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On Floquet-Fourier Realizations of Linear Time-Periodic Impulse Responses

Henrik Sandberg

***Abstract*— We show that a linear time-periodic system with a smooth impulse response can be arbitrarily well approximated by a linear time-periodic state-space representation in so-called Floquet-Fourier form. The Floquet-Fourier form has a constant state matrix and the input and output matrices have only finitely many nonzero Fourier coefficients. Such representations are easier to use for analysis and control design than the impulse response or fully general state-space forms. The construction of the Floquet-Fourier form is done using methods for model approximation of infinite-dimensional linear time-invariant systems. We also propose a method for constructing minimal realizations from impulse responses of a special structure.**

I. Introduction

Linear time-periodic (LTP) systems have been studied for a long time in the control engineering community, see, for example, [1], and the survey in [2]. In this paper, we look at the problem of constructing approximate realizations of LTP systems given in impulse-response form, using results from [3]. This problem falls into the intersection of *model reduction* and *system identification*. Finite-dimensional realizations greatly simplify both analysis and control synthesis, see, for example, [4], [5] for norm and gap computation, and [6], [7] for control design. Algebraic methods involving the Fourier coefficients of the realizations have also obtained attention, see, for example, [8]–[11]. Hence, once the methods described in this paper have been applied to a system, a wide range of tools are available for further analysis and control design.

A related problem was addressed in circuit theory in the 1960's. See, for example, [12]–[14]. There time-varying filters, so-called N-path filters, were studied. In these papers, mainly the exact realization problem of LTP systems of certain structures were addressed. In this paper, the approximate realization problem is in focus. Also, different mathematical tools are used here. However, there is some overlap in Section IV and this is commented there.

We consider LTP systems given in impulse-response form

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} g(t,\tau)u(\tau)\,d\tau + D(t)u(t), \\ g(t,\tau) &= g(t+T,\tau+T), \quad D(t) = D(t+T), \end{aligned} \tag{1}$$

where g is the impulse response, D the direct term, and T the period. We assume that the system is *uniformly exponentially stable*, meaning that there are positive constants κ_1 and κ_2 such that

$$|g(t,\tau)| \le \kappa_1 \cdot e^{-\kappa_2(t-\tau)}, \quad t > \tau, \tag{2}$$

and $\|D(t)\|_\infty < \infty$. Furthermore, the impulse response should be *causal*,

$$g(t,\tau) = 0, \quad \tau > t.$$

For simplicity, we only deal with single-input–single-output (SISO) systems in this paper. We often identify (1) with a bounded (on L_2) linear operator $G : u \mapsto y$.

If the LTP system (1) has a state-space realization, then there are T-periodic matrices such that

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}(t)\bar{x} + \bar{B}(t)u \\ y &= \bar{C}(t)\bar{x} + D(t)u, \quad \bar{x} \in \mathbb{R}^n, \end{aligned} \tag{3}$$

and $g(t,\tau) = \bar{C}(t)\Phi_{\bar{A}}(t,\tau)\bar{B}(\tau)$, see [15]. If $\bar{A}(t)$ is bounded, (3) can be transformed into Floquet form, see [15], using a T-periodic coordinate transformation $x = T(t)\bar{x}$. The Floquet form has the structure

$$\begin{aligned} \dot{x} &= Qx + B(t)u \\ y &= C(t)x + D(t)u, \quad x \in \mathbb{R}^n, \end{aligned} \tag{4}$$

where Q is constant, and $B(t), C(t), D(t)$ are T-periodic matrices. $Q, B(t), C(t)$ are not necessarily real even if $\bar{A}(t), \bar{B}(t), \bar{C}(t)$ are real. The computation of the coordinate transformation $T(t)$ may be hard in practice. Typically numerical integration of the equations is required.

