depends on the model order n. This is unfortunate but necessary since the Hilbert transform is not a bounded mapping for infinite-dimensional systems. For more details and comments about the tightness of bounds of this sort, see [1].

III. Phase Approximation

In this section, we solve the phase approximation problem. We will approximate a transfer function G with another transfer function G_a (of smaller degree) such that the phases $\arg G(j\omega)$ and $\arg G_a(j\omega)$ are close. The solution is similar to the magnitude case, but the parametrization is more complicated here. In particular, we only consider SISO problems. The reason is the splintering of the definition of phase of a MIMO system. For example, the phase of a MIMO system can be defined via phase functions, see [1], [11], or via the eigenvalues of $G(j\omega)$, see [7]. How to construct the parametrization in Section III-A for the MIMO case is also less obvious.

The approximations that we obtain here will belong to a subset Q_m of the rational transfer functions. The set is defined as

$$Q_m = \{G(s) : G(s) = \frac{f(s)}{g(s)},\ f, g \text{ are Hurwitz polynomials}, \deg f(s) + \deg g(s) = m\}.$$

If $G_a \in Q_m$, it is minimum phase since the zeros are in the left complex half plane. Because we put restrictions on $\arg G_a(j\omega)$ in this section, we can often control the stability and McMillan degree of the approximants $G_a \in Q_m$ as well. See Lemma 6.

Before stating the optimization problem, we need some assumptions on the model G and the weights w_1, w_2.

Assumption 2: $G, w_1, w_2 \in RL_\infty$ and are scalar, G, w_1, w_2 have no zeros on the imaginary axis, $\arg w_1(j\omega) \le 0, \arg w_2(j\omega) \le 0$ for all ω, and (without loss of generality) $\arg G(0) = \arg w_1(0) = \arg w_2(0) = 0$.

The argument function is usually only defined modulo 2π. To obtain a well-defined argument function, we make the argument function continuous for all ω and fix the value of the argument at $\omega = 0$. This is always possible here since zeros and poles on the imaginary axis are not allowed. This is for convenience, though it is still possible otherwise, to avoid having to consider small indentations in the left or right complex half plane like in Nyquist D-contours.

Now the optimization problem can be formulated. Given a transfer function G and weights w_1, w_2 that satisfy Assumption 2, and an integer m, find ϕ and G_a such that

$$\min_{\phi, G_a} \phi, \quad \text{subject to} \tag{19}$$

$$\begin{aligned} -\phi + \arg w_2(j\omega) &< \arg G_a(j\omega) - \arg G(j\omega) \\ &< \phi - \arg w_1(j\omega), \end{aligned} \tag{20}$$

$$0 \le \phi < \frac{\pi}{2},\ 2\phi - \pi \le \arg w_1(j\omega) + \arg w_2(j\omega), \tag{21}$$

$$G_a \in Q_m, \tag{22}$$

for all ω. This problem is almost completely analogous to magnitude approximation (5)–(7). The constraint (20) has a clear interpretation in a Bode diagram. The constant ϕ plays here the role that γ did in magnitude approximation. If $\phi > 0$ it means that the phase bounds are relaxed, see Fig. 2. Hence, in the minimization problem we will find the approximation G_a that satisfies the tightest possible bounds. It will be shown in Section III-A that (19)–(22) becomes a quasiconvex minimization problem.

There are two major differences between magnitude and phase approximation. These are contained in (21). Firstly,

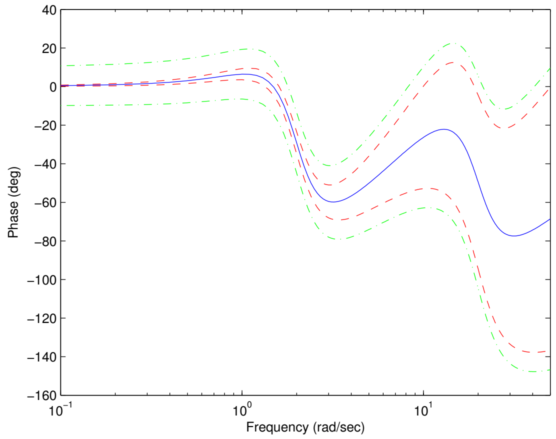

Fig. 2. The interpretation of the weights w_1, w_2, and the constant ϕ. The full line represents the model G, the dashed lines represent the bounds with $\phi = 0$, and the dash-dotted lines the relaxed bounds with $\phi = 10$ degrees. The argument of the approximation G_a should be within the bounds with the smallest possible ϕ.

there is an upper bound on the relaxation ϕ, and secondly, there is a constraint on the weights w_1, w_2 and ϕ. Hence, whereas we always find solutions in the magnitude case (for sufficiently large γ), in the phase case, there are problems that have no feasible solution. In case of an infeasible problem, one needs to increase the model order m, or modify the weights.

