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Implementation Costs and Information Flow in Kalman-Bucy Filters

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Joint work with
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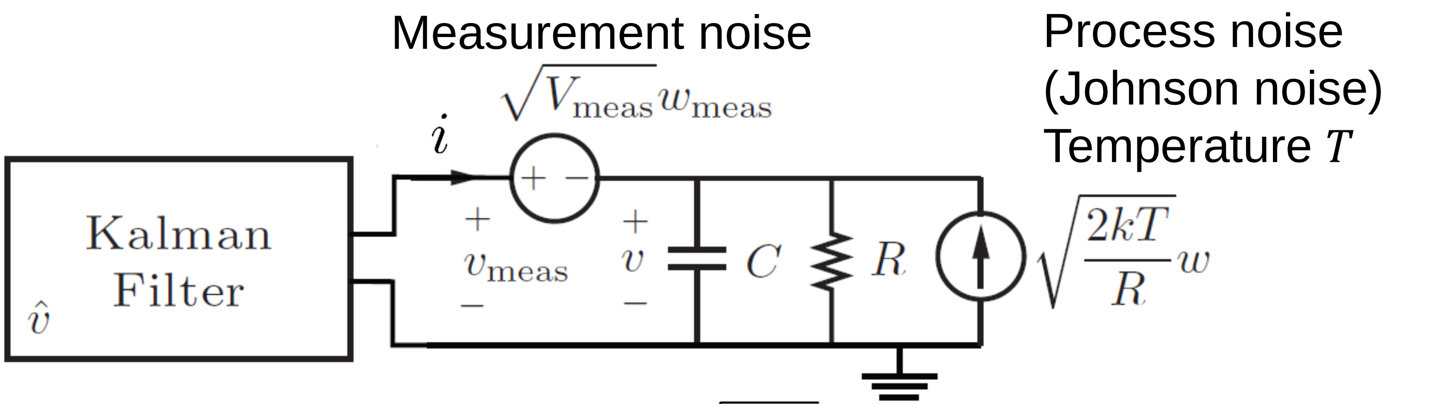
Motivating Example and Problem Formulation

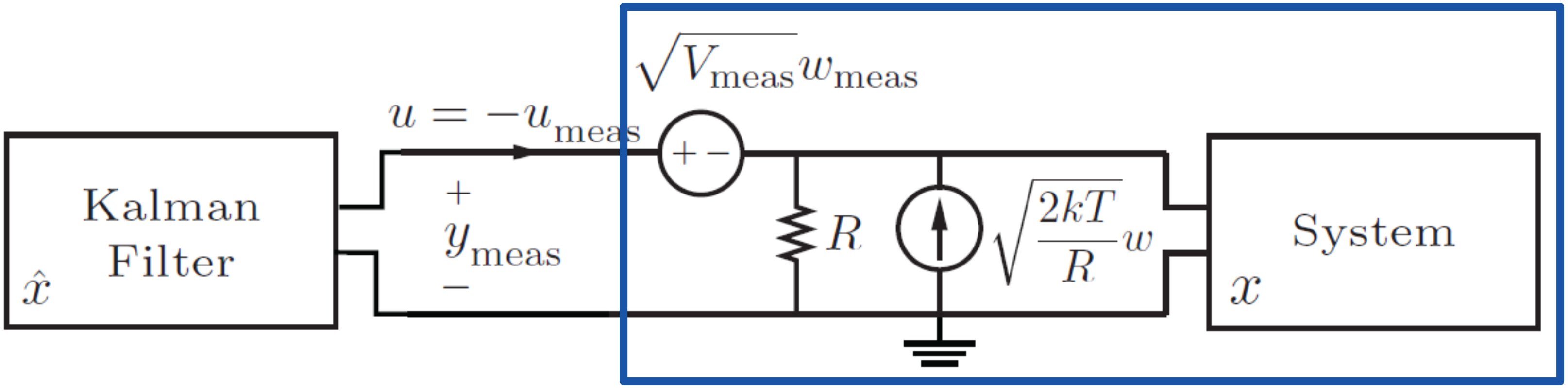


$$\dot{v} = -\frac{1}{RC}v + \frac{1}{R}i + \sqrt{\frac{2kT}{R}}w$$

$$v_{\text{meas}} = v + \sqrt{V_{\text{meas}}}w_{\text{meas}}$$

1. Is there a lower bound on external power supply to a physical implementation of the filter?
2. What is a simple cheap exact physical implementation of Kalman-Bucy filter?

