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Structured Model Order Reduction of Boiler-Header Models ⋆

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Abstract: This paper presents a model reduction of a boiler-header system. Since it is desirable that the reduced model retains the structure of the full model where the boilers are interconnected with the header, a structured model reduction technique is applied, which takes the entire system into account. This method requires the solution of two linear matrix inequalities to obtain the structured Gramians of the system, but in general it is not possible to guarantee feasibility of these linear matrix inequalities. However for stable systems that are connected in series with a negative feedback-loop with strictly positive real subsystems, we prove that solutions always exist. By showing that the boiler-header system belongs to this class of systems it follows that the structured model reduction method can be applied regardless of the system parameters.

Keywords: Model reduction; Boilers; Process models; Structural constraints; Convex optimization; Network topologies.

1. INTRODUCTION

Control applications in the industrial settings frequently involve parallel working units. Typical examples are a set of parallel boilers feeding steam to a common header (Fig. 1a), parallel working pumps, turbines, chemical reactors, etc. These parallel units are usually operated in multiple on/off configurations, where individual units are turned on/off according to process needs and optimal allocation schemes.

Design of Advanced Process Control (APC) – typically switched models Model Predictive Control (MPC) – has to consider this variability and requires a model of the full plant for each feasible on/off configuration. The number of configurations can be in the order of hundreds (plants with up to ten parallel boilers are not an exception). As the behavior of parallel units is mostly similar, it is common or even necessary (due to observability / controllability) to include them as a reduced order model (Fig. 1b). On the other hand it is favorable to retain as much physical interpretation as possible in the remaining part of the model for the ease of subsequent controller design, e.g. parameters tunning for Kalman filter, MPC, etc.

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One obvious solution is to find individual models of each parallel unit, combine them accordingly to the selected on/off configuration, apply balanced order reduction and combine the resulting low order model with the model of the remaining part of the process. However, this naive approach can give biased or even unstable result. The reason is that the reduction is done locally and not with respect to the full interconnected model. For this reason the naive approach is not recommended. Performing identification experiment for each feasible on/off configuration is possible only for a very small number of parallel units. Larger number of parallel units require a different approach.

Model reduction where various structural constraints are taken into account ("structured model reduction") has been considered in several papers. For example, in Enns (1984) frequency-weighted model reduction problems are considered, and in Anderson and Liu (1989) controller reduction is considered. More general interconnection structures have been considered in, for example, Li and Paganini (2005); Vandendorpe and Van Dooren (2007); Sandberg and Murray (2009). In this paper, we use a Linear Matrix Inequality (LMI) based approach, similar to the one presented in Sandberg and Murray (2009). This approach goes back to earlier work where generalized Gramians are used to reduce uncertain models, see Beck et al. (1996).

The main challenge in structured model reduction is to reduce models locally while ensuring a small global

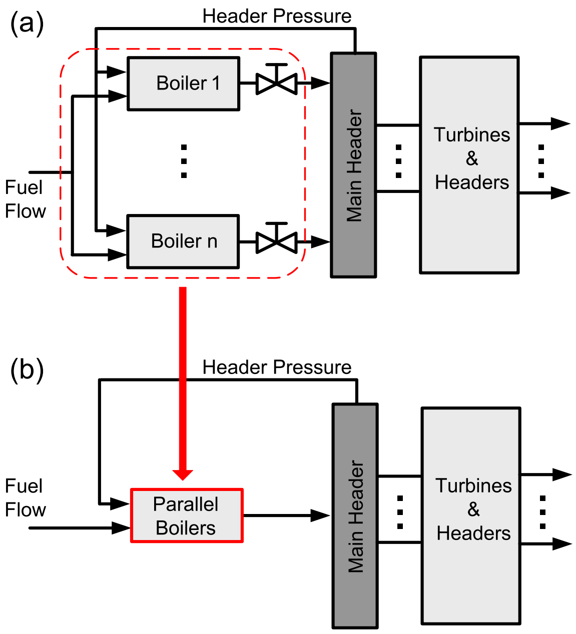

Fig. 1. Example of process with parallel working units: parallel working boilers feeding steam to common header. Notice closed-loop boilers operation by feedback from main header pressure (**a**). Model for APC design requires merging boiler models to low order model for every feasible on/off valve configuration (**b**).

model error. The advantage with the LMI-based approach is that it offers a simple a priori error bound on the global model error, much similar to the bound originally derived in Enns (1984); Glover (1984). The disadvantages are that the LMIs can be computationally expensive to solve, and they may also be infeasible. For the class of boiler-header models considered in this paper, however, we show the involved LMIs are always feasible. Hence, we prove that the structured model reduction problem can always be solved for these systems, and this is also the main contribution of the paper. The structured model order reduction is demonstrated on a set of boilers whose parameters are identified based on a high-fidelity model, see Řehoř and Havlena (2010).