Remark 6: The assumption $2\phi - \pi \le \arg w_1(j\omega) + \arg w_2(j\omega) \Leftrightarrow (\phi - \arg w_1(j\omega)) + (\phi - \arg w_2(j\omega)) \le \pi$ means that the relaxed bounds in Fig. 2 are never allowed to be more than 180 degrees apart. The reason for this assumption is that we will use Cartesian coordinates to obtain a convex feasibility problem. The Cartesian coordinates simplify the parametrization and ensure that the arguments of G and G_a are always on the same branch. Of course, this also restrict us from going into different branches.

Remark 7: If the weights $\arg w_1$ and $\arg w_2$ are chosen to be zero in (19)–(22), a solution to (2) is obtained. It should also be clear why we assume $\arg w_1 \le 0$ and $\arg w_2 \le 0$, since otherwise with $\phi = 0$ a perfect approximation $\arg G(j\omega) = \arg G_a(j\omega)$ would not be allowed.

A. Parametrization and the Semidefinite Program

In this section, we show how (19)–(22) is translated into a semidefinite program. First we introduce the parametrization that is used to obtain G_a. The parametrization is less direct in this case, but it is shown in Theorem 5 that it is appropriate. Let m be a positive integer (which will turn out to be related to the order of G_a) and define

$$T_m(s) = 1 + \sum_{i=1}^{m} t_i s^i, \tag{23}$$

where $\{t_i\}$ are the m decision variables. The polynomial T_m is used to define the real functions $c_{m,1}(\omega), c_{m,2}(\omega), d_{m,1}(\omega), d_{m,2}(\omega)$ as

$$\begin{aligned}(w_1 G^* T_m)(j\omega) &=: c_{m,1}(\omega) + j d_{m,1}(\omega)\\ (w_2 T_m^* G)(j\omega) &=: c_{m,2}(\omega) + j d_{m,2}(\omega).\end{aligned} \tag{24}$$

Notice that these functions are linear in the decision variables $\{t_i\}$, since G, w_1, w_2 are fixed. These functions $c_{m,1}, c_{m,2}, d_{m,1}, d_{m,2}$ are used to find $\{t_i\}$, which are used to construct $T_m(s)$, which in turn gives $G_a(s)$. The following theorem is the phase approximation counterpart of Theorem 3.

Theorem 5: Assume that G, w_1, w_2 satisfy Assumption 2, that ϕ is fixed and satisfies (21), and m is a fixed positive integer. Then there is a $G_a \in Q_m$ that satisfies (20) if and only if there are $c_{m,1}, c_{m,2}, d_{m,1}, d_{m,2}$, defined by (24), that satisfy

$$\begin{aligned}&d_{m,1}(\omega) < (\tan\phi) c_{m,1}(\omega)\\ \text{and}\quad &d_{m,2}(\omega) < (\tan\phi) c_{m,2}(\omega)\end{aligned} \tag{25}$$

for all ω.

The following lemma may be used to to control the stability and McMillan degree of $G_a \in Q_m$.

Lemma 6: Assume that $G \in RH_\infty$ is minimum phase and has relative degree η. Assume furthermore that

$$|\arg G_a(j\omega) - \arg G(j\omega)| < \pi/2, \quad \text{as} \quad \omega \to \infty. \tag{26}$$

Then $G_a \in Q_m$ has relative degree η, $G_a \in R_{\frac{\eta+m}{2}} H_\infty$, and $m = \eta + 2n$, for some integer $n \ge 0$.

Notice that Lemma 6 says that it only makes sense to use m equal to the relative degree of G plus $0, 2, 4, 6, \ldots$ Furthermore, each time m is increased by two, the McMillan degree of the sought approximation G_a increases by one. One can easily enforce (26) by proper choice of the weight functions w_1 and w_2. If one does not prescribe bounds such that (26) holds, the obtained approximation G_a may have different relative degree than G has. This is usually undesirable.

Theorem 5 shows that for fixed ϕ and m, (20)–(22) is a convex feasibility problem, since (25) is linear in the unknowns $\{t_i\}$. Again, this implies that (19)–(22) is a quasiconvex minimization problem. In the model reduction situation, there are infinitely many constraints in (25) because of the dependence on ω. Just as in the magnitude case, the KYP lemma can be used to tackle this. An alternative, since we here only consider SISO transfer functions, is to use sum of squares decompositions (SOS) [13], [14]. In the system identification situation, there are only finitely many data points and we only need to solve a linear program.

B. Relative H_∞ Error Bounds

In this section, we state a result about how systems that have similar phases also can have a small relative H_∞ error. Just as in Section II-B, the result is based on bounds on the Hilbert transform operator restricted in its application to rational transfer functions. These are derived in [1].