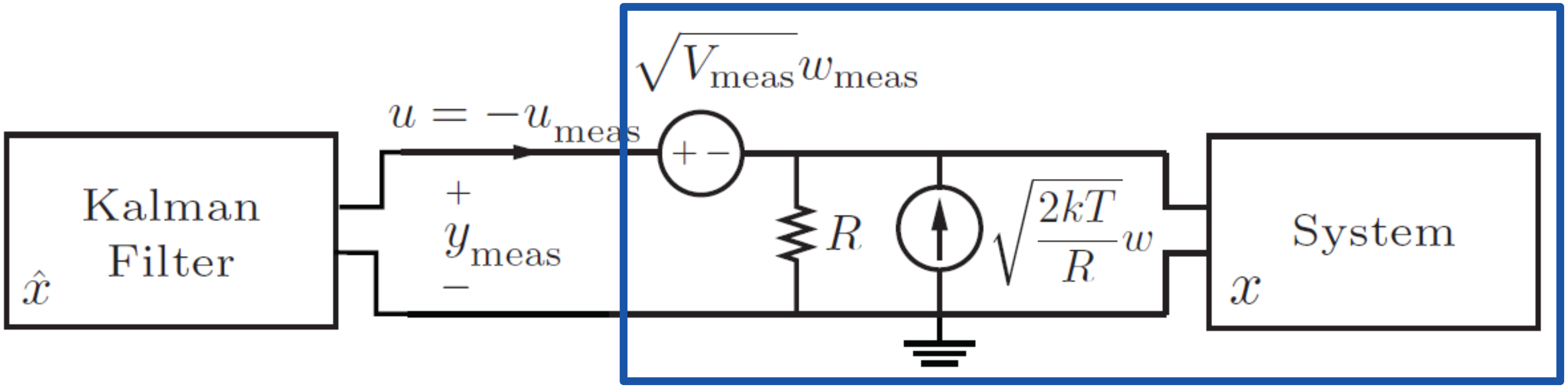
More General Class of Systems to Measure



Linear passive system

$$\dot{x} = (J - GBB^T)Mx + Bu + B\sqrt{2kTG}w$$
$$y = B^T Mx$$
$$y_{\text{meas}} = B^T Mx + \sqrt{V_{\text{meas}}}w_{\text{meas}}$$
$$(J = -J^T, \quad G := 1/R, \quad M = M^T > 0)$$

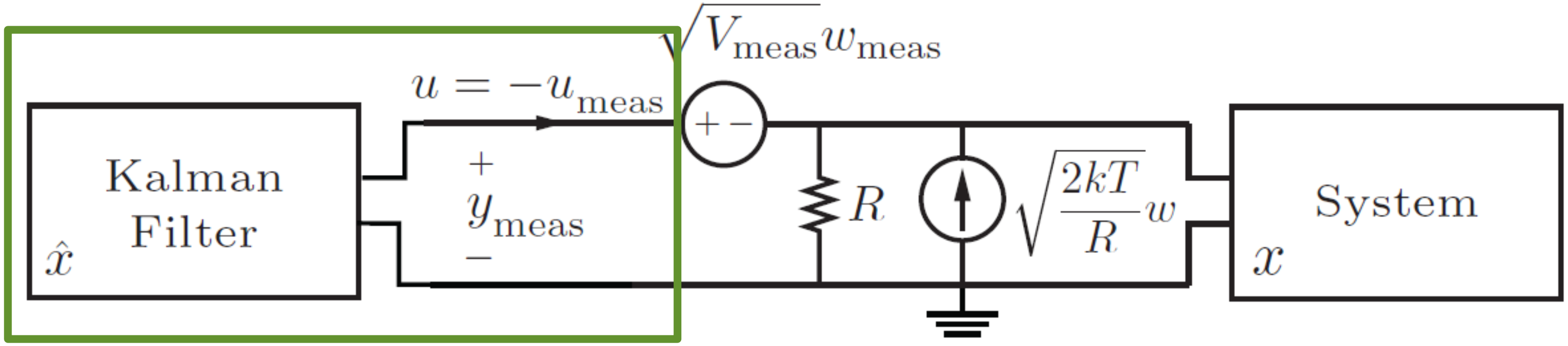
More General Class of Systems to Measure