The structure of the paper is as follows: In Section 2 the model of the boiler-header system is presented and the purpose of the paper is stated. It is followed by Section 3, giving a background of the theory used in the paper. In Section 4 the feasibility of the structured model reduction method for a particular set of systems to which the boiler-header belongs is proven and it ends with a demonstration of the method on a boiler-header system with identified parameters.

2. MODELING AND PROBLEM FORMULATION

A motivating industrial example of units working in parallel is a set of boilers feeding steam to a common header (Fig. 1). The header pressure is maintained by the boilers that typically are fed with the same fuel flow. Each boiler can with significant simplification be described by its internal boiler volume and hydrodynamic pipe resistance between the boiler and the common header. There is also dynamics from fuel flow to generated steam, which is "normalized" by a local combustion controller (to avoid oscillations / pushing among parallel units).

2.1 Boiler-Header Models

The linearized state space model of the i-th boiler is given by

$$\begin{aligned}
\mathbf{A}_i &= \begin{pmatrix} -1/T_i & 0 \\ 1/V_i & -K_i/V_i \end{pmatrix}, & \mathbf{B}_i &= \begin{pmatrix} K_s/T_i & 0 \\ 0 & K_i/V_i \end{pmatrix}, \\
\mathbf{C}_i &= (\, 0 \;\; K_i \,), & \mathbf{D}_i &= (\, 0 \;\; -K_i \,), \\
\mathbf{u}_i &= \begin{pmatrix} \mathrm{FF} \\ \mathrm{HP} \end{pmatrix}, & \mathbf{y}_i &= \mathrm{SF}_i,
\end{aligned} \tag{1}$$

where

FF	.....	Fuel Flow [t/hrs],
SF	......	Steam Flow [t/hrs],
HP	.....	Header Pressure [MPa],

and

V_i	......	i-th boiler pressure constant [kg/MPa/3.6],
K_i	......	i-th boiler pipe to header "conductivity" [kg/s/MPa/3.6],
K_s	......	units of steam from unit of fuel [−],
T_i	......	i-th boiler 1st order time constant for fuel flow to steam flow delay [s].

The header model is a simple integrator. It integrates the difference between steam inflow from the boilers and demand driven steam outflow to form the header pressure

$$\begin{aligned}
\mathbf{A}_H &= (0)\,, & \mathbf{B}_H &= (\, 1/V_H \;\; -1/V_H \,)\,, \\
\mathbf{C}_H &= (1)\,, & \mathbf{D}_H &= (\, 0 \;\; 0 \,)\,, \\
\mathbf{u} &= \begin{pmatrix} \mathrm{SF} \\ \mathrm{SD} \end{pmatrix}, & \mathbf{y} &= (\, \mathrm{HP} \,),
\end{aligned} \tag{2}$$

SD	.....	Steam Demand [t/hrs],
V_H	.....	Header pressure constant [kg/MPa/3.6].

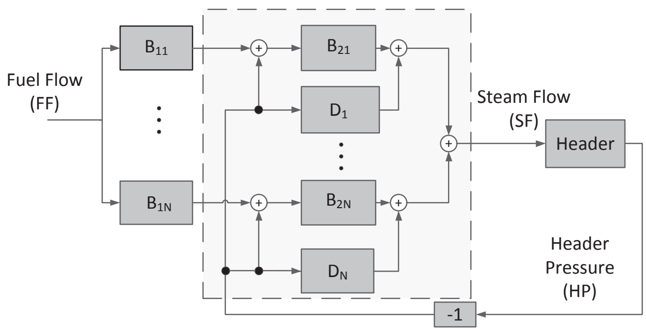

Fig. 2. The boiler-header system can be written as a negative feedback-loop in series with a parallel connection of stable systems.