The phase function is a way of representing the argument. We call Φ_G the phase function of G when

$$\Phi_G(j\omega) := \frac{G(j\omega)}{G(j\omega)^*} = \exp(2j\arg G(j\omega)).$$

Notice that phase functions are all-pass functions. We have the following counterpart to Theorem 4.

Theorem 7: Assume that $G, G_a \in RL_\infty$ have no zeros on the imaginary axis and that $\epsilon \in [0, 2)$. Then

$$-\arcsin\frac{\epsilon}{2} \le \arg G_a(j\omega) - \arg G(j\omega) \le \arcsin\frac{\epsilon}{2} \quad (27)$$

for all ω is equivalent to

$$\|\Phi_{G_a} - \Phi_G\|_\infty \le \epsilon. \quad (28)$$

If furthermore $(G_a^{-1}G)$,$(G^{-1}G_a) \in R_nH_\infty$, $(G_a^{-1}G)(\infty) = 1$, $\epsilon < 1/(2n)$, then (27) implies

$$\begin{aligned}\|G^{-1}(G - G_a)\|_\infty &\le 2n\epsilon/(1 - 2n\epsilon)\\ \|G_a^{-1}(G - G_a)\|_\infty &\le 2n\epsilon/(1 - 2n\epsilon).\end{aligned} \quad (29)$$

Hence, not only do we get a good match of the phase if the phase model reduction problem is solved, we may also get guarantees that the transfer functions G and G_a are close. This is of course provided that the minimum phase property $(G_a^{-1}G), (G^{-1}G_a) \in R_nH_\infty$ holds.

IV. Examples

MATLAB together with SeDuMi [18] and YALMIP [6] are used for the examples.

Example 1 (Model reduction—Magnitude): Using magnitude model reduction on the 35-th order model

$$G(s) = \prod_{k=1}^{35} \frac{s + 2k}{s + 2k - 1}$$

with $m = 2$ and $w_1 = w_2 = 1$ we obtain an approximation

$$G_a(s) = \frac{1.035s^2 + 42.09s + 138.2}{s^2 + 12.63s + 13.15}$$

with minimum relaxation $\gamma = 0.1$. Hence, according to Corollary 2, the magnitude curves will be no more than 0.38 dB apart for all ω. That the bound is tight can be seen in Fig. 3.

Example 2 (System identification — Phase): Here we use the phase system identification method to construct a low-pass filter that has linear phase for low frequencies. It is often desirable to have filters with linear phase, since these do not distort the signal (compare with a time delay). In discrete time, one can easily construct linear phase FIR filters. However, here we are concerned with continuous-time rational filters. We will therefore fit a rational transfer function $G_a(s)$ to frequency samples $G(j\omega_k)$ generated by a discrete-time linear phase FIR filter

$$G(j\omega) = \sum_{l=0}^{5} c_l e^{-l\pi j\omega}$$

with bandwidth 0.5 rad/s and $\{\omega_k\} = \{0, 0.01, \ldots, 0.7\}$. The coefficients $\{c_l\}$ are given by `fir1(5,0.5)` in MATLAB. We choose to look for G_a in Q_5 ($m = 5$). Then

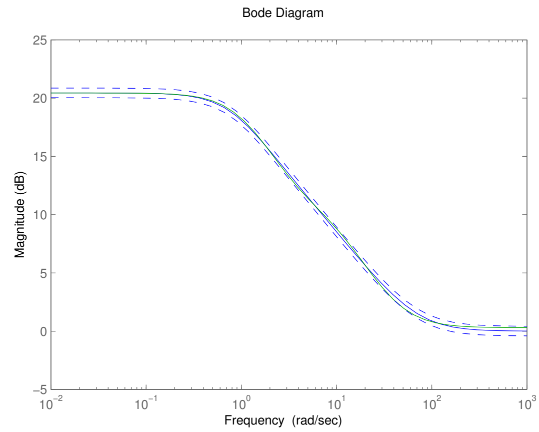

Fig. 3. The model G and the second-order approximation G_a obtained in Example 1. The uniform bounds are plotted with dashed lines. Here $\gamma = 0.1$, which gives an error bound of 0.38 dB.

