Structured Model Reduction of Networks of Passive Systems

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Joint Work With…

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Outline

• Introduction

• Part I: Clustering-based model reduction of networked passive systems

• Part II: Coherency-independent structured model reduction of power systems

• Summary
Motivation: Networked Systems

Challenges

• Dynamics dependent on subsystems and interconnection
• Large-scale interconnection complicates analysis, simulation, and synthesis

Goal. Model reduction of large-scale networked systems
Related Work

General methods
• Balanced truncation (Moore, Glover,…)
• Hankel-norm approximation (Glover,…)
• Moment matching/Krylov-subspace methods (Antoulas, Astolfi, Benner,…)

[Diagram of network and reduction process]
Related Work

Reduction of subsystems, i.e., structured reduction

• Controller reduction/closed-loop model reduction (Anderson, Zhou, De Moor, …)
• Structured balanced truncation (Beck, Van Dooren, Sandberg, …)

• Example in Part II
Related Work

Clustering-based model reduction
- Time-scale separation (Chow, Kokotovic,…)
- Graph-based clustering (Ishizaki, Monshizadeh, Trentelman…)
- Structured balanced truncation (Besselink,…)
- Example in Part I and II
Part I: Clustering-based model reduction of networked passive systems

Problem and results
• Subsystems with identical higher-order dynamics
• Controllability/observability-based cluster selection
• A priori $H_\infty$-error bound and preserved synchronization (cf. balanced truncation)

Modeling

1. *Identical* subsystem dynamics
\[ \Sigma_i : \dot{x}_i = A x_i + B v_i, \quad z_i = C x_i, \quad x_i \in \mathbb{R}^n, \quad v_i, z_i \in \mathbb{R}^m \]

2. Interconnection topology with \( w_{ij} \geq 0 \)
\[ v_i = \sum_{j=1, j \neq i}^{\bar{n}} w_{ij} (z_j - z_i) + \sum_{j=1}^{\bar{m}} g_{ij} u_j \]

3. External outputs
\[ y_i = \sum_{j=1}^{\bar{n}} h_{ij} z_j \]
Assumptions

A1. The subsystems $\Sigma_i$ are passive with storage function $V_i(x_i) = \frac{1}{2} x_i^T Q x_i$ (supply $i = v_i^T z_i$)

A2. The graph $G = (V, E)$ with graph Laplacian $L$ is such that
   a) The underlying undirected graph is a tree
   b) $G$ contains a directed rooted spanning tree

\[
(L)_{ij} = \begin{cases} 
- w_{ij}, & i \neq j \\
\sum_{j=1,j\neq i}^{n} w_{ij}, & i = j 
\end{cases}
\]
Lemma. Under A1 and A2, the subsystems of $\Sigma$ synchronize for $u = 0$, i.e., for all $(i, j) \in \mathcal{V} \times \mathcal{V}$,

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = 0$$
Problem and Approach

Goal. Approximate the input-output behavior of $\Sigma$ by a clustering-based reduced-order system $\hat{\Sigma}$

$$\Sigma : \begin{cases} \dot{x} = (I \otimes A - L \otimes BC)x + (G \otimes B)u \\ y = (H \otimes C)x \end{cases}$$

$$\hat{\Sigma} : \begin{cases} \dot{\xi} = (I \otimes A - \hat{L} \otimes BC)\xi + (\hat{G} \otimes B)u \\ \hat{y} = (\hat{H} \otimes C)\xi \end{cases}$$
Problem and Approach

Wish list for approximation method
1. Preserve synchronization and passivity
2. Identify suitable clusters
3. Provide a priori bound on $\|y - \hat{y}\|
4. Be scalable in system size ($\#\text{nodes} = \bar{n}$, state dim. $\Sigma = n \times \bar{n}$)

$\Sigma : \begin{cases} \dot{x} = (I \otimes A - L \otimes BC)x + (G \otimes B)u \\ y = (H \otimes C)x \end{cases}$
Problem and Approach

**Idea.** Find neighboring subsystems $\Sigma_i$ that are

- hard to steer individually from the inputs
- hard to distinguish from the outputs