The first contribution of this paper, presented in Section III, is to show how we directly can approximate G, given by (1), with an LTP system F in Floquet state-space form (4), where $B(t), C(t), D(t)$ are expressed with (finite) Fourier sums

$$\begin{aligned} B(t) &= \sum_k B_k e^{jk\omega_0 t} \\ C(t) &= \sum_k C_k e^{jk\omega_0 t} \\ D(t) &= \sum_k D_k e^{jk\omega_0 t} \end{aligned}$$

where $\omega_0 = 2\pi/T$. We call a realization in this form a *Floquet-Fourier realization*. The second contribution of this paper, presented in Section IV, is to show how minimal Floquet-Fourier realizations can be constructed from impulse-responses of a special structure. This is done by generalizing Gilbert's realization method for linear time-invariant (LTI) systems.

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H. Sandberg is with California Institute of Technology, Control and Dynamical Systems, M/C 107-81, Pasadena, CA 91125, USA. henriks@cds.caltech.edu

Notation

The set $\mathcal{C}^L(\Omega)$ contains the functions whose (partial) derivatives up to and including order L are continuous in the open set Ω. L_p, $p \geq 1$, is the space of p-integrable functions over $(-\infty, \infty)$ and $\|\cdot\|_p$ denotes its standard norm. $H_\infty(\Omega)$ contains the functions that are analytic and bounded in Ω, and $\|\cdot\|_\infty$ denotes the supremum of the magnitude of the functions over Ω. $\|\cdot\|$ is the induced L_2-norm of an operator. $\mathcal{L}$ is the Laplace transformation operator, and $\hat{g}(s)$ is the Laplace transform of $g(t)$. For short, derivatives are denoted by $g_t^{(q)} := \frac{\partial^q}{\partial t^q} g$, and if g has only one argument we simply write $g^{(q)}$. $\mathbb{R}$ are the real numbers, $\mathbb{Z}$ the integers, $\mathbb{C}$ the complex numbers, and $\mathbb{C}_+$ the complex numbers with strictly positive real part.

II. Preliminaries

In the following, it is assumed that $D(t) = 0$ in G given by (1). This is not a serious restriction, since if $D(t) \neq 0$ in G, one can simply copy it into the resulting approximation F afterwards. Notice, however, that even if $D(t) = 0$ in (1), D_k may still be nonzero in the approximation F we derive in Section III.

We consider systems G that have smooth impulse responses g. More precisely, they should belong to the set $\mathcal{C}_e^L$.

Definition 1 (The set $\mathcal{C}_e^L$ [3]): A causal time-periodic impulse response g belongs to the set $\mathcal{C}_e^L$ if

(E1) $g(t,\tau)$ belongs to $\mathcal{C}^L(\Omega)$, where $\Omega = \{(t,\tau) : t > \tau\}$;

(E2) The limits

$$g_x^{(q)}(t,t) := \lim_{\Omega \ni (\nu,\xi) \to (t,t)} g_x^{(q)}(\nu, \xi)$$

exist for all t, where $x = t$ or τ, and $q = 0 \dots L$;

(E3) $g(t,\tau)$ and all its partial derivatives up to and including order L have uniform exponential decay (2).

If we assume that $g \in \mathcal{C}_e^L$, then the integrals

$$g_k(t) = \frac{1}{T} \int_0^T e^{-jk\omega_0 \xi} g(\xi, \xi - t) d\xi \qquad (5)$$

are well defined for all $k \in \mathbb{Z}$ and t. The functions $g_k(t)$ have the following properties:

Proposition 1: Assume that the causal LTP impulse response g belongs to $\mathcal{C}_e^L$. Then,

(i) $g_k(t) = 0, \quad t < 0$;

(ii) $g_k(t) \in \mathcal{C}^L(0, \infty)$;

(iii) $|g_k^{(q)}(t)| \leq \kappa_1 e^{-\kappa_2 t}, \quad q = 0 \dots L$;

(iv) $\hat{g}_k(s) \in H_\infty(\mathbb{C}_+)$;

(v) $\hat{g}_k(j\omega) \in \mathcal{C}^\infty(j\mathbb{R})$.