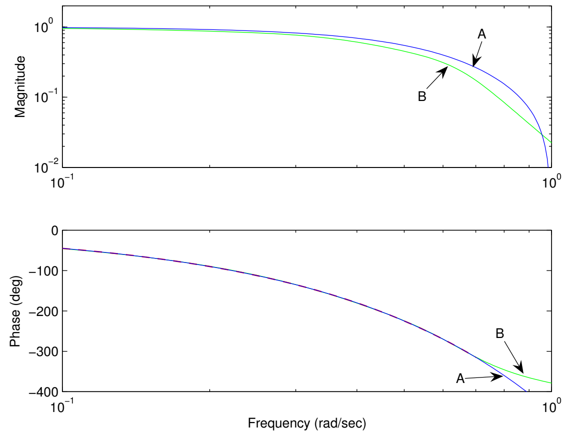

Fig. 4. The frequency data $G(j\omega_k)$ (A) and the 5th-order approximation G_a (B) from Example 2. The low-pass filter G_a, designed with the phase system identification method, has linear phase behavior up to frequency 0.7 rad/s.

the minimum relaxation becomes $\phi = 0.013$ rad, and G_a becomes

$$G_a(s) = \frac{0.01472}{s^5 + 1.011s^4 + 0.9101s^3 + 0.4047s^2 + 0.1193s + 0.01472}.$$

As can be seen in Fig. 4, the phase of the filter G_a is very close to the phase of G. The largest phase error on $\{\omega_k\}$ is 0.013 rad, and the bandwidth of G_a is approximately 0.5 rad/s.

V. Conclusions

We have shown how one can match transfer functions of given degree to magnitude and phase data using standard LMI solvers. We considered both the case where a continuum of data was available (model reduction case), and when only samples of data (system identification case) was available. In the magnitude case, we dealt with both SISO and MIMO systems, whereas in the phase case only

SISO systems were considered. We also showed that under the assumption that the systems were stable and minimum phase, systems that have bounded magnitude or phase error also satisfy a simple a priori bound on the relative H_∞ error. We illustrated the results in Bode diagrams, where the results are easily understood, and with two examples. The examples showed that the results are useful in, for example, model reduction and system identification.

An interesting topic for future research is how to combine the methods and match both magnitude and phase simultaneously. For this, methods from [12] could be useful. Furthermore, how to deal with phase approximation for MIMO systems is still an open issue. There are also many applications where the results could be useful. For example, in the design of band-pass linear phase filters.

REFERENCES

[1] B.D.O. Anderson and M. Green. Hilbert transform and gain/phase error bounds for rational functions. *IEEE Transactions on Circuits and Systems*, 35(5):528–535, May 1988.

[2] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, 2004.

[3] S.P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*. Society for Industrial and Applied Mathematics, 1997.

[4] G.E. Dullerud and F. Paganini. *A course in robust control theory — a convex approach*. Springer-Verlag, 2000.

[5] K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds. *International Journal of Control*, 39:1115–1193, 1984.

[6] J. Löfberg. YALMIP : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. Available from `http://control.ee.ethz.ch/~joloef/yalmip.php`.

[7] A.G.J. MacFarlane and I. Postlethwaite. The generalized Nyquist stability criterion and multivariable root loci. *International Journal of Control*, 25:81–127, 1977.

[8] A. Megretski. Optimal model order reduction for maximal real part norms. In Anders Rantzer and Christopher Byrnes, editors, *Directions in Mathematical Systems Theory and Optimization*, volume 286 of *Lecture Notes in Control and Information Sciences*, pages 217–226. Springer-Verlag, 2003.

[9] B.C. Moore. Principal component analysis in linear systems: controllability, observability, and model reduction. *IEEE Transactions on Automatic Control*, 26(1):17–32, February 1981.

[10] G. Obinata and B.D.O. Anderson. *Model Reduction for Control System Design*. Springer-Verlag, London, UK, 2001.

[11] P. Opdenacker and E.A. Jonckheere. State space approach to approximation by phase matching. In *Modelling, Identification and Robust Control*, pages 349–362, Stockholm, Sweden, June 1986.

[12] P.J. Parker, B.D.O. Anderson, and M. Araki. Construction of mixed Hilbert transforms. In *Proceedings of ISSPA 90, Signal Processing, Theories, Implementations and Applications*, pages 823–828, Gold Coast, Australia, August 1990.

[13] P.A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming*, 96(2):293–320, 2003.

[14] S. Prajna, A. Papachristodoulou, P. Seiler, and P. A. Parrilo. *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, 2004.

[15] A. Rantzer. On the Kalman-Yakubovich-Popov lemma. *Systems & and Control Letters*, 28(1):7–10, June 1996.

[16] H. Sandberg, A. Lanzon, and B.D.O. Anderson. Model approximation using magnitude and phase criteria: Implications for model reduction and system identification. *Submitted to International Journal of Robust and Nonlinear Control*, 2006.

[17] K.C Sou, A. Megretski, and L. Daniel. A quasi-convex optimization approach to parametrized model order reduction. In *Proceedings of Design Automation Conference*, Anaheim, CA, USA, June 2005.

[18] J.F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12:625–653, 1999. Available from `http://fewcal.kub.nl/sturm/software/sedumi.html`.