The full plant model is composed of N boilers and a single header with the following "external" inputs and outputs

$$\mathbf{u} = \begin{pmatrix} \mathrm{FF} \\ \mathrm{SD} \end{pmatrix}, \; \mathbf{y} = \begin{pmatrix} \mathrm{HP} \\ \mathrm{SF} \end{pmatrix}.$$

It can be written on the form given by Fig. 2, which is interesting since that allows for structured model order reduction to be applied if the subsystems fulfill some conditions that are given in Section 4. From the state-space description in (1), the transfer function matrix of the i-th boiler can be calculated as

$$G_{b_i}(s) = \begin{pmatrix} G_{b_{1,i}}(s) & G_{b_{2,i}}(s) \end{pmatrix} = \begin{pmatrix} \frac{K_i K_{s_i}}{V_i T_i (s+\frac{1}{T_i})(s+\frac{K_i}{V_i})} & -\frac{K_i s}{s+\frac{K_i}{V_i}} \end{pmatrix}.$$

With N different boilers connected in parallel it follows that the steam flow to the header is given by the sum

$$\mathrm{SF}(s) = \sum_{i=1}^{N} G_{b_{1,i}}(s) \cdot \mathrm{FF}(s) + \sum_{i=1}^{N} G_{b_{2,i}}(s) \cdot \mathrm{HP}(s).$$

Turning to Fig. 2, it is seen that by choosing $B_{1i}(s) = \frac{-K_{s_i}}{K_i T_i (s+\frac{1}{T_i})}$, $B_{2i}(s) = \frac{-K_i^2}{V_i(s+\frac{K_i}{V_i})}$ and $D_i = K_i$ the block diagram in Fig. 2 is equivalent to the original state-space description since the steam flow can be written as

$$\begin{aligned}\mathrm{SF}(s) &= \sum_{i=1}^{N} (B_{1i}(s) \cdot B_{2i}(s)) \cdot \mathrm{FF}(s) \\ &\quad - \sum_{i=1}^{N} (B_{2i}(s) + D_i) \cdot \mathrm{HP}(s) \\ &= \sum_{i=1}^{N} \left(\frac{K_{s_i} K_i^2}{T_i K_i V_i \left(s + \frac{1}{T_i}\right)\left(s + \frac{K_i}{V_i}\right)} \right) \cdot \mathrm{FF}(s) \\ &\quad + \sum_{i=1}^{N} \left(\frac{K_i^2}{V_i \left(s + \frac{K_i}{V_i}\right)} - K_i \right) \cdot \mathrm{HP}(s) \\ &= \sum_{i=1}^{N} G_{b_{1,i}} \cdot \mathrm{FF}(s) + \sum_{i=1}^{N} G_{b_{2,i}} \cdot \mathrm{HP}(s)\end{aligned}$$

We want to have a state-space realization of the full boiler-header system with N boilers based on the block diagram in Fig. 2 to do structured model reduction. For the systems $B_{1i}(s)$ a natural state-space realization is

$$A_1^{(i)} = -\frac{1}{T_i},\ B_1^{(i)} = -\frac{K_{s_i}}{K_i T_i},\ C_1^{(i)} = 1,\ D_1^{(i)} = 0$$

and connecting them in parallel yields the 1-input, N-output system

$$\begin{aligned} A_1 &= \mathrm{diag}\left(A_1^{(1)},\ A_1^{(2)}, ..., A_1^{(N)}\right) \\ B_1 &= \left(B_1^{(1)},\ B_1^{(2)}, ...,\ B_1^{(N)}\right)^T \\ C_1 &= \mathrm{diag}\left(C_1^{(1)},\ C_1^{(2)}, ...,\ C_1^{(N)}\right) \\ D_1 &= \left(D_1^{(1)},\ D_1^{(2)}, ...,\ D_1^{(N)}\right)^T. \end{aligned}$$

For the systems $B_{2i}(s)+D_i$ the state-space realization can be written as

$$A_{21}^{(i)} = -\frac{K_i}{V_i},\ B_{21}^{(i)} = -\frac{K_i^2}{V_i},\ C_{21}^{(i)} = 1,\ D_{21}^{(i)} = K_i$$

and the parallel connection becomes a $N+1$-input, 1-output system. Defining

$$\bar{B}_{21} = \mathrm{diag}\left(B_{21}^{(1)}, B_{21}^{(2)}, ..., B_{21}^{(N)}\right),$$

$$\tilde{B}_{21} = \left(B_{21}^{(1)},\ B_{21}^{(2)}, ...,\ B_{21}^{(N)}\right)^T,\ \tilde{D}_{21} = \sum_{i=1}^{N} D_{21}^{(i)}$$

a state-space realization is

$$\begin{aligned} A_{21} &= \mathrm{diag}\left(A_{21}^{(1)},\ A_{21}^{(2)}, ..., A_{21}^{(N)}\right) \\ B_{21} &= \left(\bar{B}_{21},\ \tilde{B}_{21}\right) \\ C_{21} &= \left(C_{21}^{(1)},\ C_{21}^{(2)}, ...,\ C_{21}^{(N)}\right) \\ D_{21} &= \left(0_{1\times N}\ \tilde{D}_{21}\right). \end{aligned}$$