Signal-to-noise ratio (SNR):

$$\frac{(\text{process noise [V]})^2}{(\text{measurement noise [V]})^2} \equiv \frac{2kT}{GV_{\text{meas}}} =: \sigma$$

Kalman-Bucy Filter



$$\frac{d}{dt}\hat{x} = (J - GBB^T)M\hat{x} + Bu + K(y_{\text{meas}} - B^T M\hat{x})$$

Lemma: Kalman gain is

$$K = (\sqrt{1+\sigma} - 1)GB \equiv g_K B$$

Kalman-Bucy Filter is Passive

Assumption: Admit linear back action current

$$\underbrace{u_{\text{meas}}}_{\text{current}} = -u = g\,\underbrace{B^T M\hat{x}}_{\text{voltage}}, \quad g = \text{gain} \in (0,\infty), \text{ free parameter}$$

Theorem: A realization of Kalman-Bucy filter is

$$\frac{d}{dt}\hat{x}_s = (J - ZB_sB_s^T)M\hat{x}_s + B_s y_{\text{meas}}$$

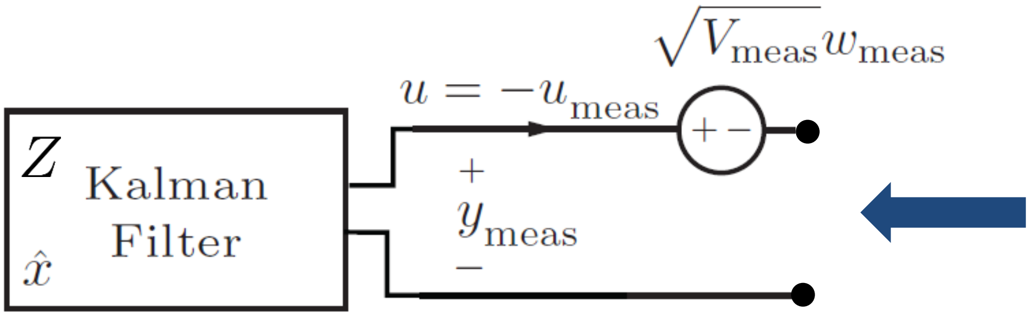
$$u_{\text{meas}} = B_s^T M\hat{x}_s$$

$$\hat{x}_s = \sqrt{g/g_K}\,\hat{x}$$

with effective resistance

$$Z = \frac{\sqrt{\sigma+1}}{(\sqrt{\sigma+1}-1)g} + \frac{1}{(\sqrt{\sigma+1}-1)G} > 0$$

Temperature of Kalman-Bucy Filter



Fluctuation-dissipation theorem: $2kT_{\text{KF}}Z = V_{\text{meas}}$

Effective filter temperature:

$$T_{\text{KF}} = \frac{T}{\sqrt{1+\sigma}+1}\frac{g}{g+g_K+G} < \frac{1}{2}T$$

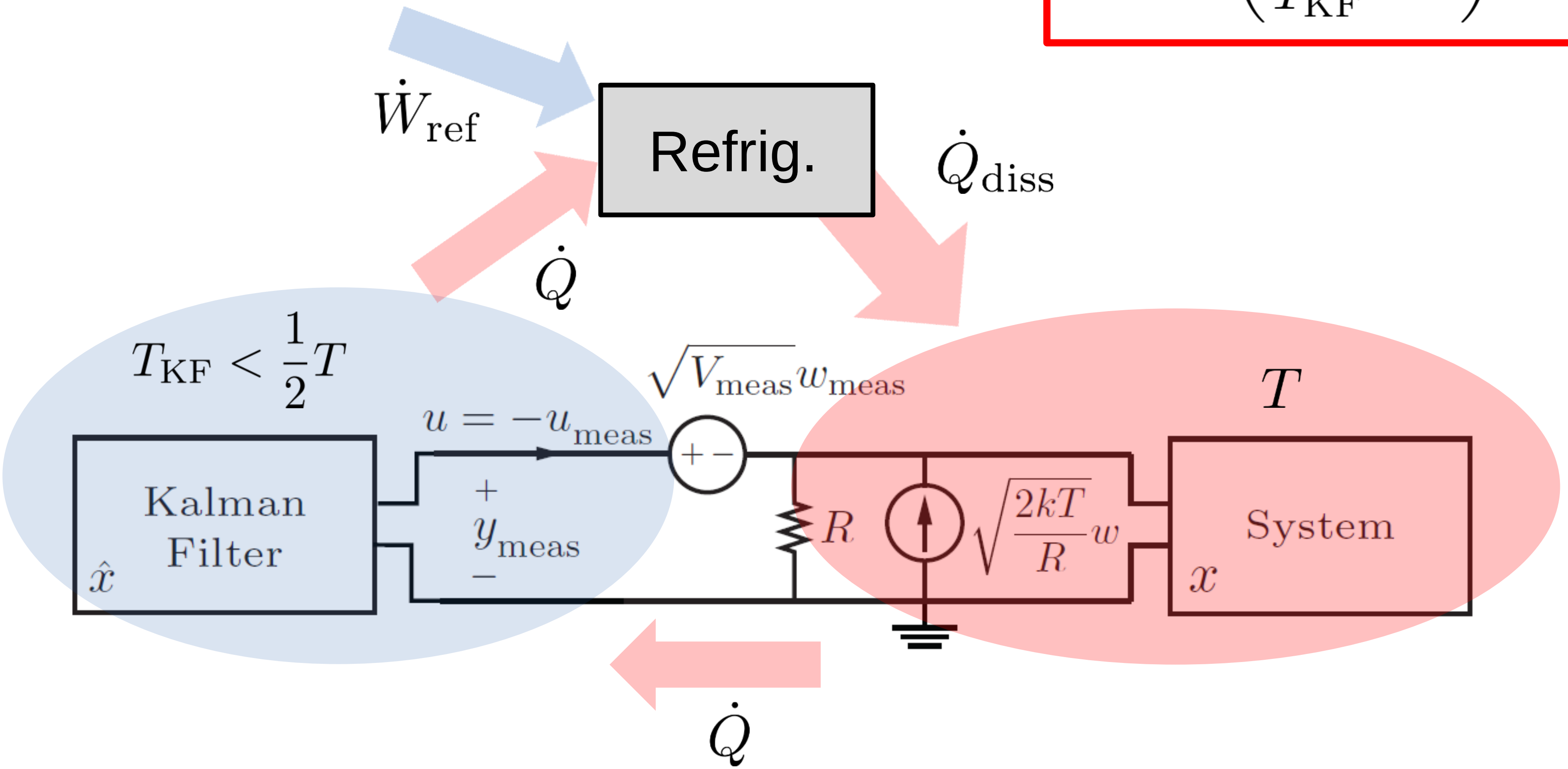
Insight:

As back action tends to zero, filter temperature tends to zero

Heat and Work Flows

2nd Law:

$$\dot{W}_{\text{ref}} \geq \left(\frac{T}{T_{\text{KF}}} - 1\right)\dot{Q}$$



(Back action heat flow)

Power Supply Required by the 2nd Law

$$\dot{W}_{\text{ref}} \geq \kappa T(1+\sigma-\sqrt{1+\sigma})\left(1-\frac{\sqrt{1+\sigma}}{1+\sqrt{1+\sigma}}\text{BA}\right)$$

No back action (BA $\rightarrow$ 0)

$$\dot{W}_{\text{ref}} \geq \kappa T(1+\sigma-\sqrt{1+\sigma}) \approx \begin{cases} \frac{\kappa T}{2} \times \text{SNR (low SNR)} \\ \kappa T \times \text{SNR (high SNR)} \end{cases}$$

Max back action (BA $\rightarrow$ 1)

$$\dot{W}_{\text{ref}} \geq \kappa T \frac{1+\sigma-\sqrt{1+\sigma}}{1+\sqrt{1+\sigma}} \approx \begin{cases} \frac{\kappa T}{4} \times \text{SNR (low SNR)} \\ \kappa T \times \sqrt{\text{SNR}} \text{ (high SNR)} \end{cases}$$

Power Supply Required by the 2nd Law

$$\dot{W}_{\mathrm{ref}} \geq \kappa T\left(1+\sigma-\sqrt{1+\sigma}\right)\left(1-\frac{\sqrt{1+\sigma}}{1+\sqrt{1+\sigma}}\mathrm{BA}\right)$$