Proof: Some of the proofs use standard results from Fourier analysis that can be found in, for example, [16]. (i): Follows from (5), since $g(\xi, \xi - t) = 0$ when $t < 0$ (g is causal). (ii): $g(t,\tau) \in \mathcal{C}^L(\Omega)$, and we are allowed to differentiate under the integral sign in (5). (iii): Essentially shown in [3]. Use (E3). (iv): $g_k(t)$ is causal from (i) and we have that $\|\hat{g}_k(s)\|_\infty \leq \|g_k(t)\|_1$, which is finite due to (iii). (v): We have $\hat{g}_k^{(q)}(j\omega) = \mathcal{L}[(-t)^q g_k(t)](j\omega)$. Because of (iii), we have that $\|(-t)^q g_k(t)\|_1 < \infty$, and (uniform) continuity of $\hat{g}_k^{(q)}(j\omega)$ follows for any $q \geq 0$. ∎

From an impulse response $g \in \mathcal{C}_e^L$, we can construct an LTP system $G_{[K]}$ with impulse response

$$G_{[K]}: \quad g_{[K]}(t,\tau) = \sum_{k=-K}^{K} g_k(t-\tau) e^{jk\omega_0 t}. \qquad (6)$$

Notice that this is just a finite sum of output-modulated LTI impulse responses $g_k(t)$. A similar expansion but with input modulation can also be derived, see [3]. If all $g_k(t)$ can be realized exactly with finite-dimensional state-space representations, then (6) can be realized exactly in Floquet-Fourier form. However, we do not assume here that $g_k(t)$ have finite-dimensional realizations.

In [3], it is shown that $G_{[K]}$ converges to G as $K \to \infty$ in induced L_p-norms. A formal statement is given in the following proposition.

Proposition 2 ([3]): Assume that the impulse response g of G belongs to $\mathcal{C}_e^L$ and $L > 1$. Then the difference between G and $G_{[K]}$ with impulse response $g_{[K]}$ is bounded by

$$\|G - G_{[K]}\| \leq \frac{C_1}{K^{L-1}},$$

for a G- and L-dependent constant C_1.

Remark 1: The approximation $G_{[K]}$ corresponds to a skew truncation of the harmonic transfer function of G, see, for example [3], [11]. The N-path filters in [12], [14] are parallel connections of N input- and output-modulated LTI systems. Thus they have similarities with $G_{[K]}$.

Remark 2: Since we measure the quality of approximation in an induced norm, we cannot approximate unstable systems in this framework. If there is a stable/anti-stable decomposition of a model, one could apply the methods to each term separately, by reversing time for the anti-stable term.

III. Floquet-Fourier approximations

In the previous section, we saw that a smooth stable LTP impulse response $g(t,\tau)$ can be arbitrarily well approximated by a finite sum of modulated LTI impulse responses $g_k(t)$. The purpose of this section is to show that the possibly infinite-dimensional transfer functions $\hat{g}_k(s)$ can be approximated arbitrarily well in H_∞ by finite-dimensional transfer functions $\hat{f}_k(s)$. Since this is the case, G itself can be approximated arbitrarily well in induced L_2-norm by an LTP system F in Floquet-Fourier form.

Remark 3 (Real g): When $g(t,\tau)$ is real, then $g_{-k}(t) = \overline{g_k(t)}$. Then it is only necessary to use the following approximation techniques for $g_0(t), \dots, g_K(t)$, since the others are obtained from the conjugates.

We will use a bilinear transformation that is a bijective map between the closed complex right half-plane $s \in \overline{\mathbb{C}_+} \cup \{\infty\}$ and $z \in (\mathbb{C} \setminus \mathbb{D}) \cup \{\infty\}$, where $\mathbb{D} = \{z : |z| < 1\}$ is the open unit disc,

$$z = \frac{s+\alpha}{\alpha - s}, \quad s = \alpha \frac{z-1}{z+1},$$

for all $\alpha > 0$. In particular, the unit circle in the z-plane is mapped to the imaginary axis in the s-plane.

Remark 4: The bilinear map is introduced to be able to do a Fourier expansion on the unit circle. It has nothing to do with sampling here. Since the map is a bijection, nothing is lost in the transformation. The choice of the parameter α is discussed below.