To perform structured model order reduction on the system we require that the closed-loop system is stable, see Sandberg and Murray (2009). This is not the case when the header is modeled as an integrator according to (2), but if the header pressure is fed back to the steam demand input with a constant gain K_H, the pole is moved to the left half-plane. This remedy is undone after the model reduction, which is possible since the header is not reduced. The feedback can be thought of as modeling a pressure dependent header load of the system, see Fig. 1. This gives the new state-space realization of the header

$$A_{22} = -\frac{K_H}{V_H},\ B_{22} = \frac{1}{V_H},\ C_{22} = 1,\ D_{22} = 0. \tag{3}$$

Combining the N boilers and the header, the full boiler-header system can be realized as

$$\begin{aligned} A &= \begin{pmatrix} A_1 & 0 & 0 \\ \bar{B}_{21} C_1 & A_{21} & -\tilde{B}_{21} C_{22} \\ 0 & B_{22} C_{21} & A_{22} - B_{22} \tilde{D}_{21} C_{22} \end{pmatrix} \\ B &= \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix},\ C = \begin{pmatrix} 0_{1\times N} & 0_{1\times N} & 1 \\ 0_{1\times N} & 1_{1\times N} & -\tilde{D}_{21} \end{pmatrix},\ D = 0, \end{aligned} \tag{4}$$

with the input $\mathbf{u} = \mathrm{FF}$ and the output $\mathbf{y} = (\mathrm{HP}\ \mathrm{SF})^T$.

2.2 Problem Formulation

The purpose of this paper is to present a model reduction of the boiler-header system with the constraint that the header is not subject to reduction. The transfer function matrix of the system is given by $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, using the state space realization (4) and the objective is to find the reduced system $\hat{G}$, where the header $(A_{22}, B_{22}, C_{22}, D_{22})$ has not been reduced, such that a small upper bound for the closed-loop model error $\|G - \hat{G}\|_\infty$ is attained.

3. STRUCTURED MODEL ORDER REDUCTION

Structured model order reduction is a model reduction technique that can be applied to systems composed of subsystems that are connected with some network dynamics. The idea is to reduce the subsystems while retaining the interconnection structure and keeping the global model error small. It is based on the idea of balancing the system. Two methods for this were proposed in Sandberg and Murray (2009). Method 1 uses the controllability and observability Gramians P and Q given by the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^TQ + QA + C^TC = 0,$$

similarly to what is done in Vandendorpe and Van Dooren (2007). It is helpful to use a partition with a block Q_{11}, P_{11} for the subsystem that should be reduced and another block Q_{22}, P_{22} for the header that is not reduced

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix},\ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}.$$

The method balances the system by the coordinate transformation $x_1 = T\bar{x}_1$ that makes the transformed Gramians $\bar{Q}_{11} = T^TQ_{11}T$ and $\bar{P}_{11} = T^{-1}P_{11}T^{-T}$ subsystem balanced, which means that

$$\bar{P}_{11} = \bar{Q}_{11} = \Sigma = \operatorname{diag}\{\sigma_1, ..., \sigma_n\},$$
$$\sigma_1 \geq ... \geq \sigma_n,\ \sigma_i = \sqrt{\lambda_i(P_{11}Q_{11})} = \sqrt{\lambda_i(\bar{P}_{11}\bar{Q}_{11})}.$$

This heuristic often works well, but it gives no a priori error bound. By imposing a block-diagonal structure on the Gramians P and Q, which is done with Method 2, this can be remedied. Instead of solving the Lyapunov equations, the following Linear Matrix Inequalities (LMIs) are formed

$$\min_{\substack{AP+PA^T+BB^T<0 \\ P=\operatorname{diag}\{P_{11},P_{22}\}}} \operatorname{trace} P \qquad \min_{\substack{A^TQ+QA+C^TC<0 \\ Q=\operatorname{diag}\{Q_{11},Q_{22}\}.}} \operatorname{trace} Q \quad . \tag{5}$$

The solutions to the LMIs are called *structured controllability and observability Gramians*. Just as for Method 1, a coordinate transformation of the system can be found by making the Gramians subsystem balanced.