Heat conductivity of system

$$kGB^{T}MB$$

Normalized back action

$$\frac{g}{g+G}$$

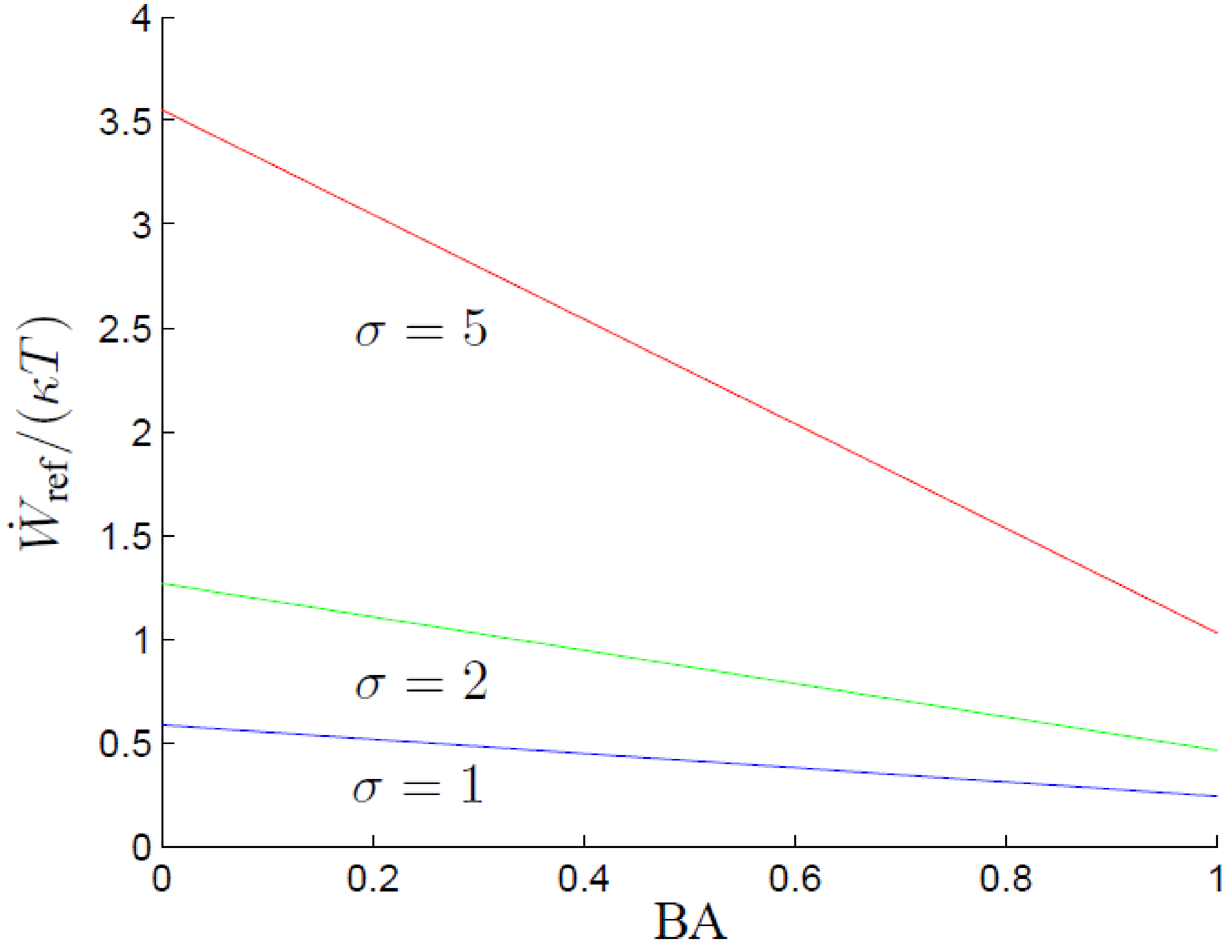
N

$$\ldots) \approx \begin{cases} \frac{\kappa T}{2} \times \mathrm{SN}\ldots \\ \kappa T \times \mathrm{SN}\ldots \end{cases}$$