Mathematically, the system is described by:

$$\Sigma : \begin{cases} \dot{x} = (I \otimes A - L \otimes BC)x + (G \otimes B)u \\ y = (H \otimes C)x \end{cases}$$
Edge Laplacian $L_e$

Lemma. Consider $L$ and let $E$ be an oriented incidence matrix of the underlying undirected graph. Then,

$$L = FE^T \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$$

Lemma. Under A2, the edge Laplacian

$$L_e = E^T F \in \mathbb{R}^{(\tilde{n}-1) \times (\tilde{n}-1)}$$

has all eigenvalues in the open right-half complex plane

$$E = [\ast e_i - e_j \ast], \quad F = [\ast w_{ij} e_i - w_{ji} e_j \ast]$$

# nodes = \tilde{n}
**Edge Dynamics and Controllability**

\[
\begin{align*}
\dot{x}_i &= x_j - x_j \\
\Sigma_i &\quad \rightarrow \\
\Sigma_j \\
\end{align*}
\]

**Edge system in coordinates** 
\[x_e = (E^T \otimes I)x\]

\[\Sigma_e: \quad \dot{x}_e = (I \otimes A - L_e \otimes BC)x_e + (E^T G \otimes B)u, \quad y_e = (H_e \otimes C)x_e\]

\(\Sigma_i\) and \(\Sigma_j\) are hard to steer individually \(\longleftrightarrow\) weakly controllable coordinate in \(\Sigma_e\)

**Edge controllability gramian** \(P_e\) characterizes controllability

\[x_e^T P_e^{-1} x_e = \inf_{\{u \mid 0 \sim x_e\}} \int_{-\infty}^{0} |u(t)|^2 dt\]

**Challenges**

- \(P_e\) dependent on subsystems and interconnection topology
- Role of individual edges not apparent from \(P_e\)
Edge Dynamics and Controllability

**Theorem.** The edge controllability Gramian $P_e$ can be bounded as

$$P_e \preceq \Pi^c \otimes Q^{-1}$$

if there exists $\Pi^c = \text{diag}\{\pi^c_1, \ldots, \pi^c_{\bar{n}-1}\} \succeq 0$ such that

$$L_e \Pi^c + \Pi^c L_e^T - E^T GG^T E \succeq 0$$

**Properties**

- Gramian can be defined as $\Sigma_e$ is asymptotically stable
- $\Pi_c \in \mathbb{R}^{(\bar{n}-1) \times (\bar{n}-1)}$ only dependent on interconnection properties
- Measure of controllability for each individual edge

**Lemma.** $\Pi^c \succeq 0$ exists if $w_{ij} > 0 \iff w_{ji} > 0$
Edge Singular values

Generalized edge controllability Gramian
\[ \Pi^c = \text{diag}\{\pi_1^c, \ldots, \pi_{\bar{n}-1}^c\}, \quad L_e \Pi^c + \Pi^c L_e^T - E^T GG^T E \succeq 0 \]

Generalized edge observability Gramian
\[ \Pi^o = \text{diag}\{\pi_1^o, \ldots, \pi_{\bar{n}-1}^o\}, \quad L_e^T \Pi^o + \Pi^o L_e - F^T H^T HF \succeq 0 \]

Generalized squared edge singular values
\[ (L_e^{-1})_{ii}^2 \pi_i^c \pi_i^o \geq (L_e^{-1})_{i+1,i+1}^2 \pi_{i+1}^c \pi_{i+1}^o \geq 0, \quad i = 1, \ldots, \bar{n} - 1 \]

Note. Minimize trace of \( \Pi_c \) and \( \Pi_o \) to obtain unique Gramians and small singular values
One-step Clustering

\[ \Sigma_i \xrightarrow{W_{ji}} \Sigma_j \xleftarrow{W_{ij}} \Sigma_{i,j} \]

\[ V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 & w_{ji} \\ 0 & w_{ij} + w_{ji} \end{bmatrix} \]