Finally either truncation of the balanced states, which gives a good approximation at high frequencies or singular perturbation, which is good for low frequency approximation, is used to reduce the model order. Structured model reduction of networked systems using structured controllability and observability Gramians has the advantage of providing an a priori error bound, namely

$$\| G - \hat{G} \|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i, \tag{6}$$

where r is the order of the reduced interconnected system and by minimizing the trace in (5) an upper bound on the sum of the structured Hankel singular values is guaranteed. However it is not always so that the LMIs given by (5) are feasible. See Li and Paganini (2005); Vandendorpe and Van Dooren (2007); Sandberg and Murray (2009) for further details on structured model reduction.

4. EXISTENCE OF STRUCTURED GRAMIANS

4.1 Theory

Because the LMI (5) does not always admit solutions, it is interesting to identify special classes of systems for which feasibility can be guaranteed. A cascade connection of two stable systems is one such case. The dynamics matrix A for such a system can be written on lower triangular form, which motivates Proposition 4.1.

Proposition 4.1. Assume $A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are stable. Then there exists a structured controllability Gramian $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} > 0$ and a structured observability Gramian $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} > 0$ satisfying $AP + PA^T + BB^T < 0$ and $A^TQ + QA + C^TC < 0$ respectively.

Proof: The proof is only given for the structured controllability Gramian, since it is analogous for the structured observability Gramian. The Lyapunov equation gives the conditions

$$AP + PA^T + BB^T = \begin{pmatrix} A_{11}P_1 & 0 \\ A_{21}P_1 & A_{22}P_2 \end{pmatrix}$$
$$+ \begin{pmatrix} P_1A_{11}^T & P_1A_{21}^T \\ 0 & P_2A_{22}^T \end{pmatrix} + \begin{pmatrix} B_1B_1^T & B_1B_2^T \\ B_2B_1^T & B_2B_2^T \end{pmatrix} < 0 \Leftrightarrow$$

(i) $A_{11}P_1 + P_1A_{11}^T + B_1B_1^T < 0$
(ii) $A_{22}P_2 + P_2A_{22}^T + B_2B_2^T$
$- \left(A_{21}P_1 + B_2B_1^T\right)\left(A_{11}P_1 + P_1A_{11}^T + B_1B_1^T\right)^{-1}$
$\cdot \left(P_1A_{21}^T + B_1B_2^T\right) < 0$

(i) follows directly since there is always a $P_1 > 0$ solving $A_{11}P_1 + P_1A_{11}^T + B_1B_1^T < 0$ when A_{11} is stable.

Next try to find a P_2 solving (ii).

$$\left(A_{11}P_1 + P_1A_{11}^T + B_1B_1^T\right)^{-1} < 0 \Rightarrow$$
$$D \equiv \left(A_{21}P_1 + B_2B_1^T\right)\left(A_{11}P_1 + P_1A_{11}^T + B_1B_1^T\right)^{-1}$$
$$\cdot \left(P_1A_{21}^T + B_1B_2^T\right) \leq 0.$$

This means $-D \geq 0$ and thus $B_2B_2^T - D \geq 0$ which implies the existence of a $P_2 > 0$ satisfying (ii).

■

Another case which admits solutions to the LMIs given by (5) is when two strictly positive real systems are connected in negative feedback. Strictly positive real systems are of interest because of their importance in passivity theory, see for example Khalil (2002).

Proposition 4.2. The negative feedback interconnection of two strictly positive real (SPR) systems with dimensions n_1, n_2 and with minimal realizations (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) respectively admits a structured controllability Gramian $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ and a structured observability Gramian $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, where $P_1, Q_1 \in \mathbb{R}^{n_1 \times n_1}$ and $P_2, Q_2 \in \mathbb{R}^{n_2 \times n_2}$.

Proof: The proof is only given for the structured controllability Gramian. The Kalman-Yakubovich-Popov lemma can be used to prove the proposition. It states that for strictly positive real systems of minimal realization there exists matrices $\bar{P}_i = \bar{P}_i^T > 0$, L_i, W_i and positive constants ϵ_i with $i \in \{1, 2\}$ such that

$$\begin{aligned} \bar{P}_i A_i + A_i^T \bar{P}_i &= -L_i^T L_i - \epsilon_i \bar{P}_i \\ \bar{P}_i B_i &= C_i^T - L_i^T W_i \\ W_i^T W_i &= D_i + D_i^T. \end{aligned}$$