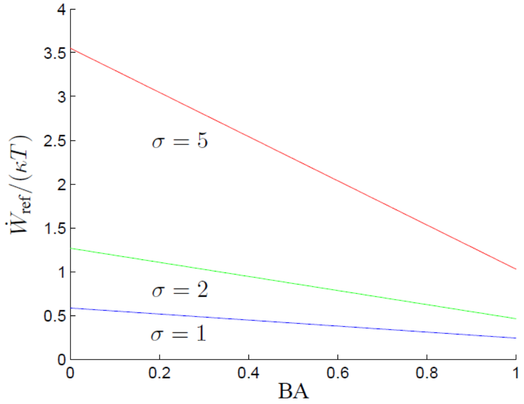
<u>Max back action (BA $\rightarrow$ 1)</u>

$$\dot{W}_{\mathrm{ref}} \geq \kappa T \frac{1+\sigma-\sqrt{1+\sigma}}{1+\sqrt{1+\sigma}} \approx \begin{cases} \frac{\kappa T}{4} \times \mathrm{SNR}\ (\text{low SNR}) \\ \kappa T \times \sqrt{\mathrm{SNR}}\ (\text{high SNR}) \end{cases}$$

Trade-Off: Power Supply vs. Back Action



Observations



- Large back action ⇒ lower power supply

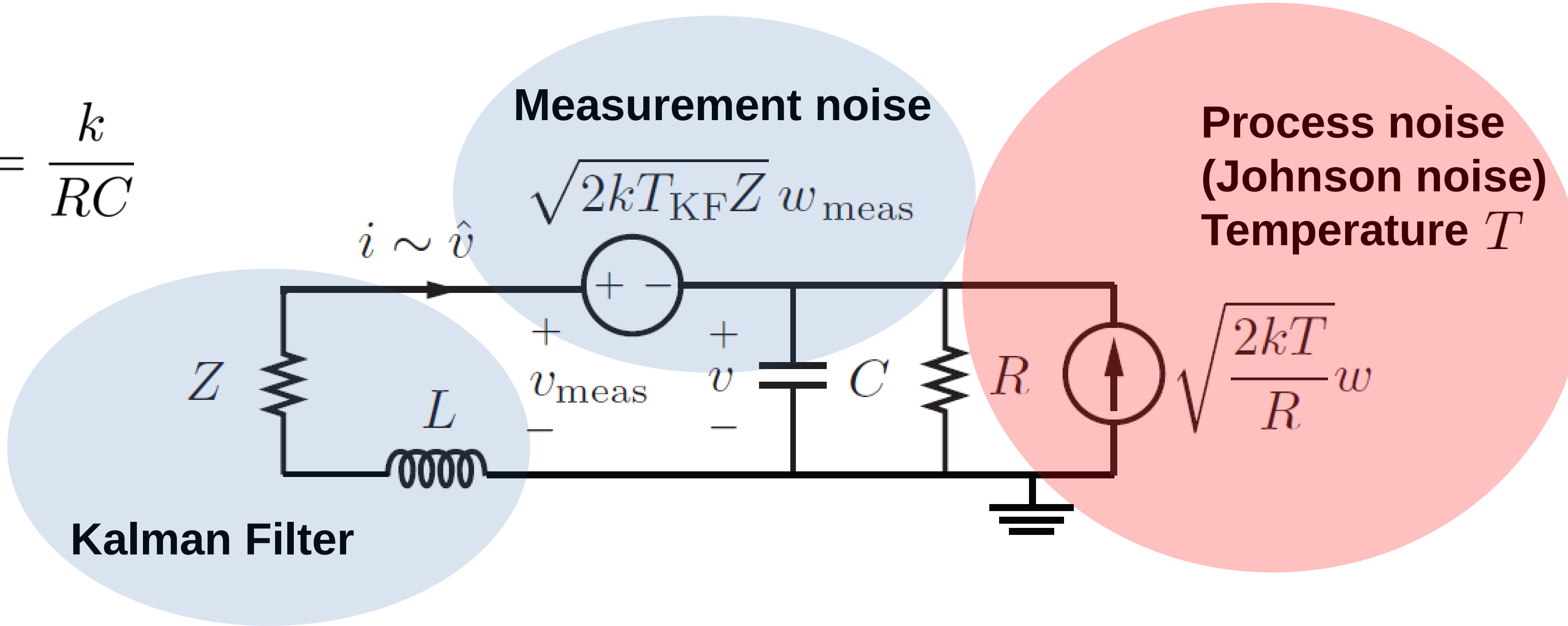
- **Explanation**: Temperature ratio T/T_{KF} smaller ⇒ 2nd law less restrictive

- Trade-off more significant for high SNRs

- No back action costs a factor $\sqrt{\rm SNR}$ more than max back action in high SNR regime