Let us define

$$G_k(z) := \hat{g}_k\left(\alpha\frac{z-1}{z+1}\right).$$

Since $\hat{g}_k(s) \in H_\infty(\mathbb{C}_+)$, we have that $G_k(z) \in H_\infty(\mathbb{C}\setminus\bar{\mathbb{D}})$, and $G_k(z)$ can be expanded in a Laurent series

$$G_k(z) = \sum_{l=0}^{\infty} G_{k,l} z^{-l}, \quad |z| > 1, \tag{7}$$

where

$$G_{k,l} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_k(e^{j\theta})e^{jl\theta}d\theta. \tag{8}$$

The integrals and functions are well defined due to Proposition 1(iv)–(v). In particular, $G_k(e^{j\theta})$ is well defined and bounded for all θ. To compute $G_{k,l}$, the integration in (8) can be done *on* the unit circle because $G_k(z) \in H_\infty(\mathbb{C}\setminus\bar{\mathbb{D}})$. See, for example, Theorem 13.11 in [17]. The series (7) converges at least in L_2-sense on the unit circle $|z| = 1$. On the unit circle the series becomes a Fourier series.

We denote a truncated series by

$$G_k^n(z) := \sum_{l=0}^{n} G_{k,l} z^{-l}. \tag{9}$$

Truncated Laurent series have been suggested in [18], [19] for the approximation of infinite-dimensional and large-scale LTI systems. Since we have decomposed the LTP system into a sum of LTI systems, we can use the same approach here. In [19], it is shown that $G_k^n(z)$ converges uniformly to $G_k(z)$ for $|z| = 1$, provided $G_k(e^{j\theta})$ is smooth enough. We have the following lemma:

Lemma 1 ([19]): For $q = 1, 2, \dots$ it holds that

$$\|G_k(z) - G_k^n(z)\|_\infty^2 \le \frac{n^{1-2q}}{2\pi(2q-1)}\int_{-\pi}^{\pi}\left|\frac{d^q}{d\theta^q}G_k(e^{j\theta})\right|^2 d\theta.$$

The more derivatives of $G_k(e^{j\theta})$ that are bounded, the faster asymptotic convergence rate can we thus expect for the truncated series. We will here only use Lemma 1 in the case $q = 1$. The next lemma show boundedness of the first derivative.

Lemma 2: Assume that $g \in \mathcal{C}_e^2$ ($L = 2$). Then there is a G-dependent constant C_2 such that

$$\left|\frac{d}{d\theta}G_k(e^{j\theta})\right| \le C_2,$$

for all θ and $k \in \mathbb{Z}$.

Proof: Using the chain rule we obtain

$$\frac{d}{d\theta}G_k(e^{j\theta}) = \frac{d}{d\theta}\hat{g}_k\left(\alpha\frac{e^{j\theta}-1}{e^{j\theta}+1}\right) = \frac{2\alpha j e^{j\theta}}{(e^{j\theta}+1)^2}\hat{g}_k'\left(\alpha\frac{e^{j\theta}-1}{e^{j\theta}+1}\right). \tag{10}$$

First we bound $\hat{g}_k'$. Basic Fourier transform identities give

$$\hat{g}_k'(j\omega) = \int_0^\infty (-t)g_k(t)e^{-j\omega t}dt,$$
$$(j\omega)^2\hat{g}_k'(j\omega) = \int_0^\infty e^{-j\omega t}\left(\frac{d^2}{dt^2}(-t)g_k(t)\right)dt.$$

Using Proposition 1(ii)–(iii) with $L = 2$, we have

$$|\hat{g}_k'(j\omega)| \le \|(-t)g_k(t)\|_1 \le M_1 < \infty$$
$$\omega^2|\hat{g}_k'(j\omega)| \le \|\frac{d^2}{dt^2}(-t)g_k(t)\|_1 \le M_2 < \infty,$$

for all ω and k. Hence,

$$|\hat{g}_k'(j\omega)| \le \frac{M_1+M_2}{1+\omega^2} \tag{11}$$

for all ω and k. Returning to (10) and using (11), we have

$$\left|\frac{d}{d\theta}G_k(e^{j\theta})\right| \le \frac{2\alpha}{|e^{j\theta}+1|^2}\frac{M_1+M_2}{1+\frac{\alpha^2|e^{j\theta}-1|^2}{|e^{j\theta}+1|^2}} = \frac{2\alpha(M_1+M_2)}{|e^{j\theta}+1|^2+\alpha^2|e^{j\theta}-1|^2}$$

which is clearly bounded for all θ and we can pick its maximum value as C_2. ∎

Remark 5: One can prove boundedness of higher derivatives of $G_k(e^{j\theta})$ by using $L > 2$. Since we are mostly interested in an existence proof, and not in the best possible convergence bound, we do not use higher derivatives here.