Using this, we can define the storage functions $V_i(x) = x_i^T \bar{P}_i x_i$ that satisfy $\dot{V}_i \leq 2u_i^T y_i - \epsilon_i x_i^T \bar{P}_i x_i$, which means that the systems are strictly passive. By connecting these two systems in negative feedback it follows that the feedback system is asymptotically stable with the Lyapunov function $V(x) = V_1(x_1) + V_2(x_2)$ satisfying $\dot{V}(x) \leq -\epsilon_1 x_1^T \bar{P}_1 x_1 - \epsilon_2 x_2^T \bar{P}_2 x_2$. The time derivative of the Lyapunov function can also be expanded using the state-space realizations as $\dot{V}(x) = x^T \left(\bar{P}\bar{A} + \bar{A}^T \bar{P}\right) x$, where $\bar{A} = \begin{pmatrix} A_1 - B_1 L D_2 C_1 & -B_1 L C_2 \\ B_2 M C_1 & A_2 - B_2 M D_1 C_2 \end{pmatrix}$ is the dynamics matrix of the feedback system with $L = (I + D_2 D_1)^{-1}$, $M = (I + D_1 D_2)^{-1}$ and $\bar{P} = \begin{pmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{pmatrix}$. This implies that $\bar{P}\bar{A} + \bar{A}^T \bar{P} \leq \begin{pmatrix} -\epsilon_1 \bar{P}_1 & 0 \\ 0 & -\epsilon_2 \bar{P}_2 \end{pmatrix}$ which we will now use to prove the existence of a structured controllability Gramian for the feedback system.

The negative feedback system is given by $\bar{A}$ and $\bar{B} = \left(B_1^T \; 0\right)^T$, which means that we want to show that there exists a $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ satisfying $P\bar{A} + \bar{A}^T P + \bar{B}\bar{B}^T < 0$

We can verify that the LMI holds if we let P_1 and P_2 be scaled versions of the positive definite matrices $\bar{P}_1$ and $\bar{P}_2$ given by the KYP-lemma since,

$$\bar{P}\bar{A} + \bar{A}^T \bar{P} \leq \begin{pmatrix} -\epsilon_1 \bar{P}_1 & 0 \\ 0 & -\epsilon_2 \bar{P}_2 \end{pmatrix} \leq -\eta_1 I < 0 \qquad (7)$$

for some η_1 belonging to
$0 < \eta_1 < |\max\left\{\lambda_i(-\epsilon_1 \bar{P}_1),\, \lambda_i(-\epsilon_2 \bar{P}_2)\right\}|$. Since there exists an η_2, $0 \leq \lambda_{max}\left(\bar{B}\bar{B}^T\right) \leq \eta_2$, such that $0 \leq \bar{B}\bar{B}^T \leq \eta_2 I$, it follows that

$$\begin{aligned} &\frac{\eta_2 + 1}{\eta_1}\left(\bar{P}\bar{A} + \bar{A}^T \bar{P}\right) + \bar{B}\bar{B}^T \leq \\ &\leq -(\eta_2 + 1)I + \bar{B}\bar{B}^T \leq -(\eta_2 + 1)I + \eta_2 I < 0 \end{aligned}$$

and thus we have found a $P = \frac{(\eta_2+1)}{\eta_1}\begin{pmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{pmatrix}$, which is a structured controllability Gramian for the negative feedback system.

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For SPR systems in a negative feedback loop connected in series with a stable system Proposition 4.1 and Proposition 4.2 can be combined to give the following Proposition.

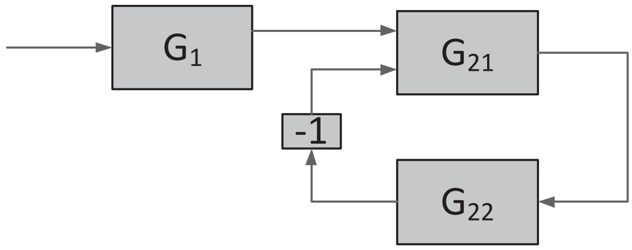

Fig. 3. Block diagram of two SPR systems G_{21} and G_{22} connected with negative feedback in series with a stable system G_1.

Proposition 4.3. A stable LTI system of dimension n_1 connected in series with two strictly positive real systems of order n_{21} and n_{22} connected with negative feedback (Fig. 3) admits a structured controllability Gramian $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ and a structured observability Gramian $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, where $P_1, Q_1 \in \mathbb{R}^{(n_1+n_{21})\times(n_1+n_{21})}$ and $P_2, Q_2 \in \mathbb{R}^{n_{22}\times n_{22}}$.

Proof: The proof is only given for the structured controllability Gramian. The system will have a state-space representation of the form $A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, where index 1 is associated with system 1 and index 2 is associated with the feedback-loop.