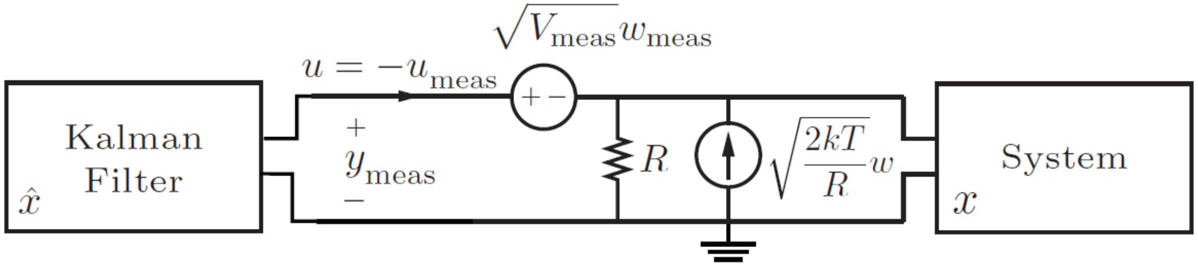
Exact Filter Circuit for Motivating Example

$$\kappa = \frac{k}{RC}$$



- Implementation using passive components
- $T_{KF} < T$ (non-equilibrium thermodynamic system)

Landauer's Principle and (Directed) Information Flow



- **Landauer's principle (1961):** Need to spend at least work $kT \ln 2$ to erase one bit of information
- Directed information flow (system to filter):

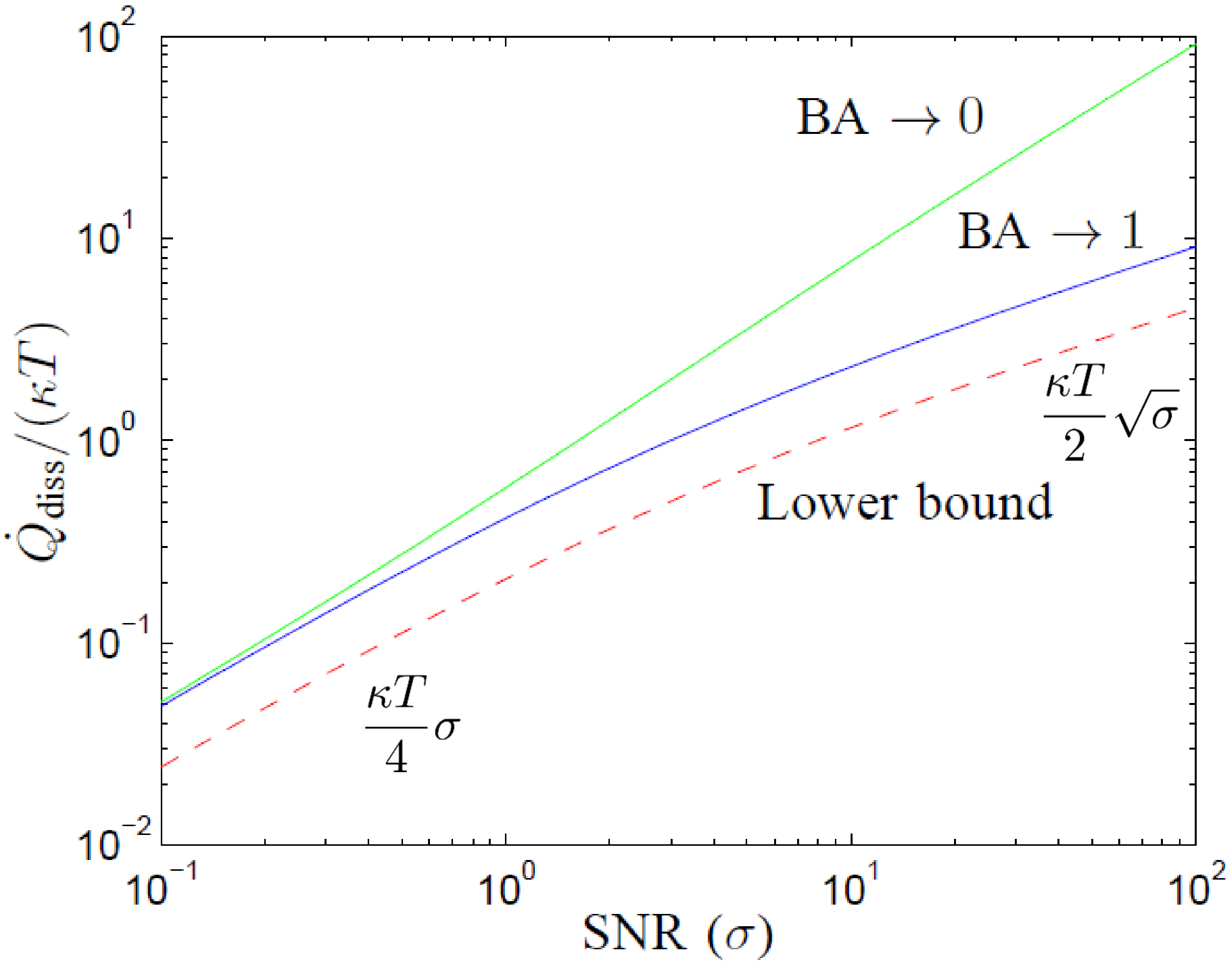
$$\dot{I}_c := \frac{d}{dt} I((w_0^t, x(0));\ (y_{\mathrm{meas}})_0^t)$$