We are now ready to state the first theorem of the paper.

Theorem 1: Assume that the causal LTP system G has an impulse response $g \in C_e^2$, and fix an $\alpha > 0$. Then for all (fixed) $\epsilon > 0$ there are integers n and K such that $\|G-F\| < \epsilon$ where F has impulse response

$$f(t,\tau) = \sum_{k=-K}^{K} e^{jk\omega_0 t}f_k(t-\tau)$$

and f_k have transfer functions of McMillan degree less or equal to n given by

$$\hat{f}_k(s) = \sum_{l=0}^{n} G_{k,l}\cdot\left(\frac{\alpha-s}{s+\alpha}\right)^l, \tag{12}$$

where $G_{k,l}$ is defined in (8).

Proof: We have that

$$\|G-F\| \le \|G-G_{[K]}\| + \|G_{[K]}-F\|.$$

First we fix a K such that $\|G-G_{[K]}\| \le \epsilon/2$ using Proposition 2. It is clear that a $K \ge 2C_1/\epsilon$ is sufficient.

Next we fix n such that $\|G_{[K]}-F\| \le \epsilon/2$. This is done by approximating $\hat{g}_k(s)$, for $k = -K\dots K$ with truncated Laurent series (9). From Lemmas 1 and 2 we have

$$\|\hat{g}_k(s) - \hat{f}_k(s)\|_\infty \le \frac{C_2}{\sqrt{n}},$$

where $\hat{f}_k(s)$ is defined in (12). Let us now choose an n such that $C_2/\sqrt{n} \leq \epsilon/(2(2K+1))$. Using the triangle inequality we then have

$$\|G_{[K]} - F\| \leq \sum_{k=-K}^{K} \|\hat{g}_k(s) - \hat{f}_k(s)\|_\infty \leq \frac{\epsilon}{2}.$$

This concludes the proof. ■

Remark 6 (To measure $\hat{g}_k(j\omega)$): To compute $G_{k,l}$, it is necessary to know $\hat{g}_k(j\omega)$. The functions $\hat{g}_k(j\omega)$ can be obtained from a Fourier transform of (5). However, they can also be directly obtained from experiments since the steady-state response to harmonics $e^{j\omega t}$ is

$$y(t) = Ge^{j\omega t} = \left(\sum_{k=-\infty}^{\infty} \hat{g}_k(j\omega)e^{jk\omega_0 t}\right) e^{j\omega t},$$

as shown in [1], [8], [10]. (See [3] for a derivation using the current formalism.)

The transfer functions $\hat{f}_k(s)$ in (12) all have poles at $-\alpha$, and can be written in the form

$$\hat{f}_k(s) = \frac{c_{k,n-1}s^{n-1} + \ldots + c_{k,0}}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + d_k,$$

for some (in general complex) numbers $c_{k,l}$, d_k, and

$$a_k = \alpha^{n-k} \begin{pmatrix} n \\ k \end{pmatrix}.$$

It is now easy to construct a Floquet-Fourier realization of F.

Corollary 1: A Floquet-Fourier realization of F is given by

$$Q = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \ldots & -a_1 & -a_0 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & & 1 & 0 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C_k = \begin{pmatrix} c_{k,n-1} & c_{k,n-2} & \ldots & c_{k,0} \end{pmatrix},$$

$$D_k = d_k,$$

for $k = -K \ldots K$. The realization has state-space dimension n and is controllable, but is not necessarily minimal.