Reduced-order system

Petrov-Galerkin projection of graph Laplacian

\[ \hat{\Sigma}_{n-1} : \dot{\xi} = (I \otimes A - \hat{L} \otimes BC)\xi + (\hat{G} \otimes B)u, \quad \hat{y} = (\hat{H} \otimes C)\xi \]

with \( \hat{L} = W^T L V \), \( \hat{G} = W^T G \), \( \hat{H} = HV \)
One-step Clustering

\[ V = \begin{bmatrix} I & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} I & 0 \\ 0 & \frac{w_{ji}}{w_{ij} + w_{ji}} \\ 0 & \frac{w_{ij}}{w_{ij} + w_{ji}} \end{bmatrix} \]

**Theorem.** Consider \( \Sigma \) and the one-step clustered \( \hat{\Sigma}_{\bar{n}-1} \). Then,

1. The edge controllability Gramian of \( \hat{\Sigma}_{\bar{n}-1} \) satisfies
   \[ \hat{P}_e \preceq \hat{\Pi}^c \otimes Q^{-1}, \quad \hat{\Pi}^c = \text{diag}\{\pi_1^c, \ldots, \pi_{\bar{n}-2}^c\} \]

2. The edge observability Gramian of \( \hat{\Sigma}_{\bar{n}-1} \) satisfies
   \[ \hat{Q}_e \preceq \hat{\Pi}^o \otimes Q, \quad \hat{\Pi}^o = \text{diag}\{\pi_1^o, \ldots, \pi_{\bar{n}-2}^o\} \]

**Opens up for repeated one-step clustering!**
**Performance Guarantees**

**Theorem.** The subsystems of $\hat{\Sigma}_k$ synchronize for $u = 0$, i.e.,

$$\lim_{t \to \infty} (\xi_i(t) - \xi_j(t)) = 0, \quad (i, j) \in \hat{\nu} \times \hat{\nu}$$

**Theorem.** For trajectories $x(\cdot)$ of $\Sigma$ and $\xi(\cdot)$ of $\hat{\Sigma}_k$ for the same input $u(\cdot)$ and $x(0) = 0$, $\xi(0) = 0$, the output error is bounded as

$$\|y - \hat{y}\|_2 \leq 2 \left( \sum_{l=\vec{k}}^{\vec{n}-1} (L_{\vec{l}}^{-1})_{\|\|} \sqrt{\pi_l^c \pi_l^0} \right) \|u\|_2$$

**Generalized edge singular values**
Summary So Far

Wish list for approximation method
1. Preserve synchronization and passivity
   • OK
2. Identify suitable clusters
   • Use generalized edge singular values
3. Provide a priori bound on $\|y - \hat{y}\|$ 
   • Generalized edge singular values provide bounds
4. Be scalable in system size ($\#$nodes = $\tilde{n}$, state dim. $\Sigma = n \times \tilde{n}$)
   • Solve two LMIs of size $\tilde{n}$ (independent of subsystem size $n$) [and possibly one Riccati equation of size $n$ to verify passivity]
Example: Thermal Model of a Corridor of Six Rooms

\[ T_{\text{env}} \]

\[ \Sigma_1 \quad \Sigma_2 \quad \Sigma_3 \quad \Sigma_4 \quad \Sigma_5 \quad \Sigma_6 \]

\[ P_h \quad T_1^3 \]

Subsystems: thermal dynamics within a room

\[ C_1 \dot{T}_1^i = R_{\text{int}}^{-1} (T_2^i - T_1^i) - R_{\text{out}}^{-1} T_1^i + P_i \]

\[ C_2 \dot{T}_2^i = R_{\text{int}}^{-1} (T_1^i - T_2^i) \]

Edges: thermal resistances of walls, \( u_j = [P_h \ T_{\text{env}}]^T \)

\[ P_i = \sum_{j=1, j \neq i}^{\tilde{n}} R_{\text{wall}}^{-1} (T_1^j - T_1^i) + \sum_{j=1}^{\tilde{m}} g_{ij} u_j \]

Reduction from \( \tilde{n} = 6 \) to \( \tilde{k} = 3 \)
Example: Thermal Model of a Corridor of Six Rooms

Edge singular values: \( \tilde{\pi}_l := (L^{-1}_e)_{ll} \sqrt{\pi^c_l \pi^o_l} \)