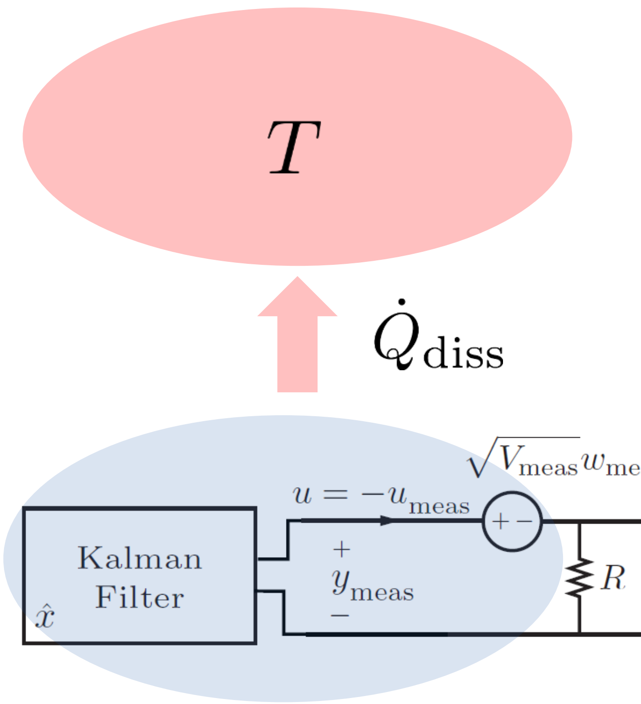
$$= \frac{\kappa T}{2}(\sqrt{1+\sigma} - 1)$$

[Sandberg *et al*., Phys. Rev. E, 2014]

$$\dot{W}_{\mathrm{memo}} \geq kT\dot{I}_c = \frac{\kappa T}{2}(\sqrt{1+\sigma} - 1)$$

Absolute Lower Bound Compared to Physical Implementations





Observations

- “Passive” implementation at least a factor 2 more dissipation than required by lower bound

- **Explanations:**
  - Landauer’s principle holds for **infinitely slow erasure**. Here finite erasure rate, which costs more

  - Directed information rate is a **lower bound** on **entropy rate** of memory in filter. Entropy rate can be a factor 2 larger

[Sandberg *et al*., Phys. Rev. E, 2014]

Summary

- Class of systems with “passive” Kalman-Bucy filters found. Passive but active cooling required (unless we own a cold heat bath…)
- Trade-off identified: Allow for back action to reduce required power supply
- Physical implementations are a factor 2 away from absolute lower bound. In fact optimal?
- Possible applications: nonequilibrium thermodynamics, synthetic biology, energy harvesting…

Related References

- H. Sandberg, J.-C. Delvenne, N.J. Newton, S.K. Mitter: "Thermodynamic Costs in Implementing Kalman-Bucy Filters". In Proceedings of the 52nd Annual Allerton Conference on Communication, Control, and Computing, Monticello, Illinois, October 2014.

- H. Sandberg, J.-C. Delvenne, N.J. Newton, S.K. Mitter: “Maximum work extraction and implementation costs for nonequilibrium Maxwell's demons”. *Physical Review E*, 90, 042119, 2014.

- S.K. Mitter and N.J. Newton: “Information and entropy flow in the Kalman-Bucy filter”. *Journal of Statistical Physics*, 2005.