Proof: Use the controllable canonical form to realize $\hat{f}_k(s)$. Since $f(t,\tau)$ has modulation on the output, we can just multiply the outputs of the realizations of $\hat{f}_k(s)$ with $e^{jk\omega_0 t}$ and add them. The input-to-state dynamics is time invariant and identical for all realizations of $\hat{f}_k(s)$, and is realized with Q and B_0 above ($B_k = 0, k \neq 0$). Since the realization is in controllable canonical form, the controllability follows. ■

We have reasons to expect that the state dimension n in Corollary 1 needs to be large in general. The reason is that all the poles are located in $-\alpha$, and hence many states should be needed to approximate dynamics with poles far from $-\alpha$. Following the recommendations given in [18], α should be chosen to be around the bandwidth of the LTI system, for fast convergence of the truncated series. But since we here have $2K+1$ different LTI systems $\hat{g}_k(s)$ with possibly different bandwidths, such a choice may not be possible.

Of course, there is nothing that prevent us from choosing different α in the approximation of different $\hat{g}_k(s)$. Then the number of terms in (12) depends on k, call it n_k, and it is generally no longer possible to use the same states to realize all $\hat{f}_k(s)$. In the worst case, we end up with a Floquet-Fourier realization with $\sum_{k=-K}^{K} n_k$ states.

In [19], [20], a two-step approximation technique is suggested. First a large number of terms n is used in the truncated series. Then balanced truncation is applied to bring down the order of the realization. We can use this method here as well, and apply standard balanced truncation separately to each $\hat{f}_k(s)$. Generically, the poles will then move away from $-\alpha$.

In the next section, we show how a minimal Floquet-Fourier realization can be constructed for $\hat{f}_k(s)$ with different pole locations.

IV. A MINIMAL FLOQUET-FOURIER REALIZATION

It is instructive to look at the structure of $g_k(t)$ for a system in Floquet-Fourier form (4). We then have [3]

$$g(t,\tau) = \sum_k \left(\sum_l C_{k-l}e^{(Q-jIl\omega_0)\cdot(t-\tau)}B_l\right) e^{jk\omega_0 t},$$

and

$$g_k(t) = \sum_l C_{k-l}e^{(Q-jIl\omega_0)t}B_l, \tag{13}$$

where the indices k and l run over (finite) intervals that take all nonzero Fourier coefficients of $B(t)$ and $C(t)$ into account. Next we assume that Q can be diagonalized and has the form

$$Q = \text{diag}\{\lambda_1 I_{q_1}, \ldots, \lambda_m I_{q_m}\}, \tag{14}$$

where $q_1 + \ldots + q_m = n$ and $\lambda_1, \ldots, \lambda_m$ are distinct.

Remark 7: The assumption about diagonalizability of Q simplifies matters next. It can be relaxed by considering Jordan canonical forms. Notice that for a minimal realization of a (SISO) LTI system, $q_i = 1$. This need not be true in the LTP case.

Using the diagonal form of Q, $g_k(t)$ can be expressed as

$$g_k(t) = \sum_{i=1}^{m} \sum_l \gamma_{k,l}^{[i]} e^{(\lambda_i - jl\omega_0)t}, \tag{15}$$

for some complex numbers $\gamma_{k,l}^{[i]}$. We notice that if there are eigenvalues λ_i and λ_o that satisfy $\lambda_i = \lambda_o + pj\omega_0$ for some integer p, the realization may not be minimal. It is seen from the structure of (13) and (15) that the mode λ_o can be described by the mode λ_i by changing its modulation. Hence, we make the following definition.

Definition 2: The modes $\lambda_1, \ldots, \lambda_m$ in an impulse response (15) are only considered distinct if there are no integers i, o, p, $i \neq o$, such that $\lambda_i = \lambda_o + pj\omega_0$.

This definition should not be surprising. It is well known that the characteristic exponents of an LTP system are only unique modulo $j\omega_0$, see for example, [15].

Next, we consider the realization problem (the converse problem): Given a set of LTI impulse responses in the form (all sums are finite)

$$g_k(t) = \sum_l \gamma_{k,l} e^{\lambda_l t}, \tag{16}$$

construct a minimal Floquet-Fourier realization of

$$g(t,\tau) = \sum_k g_k(t-\tau) e^{jk\omega_0 t}. \tag{17}$$

The first step is to eliminate “unnecessary” modes in (16) and pick out the smallest number of m distinct modes among λ_l. Then (16) can be written in the form (15).