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<td>( 10^2 \tilde{\pi}_l )</td>
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<td>0.951</td>
<td>1.267</td>
<td>0.337</td>
<td>0.092</td>
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\[ T_{env} \]

\[ \Sigma_1 \rightarrow \Sigma_2 \rightarrow \Sigma_3 \rightarrow \Sigma_4 \rightarrow \Sigma_5 \rightarrow \Sigma_6 \]

\( P_h \rightarrow T_1^3 \)

75% reduction of wall 3 resistance

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<td>0.308</td>
<td>0.373</td>
<td>0.109</td>
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</tbody>
</table>

\[ T_{env} \]

\[ \Sigma_1 \rightarrow \Sigma_2 \rightarrow \Sigma_3 \rightarrow 0.25 \rightarrow \Sigma_4 \rightarrow \Sigma_5 \rightarrow \Sigma_6 \]

\( P_h \rightarrow T_1^3 \)
Example: Thermal Model of a Corridor of Six Rooms

Frequency response function from input $P_h$ to output $T_{1}^{3}$

Error bound: $2 \sum_{i=3}^{5} (L_e^{-1})_{ii} \sqrt{\pi_i \pi'_i} = 11.4 \cdot 10^{-3}$

Actual error: $0.849 \cdot 10^{-3}$
Summary Part I

- Clustering-based reduction procedure
- Edge controllability and observability properties
- Preservation of synchronization and error bound

Possible extensions
- Arbitrary network topology
- Non-identical subsystems
- Nonlinear networked systems
- Lower bounds

Part II: Coherency-independent structured model reduction of power systems

Problem and results

- Model reduction of nonlinear large-scale power system
- Clustering, linearization, and reduction of external area
- Application of structured balanced truncation

Background

- Increasingly interconnected power systems
- New challenges for dynamic simulation, operation, and control of large-scale power systems
- Coherency-based power system model reduction not always suitable
Approach

Divide system into a study area and an external area

Objective: Reduce the external area so that the effect of the approximation error in the study area is as small as possible
Approach

Divide system into a study area and an external area

- Study area $N$ often set by utility ownership or market area. Nonlinear model will be retained here
- External area $G$ denotes other utilities. Will be linearized and reduced here
- Insight from structured/closed-loop model reduction: Reduction of $G$ should depend on $N$!
Four-Step Procedure

1. Define the model (DAE)

\[
\begin{align*}
\dot{x} &= f(x, x_{\text{alg}}, u) \\
0 &= g(x, x_{\text{alg}}, u).
\end{align*}
\]

\[
\begin{align*}
\dot{x}^G &= f^G (x^G, x_{\text{alg}}^G, u^G) \\
0 &= g^G (x^G, x_{\text{alg}}^G, u^G) \\
\dot{x}^N &= f^N (x^N, x_{\text{alg}}^N, u_1^N, u_2^N) \\
0 &= g^N (x^N, x_{\text{alg}}^N, u_1^N, u_2^N)
\end{align*}
\]

2. Linearizing

\[
\begin{align*}
\begin{pmatrix}
\dot{x}^G \\
0
\end{pmatrix}
&= \begin{pmatrix}
A_{11}^G & A_{12}^G \\
A_{21}^G & A_{22}^G
\end{pmatrix}
\begin{pmatrix}
x^G \\
x_{\text{alg}}^G
\end{pmatrix}
+ \begin{pmatrix}
B_{11}^G \\
B_{21}^G
\end{pmatrix} u^G \\
\begin{pmatrix}
\dot{x}^N \\
0
\end{pmatrix}
&= \begin{pmatrix}
A_{11}^N & A_{12}^N \\
A_{21}^N & A_{22}^N
\end{pmatrix}
\begin{pmatrix}
x^N \\
x_{\text{alg}}^N
\end{pmatrix}
+ \begin{pmatrix}
B_{11}^N \\
B_{21}^N
\end{pmatrix} u_1^N \\
+ \begin{pmatrix}
B_{12}^N \\
B_{22}^N
\end{pmatrix} u_2^N.
\end{align*}
\]

\[
x_{\text{alg}}^G = -A_{22}^{G-1} (A_{21}^G x^G + B_2^G u^G) \\
x_{\text{alg}}^N = -A_{22}^{N-1} (A_{21}^N x^N + B_{21}^N u_1^N + B_{22}^N u_2^N)
\]
Four-Step Procedure