With the same reasoning as in Proposition 4.1 we want to show that there exists matrices P_1 and P_2 satisfying

(i) $A_{11}P_1 + P_1 A_{11}^T + B_1 B_1^T < 0$
(ii) $A_{22}P_2 + P_2 A_{22}^T + B_2 B_2^T$
$- \left(A_{21}P_1 + B_2 B_1^T\right)\left(A_{11}P_1 + P_1 A_{11}^T + B_1 B_1^T\right)^{-1}$
$\cdot \left(P_1 A_{21}^T + B_1 B_2^T\right) < 0$

It is always possible to find a $P_1 > 0$ solving (i), since A_{11} is assumed to be stable. To find a $P_2 > 0$ solving (ii), begin by noting that there exists a block diagonal $\bar{P}_2 = \begin{pmatrix} \bar{P}_{21} & 0 \\ 0 & \bar{P}_{22} \end{pmatrix} > 0$ satisfying $A_{22}\bar{P}_2 + \bar{P}_2 A_{22}^T < 0$ according to Proposition 4.2. We know that $B_2 B_2^T - \left(A_{21}P_1 + B_2 B_1^T\right)\left(A_{11}P_1 + P_1 A_{11}^T + B_1 B_1^T\right)^{-1} \cdot \left(P_1 A_{21}^T + B_1 B_2^T\right) \geq 0$. This means that we can rescale $\bar{P}_2$ with an $\eta > 0$ such that $P_2 = \eta \bar{P}_2$ solves (ii) by using the same reasoning as in Proposition 4.2. Thus we have found a structured controllability Gramian $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & P_{21} \end{pmatrix} & 0 \\ 0 & P_{22} \end{pmatrix}$ with dimensions $P_1 \in \mathbb{R}^{(n_1+n_{21})\times(n_1+n_{21})}$ and $P_2 \in \mathbb{R}^{n_{22}\times n_{22}}$. ■

4.2 The Boiler-Header System

In Section 2 it was shown that the boiler-header plant can be written as a subsystem connected in series with a negative feedback loop, i.e. it has the same structure as in Proposition 4.3. The parallel connection of the subsystems $B_{1i}(s)$ corresponds to system G_1 in Proposition 4.3, the parallel connection of the subsystems $B_{2i}(s) + D_i$ corresponds to system G_{21} and the header corresponds to system G_{22}. We now want to show that the conditions for Proposition 4.3 are fulfilled for the plant. Beginning with the header it follows from the state-space description in (3) that its transfer function is $G_H(s) = \frac{1}{V_H(s+K_H/V_H)}$, i.e. it is SPR.

Further if the transfer functions $B_{2i}(s)+D_i$ were SPR, the parallel connection of them would also be SPR, since the parallel system would be stable, $\sum_i G_i(i\omega) + G_i^T(-i\omega) > 0\ , \forall\omega$ if it was true for each individual subsystem and $\sum_i G_i(\infty) + G_i^T(\infty) > 0$ since $D_i > 0$. And further since the transfer functions $B_{1i}(s)$ are stable, it would follow that all conditions for Proposition 4.3 are fulfilled which proves the existence of a structured controllability and observability Gramian P and Q respectively for the boiler-header system. However it turns out that the subsystems $B_{2i}(s) + D_i$ are not SPR, but only output strictly passive. Even so, Proposition 4.4 shows that the conclusion still holds.

Proposition 4.4. A negative feedback of a parallel connection of n subsystems $B_{2i}(s) + D_i = \frac{K_i s}{s+\frac{K_i}{V_i}}$, denoted as subsystem 1, and the SPR header, denoted as subsystem 2, admits structured controllability and observability Gramians P and Q.

Proof: The structure of the proof is the same as that of Proposition 4.2 with the difference that subsystem 1 is output strictly passive instead of SPR. For the parallel system it holds that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}x_1^T P_1 x_1 + \sum_{i=1}^{n} \frac{1}{K_i}\left(y_1^{(i)}\right)^2 \le u_1^T y_1 \tag{8}$$

where $P_1 = \mathrm{diag}\left(\frac{V_1}{K_1^2}, \frac{V_2}{K_2^2}, ..., \frac{V_n}{K_n^2}\right)$ and $y_1^{(i)}$ is the output of subsystem $B_{2i}(s) + D_i$. For the SPR header