Next, we introduce the notation

$$\begin{aligned} B_k^T &= \begin{pmatrix} (B_k^{[1]})^T & \dots & (B_k^{[m]})^T \end{pmatrix} \\ C_k &= \begin{pmatrix} C_k^{[1]} & \dots & C_k^{[m]} \end{pmatrix} \end{aligned} \tag{18}$$

where $B_k^{[i]}$ and $C_k^{[i]}$ correspond to the distinct mode λ_i and whose sizes are to be determined. By equating (13) and (15), we see that $B_k^{[i]}$ and $C_k^{[i]}$ should satisfy

$$\gamma_{k,l}^{[i]} = C_{k-l}^{[i]} B_l^{[i]}, \tag{19}$$

assuming we have chosen m distinct modes and a Q in the form (14). We can arrange (19) into the matrix equation

$$C^{[i]}B^{[i]} = \begin{pmatrix} \vdots \\ C_1^{[i]} \\ C_0^{[i]} \\ C_{-1}^{[i]} \\ \vdots \end{pmatrix} \begin{pmatrix} \dots & B_1^{[i]} & B_0^{[i]} & B_{-1}^{[i]} & \dots \end{pmatrix}$$

$$= \begin{pmatrix} \ddots & & & & \\ & \gamma_{2,1}^{[i]} & \gamma_{1,0}^{[i]} & \gamma_{0,-1}^{[i]} & \\ & \gamma_{1,1}^{[i]} & \gamma_{0,0}^{[i]} & \gamma_{-1,-1}^{[i]} & \\ & \gamma_{0,1}^{[i]} & \gamma_{-1,0}^{[i]} & \gamma_{-2,-1}^{[i]} & \\ & & & & \ddots \end{pmatrix} =: \Gamma^{[i]} \in \mathbb{C}^{M\times N}, \tag{20}$$

where the size of $\Gamma^{[i]}$ is the smallest possible to include all nonzero elements $\gamma_{k,l}^{[i]}$. From the above expression, it follows that up to M harmonics are needed in the output modulation, and up to N harmonics are needed in the input modulation to realize the mode λ_i. The elements $C_k^{[i]}$ and $B_k^{[i]}$ can be found by means of a singular value decomposition (SVD) or QR factorization of $\Gamma^{[i]}$. The realization procedure is summarized in the following algorithm, and can be seen as a generalization of *Gilbert's realization* of LTI systems, see, for example [21].

Algorithm 1: Input: $g_k(t)$ in the the form (16), and period T. Output: Floquet-Fourier realization of (17).

1. Pick the smallest number m of distinct modes in (16) and put $g_k(t)$ into the form (15). Determine $\gamma_{k,l}^{[i]}$.
2. For $i = 1\dots m$: construct the smallest possible $\Gamma^{[i]}$ to include all the nonzero $\gamma_{k,l}^{[i]}$, and factorize it into full-rank factors $B^{[i]}$ and $C^{[i]}$ as in (20). Put $q_i = \operatorname{rank}\Gamma^{[i]}$.
3. Construct Q, B_k, and C_k as in (14) and (18).

Theorem 2: The Floquet-Fourier realizations from Algorithm 1 are minimal.

Proof: We prove the theorem by contradiction: Assume there is a (minimal) realization of (16)–(17) with $\tilde{n} < n$ states. Without loss of generality we can assume the minimal realization has a Floquet-Fourier form $(\tilde{Q}, \tilde{B}_k, \tilde{C}_k)$. This is because we have shown there exist Floquet-Fourier realizations of (16)–(17), and removing uncontrollable and unobservable states are just a matter of time-invariant coordinate projections.

$\tilde{Q}$ can always be put in Jordan canonical form. Since Jordan blocks with α $'1'$s on the superdiagonal may lead to terms $t^\alpha e^{\lambda_i t}$ in the expansion of $g_k(t)$, but no such terms are present by assumption, we can without loss of generality assume that $\tilde{Q}$ is diagonal.

Assume that $\tilde{Q}$ has $\tilde{m}$ distinct eigenvalues $\tilde{\lambda}_i$. For $(\tilde{Q}, \tilde{B}_k, \tilde{C}_k)$ to be able to realize $g_k(t)$, we must have $\tilde{m} = m$, because of the choice of m in Algorithm 1. Furthermore, we can always put $\tilde{\lambda}_i = \lambda_i$ by re-arranging the states and/or changing the modulation $\tilde{B}_k$ and $\tilde{C}_k$.

Because $(\tilde{Q}, \tilde{B}_k, \tilde{C}_k)$ is minimal, $\tilde{B}^{[i]}$ and $\tilde{C}^{[i]}$ must be full rank. Otherwise, more states can be removed. Then $\tilde{n} = \sum_{i=1}^{\tilde{m}} \operatorname{rank}\tilde{B}^{[i]} = \sum_{i=1}^{\tilde{m}} \operatorname{rank}\tilde{C}^{[i]}$. Because $\tilde{\lambda}_i = \lambda_i$, we have $\tilde{C}^{[i]}\tilde{B}^{[i]} = \Gamma^{[i]}$. Then $\operatorname{rank}\tilde{B}^{[i]} = \operatorname{rank}\tilde{C}^{[i]} = q_i$, and $\tilde{n} = n$. This gives the contradiction. ∎

Example 1: Consider the LTI impulse responses

$$\begin{aligned} g_1(t) &= e^{(-1+j)t} \\ g_0(t) &= e^{-t} \\ g_{-1}(t) &= e^{(-1-j)t} \end{aligned}$$

and let us use Algorithm 1 to find a minimal realization of $g(t,\tau) = \sum_{k=-1}^{1} e^{jkt} g_k(t-\tau)$, where $\omega_0 = 1$. We have $m = 1$ and pick $\lambda_1 = -1$. Then $\gamma_{1,-1}^{[1]} = 1$, $\gamma_{0,0}^{[1]} = 1$, and $\gamma_{-1,1}^{[1]} = 1$. Since

$$\Gamma^{[1]} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

has rank 3, we put $q_1 = 3$. A Floquet-Fourier realization can now be obtained by factorizing $\Gamma^{[1]}$. One such realization is

$$\begin{aligned} C_2 &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} & B_1 &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T \\ C_0 &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} & B_0 &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T \\ C_{-2} &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} & B_{-1} &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \\ Q &= \operatorname{diag}\{-1,-1,-1\}, \end{aligned}$$

and it can be verified to be minimal.

Example 2: Consider the same problem as in Example 1 but with

$$g_1(t) = e^{(-1-j)t}$$
$$g_0(t) = e^{-t}$$
$$g_{-1}(t) = e^{(-1+j)t}.$$

Again we have $m = 1$ and pick $\lambda_1 = -1$. This gives

$$\Gamma^{[1]} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

Since $\Gamma^{[1]}$ has rank 1, we have $q_1 = 1$, and a minimal realization is given by

$$C_0 = 1 \qquad B_1 = 1 \qquad B_0 = 1 \qquad B_{-1} = 1$$
$$Q = -1.$$

Hence, a seemingly small change in the modulation can give rise to a large change in the degree of the minimal realization.

Remark 8: In [14], a realization method similar to the one presented here in Section IV is derived. The framework is different though, because in [14] the parametric transfer function formalism introduced in [1] is used. Another difference is that we here prove that the method generates minimal realizations.

V. Conclusion

We have studied so-called Floquet-Fourier realizations of LTP systems. These are realizations with constant state matrix Q and where $B(t)$, $C(t)$, and $D(t)$ only have a finite number of nonzero Fourier coefficients. These realizations are convenient both to store and to use for analysis and control design. The first contribution of the paper was to show that LTP systems with sufficiently smooth impulse responses can be arbitrarily well approximated by LTP systems in Floquet-Fourier form. The proof was constructive and gave a method to build such realizations via approximation of infinite-dimensional LTI transfer functions. The realizations were not necessarily minimal, and the second contribution of the paper was to show how minimal Floquet-Fourier realizations can be constructed from a certain class of impulse responses. More general classes of impulse responses can be considered by using the Jordan canonical form, but details were not given here.

Future work could include design of efficient algorithms that directly realizes a set of measured $\hat{g}_k(j\omega)$ with a minimal Floquet-Fourier form.

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