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}x_2^T P_2 x_2 + \frac{1}{2}\epsilon_2 x_2^T P_2 x_2 \le u_2^T y_2, \tag{9}$$

where P_2 is given by the KYP-lemma. Given that subsystem $B_{2i}(s)+D_i = \left[\begin{array}{c|c} A_1^i & B_1^i \\ \hline C_1^i & D_1^i \end{array}\right]$ and introducing the notation

$$\bar{C}_1 = \begin{pmatrix} C_1^1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & C_1^n \end{pmatrix}, \bar{D}_1 = \begin{pmatrix} D_1^1 \\ \vdots \\ D_1^n \end{pmatrix}, \bar{K} = \begin{pmatrix} \frac{1}{K_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{K_n} \end{pmatrix}$$

it follows that by summing (8) and (9)

$$\bar{A}^T \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} + \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \bar{A}$$
$$\le -2 \begin{pmatrix} \bar{C}_1^T \bar{K} \bar{C}_1 & \bar{C}_1^T \bar{K} \bar{D}_1 C_2 \\ C_2^T \bar{D}_1^T \bar{K} \bar{C}_1 & \frac{1}{2}\epsilon_2 P_2 + C_2^T \bar{D}_1^T \bar{K} \bar{D}_1 C_2 \end{pmatrix} < 0,$$

where $\bar{A} = \begin{pmatrix} A_1 & -B_1 C_2 \\ B_2 C_1 & A_2 - B_2 D_1 C_2 \end{pmatrix}$.

To arrive at the first inequality it can be noted that $u_1^T y_1 = -u_2^T y_2$ and the last inequality can be derived by looking at the Schur complement. The rest of the proof is identical to the last part of the proof of Proposition 4.2 beginning at (7). ■

4.3 Algorithm Demonstration

The algorithm will be demonstrated on a system of 3 parallel boilers feeding a single header. It will be demonstrated on a model identified from simulated data in order to have a "correct model" for comparison. To maintain the relation with practical problems the models of the individual boilers and the header were obtained as follows. Data sets for identification were obtained by step-testing a Simulink model with high-fidelity boiler models. The obtained data was burdened by additive noise with the same spectrum as the noise estimated from industrial data. The models were identified by a state-of-the-art grey-box identification algorithm based on iterative re-optimization of prediction and simulation criterion mixture, see Řehoř and Havlena (2010). The identified parameters are:

i	1	2	3
V_i	61129	54412	50645
K_i	202.57	271.29	313.39
T_i	108.48	101.22	99.92
K_s	4.59	4.58	4.57
V_h	7000		

In Fig. 4 and 5, the step response and frequency response of the original full system is compared with the reduced systems of order 3 obtained by applying structured model order reduction and locally balanced model order reduction respectively. The LMIs (5) were solved using Yalmip and SeDuMi, see Löfberg (2004); Sturm (1999). It is clear that the local model reduction which naively balances the boilers without taking the header into account is outperformed by the structured reduction which bases the reduction on the behavior of the full system. Calculating the H_∞-norm of the model error $\|G-\hat{G}\|_\infty$ when the steam demand is set to be proportional to the header pressure, it is found to be 0.3146 and the upper bound (6) becomes 0.6274, which is indeed relatively tight.

5. CONCLUSION

This paper has shown that structured model order reduction can always be applied to a class of systems to which the boiler-header system belongs. It has also demonstrated that the proposed method outperforms balanced model reduction that is done locally on the boilers and therefore ignores the header interconnection.

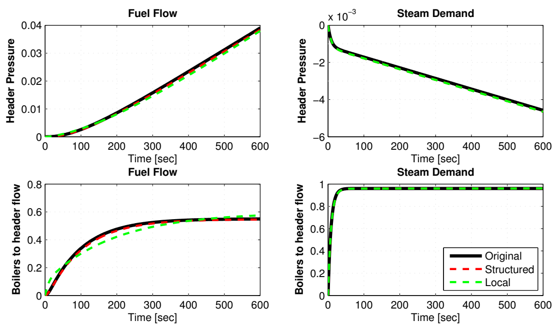

Fig. 4. Step response comparison.

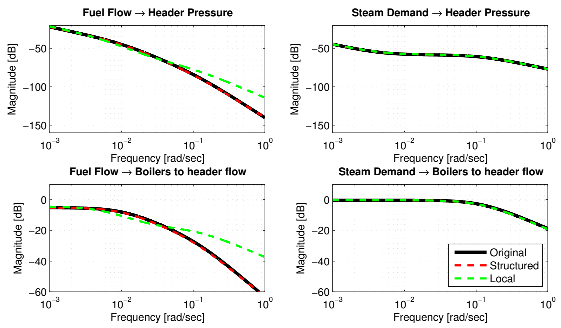

Fig. 5. Frequency response comparison.

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