3. Structured/closed-loop model reduction of external area model, $G \rightarrow \hat{G}$ (details next)

4. Nonlinear complete reduced model

\[
\begin{align*}
\dot{x}^G &= A^G x^G + B^G u^G \\
u_2^N &= y^G = C^G x^G + D^G u^G \\
\dot{x}^N &= f^N (x^N, x_{\text{alg}}^N, u_1^N, u_2^N) \\
0 &= g^N (x^N, x_{\text{alg}}^N, u_1^N, u_2^N) \\
u^G &= y_2^N = M^N x_{\text{alg}}^N.
\end{align*}
\]

Reduced linear external area

Unreduced nonlinear study area
Structured Model Reduction of $G$

(Following Schelfhout/De Moor, Vandendorpe/Van Dooren, Sandberg/Murray): $(N, G) = \Sigma(A, B, C, D)$

$$AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0$$

$$P = \begin{bmatrix} P_N & P_{NG} \\ P_{NG}^T & P_G \end{bmatrix}, \quad Q = \begin{bmatrix} Q_N & Q_{NG} \\ Q_{NG}^T & Q_G \end{bmatrix}$$

Local balancing of $G$ only:

$$\Sigma_G = T_{G}^{-1}P_GT_{G}^{-T} = T_{G}^TQ_GT_G$$

Structured (Hankel) singular values of $G$:

$$\Sigma_G = \text{diag}\{\sigma_{G,1}, \sigma_{G,2}, \ldots, \sigma_{G,n}\}$$

Truncation or singular perturbation of $G$ yields $\hat{G}$

**Note 1.** $\hat{G}$ depends on study area $N$
**Note 2.** Error bound and stability guarantee require generalized Gramians (LMIs) [Sandberg/Murray]
Model Reduction of Non-Coherent Areas: KTH-Nordic32 System

Study area: Southern Sweden. Keep detailed model

External area: Simplify as much as possible

Model info:
- 52 buses
- 52 lines
- 28 transformers
- 20 generators (12 hydro gen.)
Model Reduction of Non-Coherent Areas: KTH-Nordic32 System

- External area $G$ has 246 dynamic states.
- Reduced external area $\hat{G}$ has 17 dynamic states.

Fig. 8. Responses of $\theta_{49} - \theta_{50}$ after a 2% perturbation to $V_{\text{ref},18}$.

Fig. 9. Responses of $\theta_{49} - \theta_{50}$ after a 5% perturbation to $V_{\text{ref},18}$.
Model Reduction of Non-Coherent Areas: KTH-Nordic32 System

- External area $G$ has 246 dynamic states.
- *Reduced* external area $\hat{G}$ has 17 dynamic states.

Fig. 12. Responses of $\theta_{49} - \theta_{50}$ after opening a line for 80 ms.

Fig. 13. Responses of $\theta_{49} - \theta_{50}$ after opening a line for 200 ms.
What If Open-Loop Reduction Used to Simplify External Area $G$?

Fig. 3. The WSCC 3-machine, 9-bus system with an infinite bus.


Fig. 5. Transients of $V_7$ at the tie-line bus with a third order system $\hat{G}$.

Fig. 6. Transients of $\theta_7$ at the tie-line bus with a third order system $\hat{G}$. Structured model reduction and ordinary balanced truncation are compared with the full system.
Summary Part II

• Clustering, linearization, and reduction of external power system area
• Application of structured balanced truncation: Closed-loop behavior matters!
• Verification on a model of the Nordic grid

Possible extensions
• Nonlinear model reduction with error bounds and stability guarantees

Concluding Remarks

• Model reduction of networked systems. Dynamics dependent on subsystems and interconnection. Many applications!

• Model reduction methods could reduce topology and/or dynamics

**Challenge.** Many heuristics possible. We want rigorous scalable methods with performance guarantees.
• Balanced truncation and Hankel-norm approximation do not preserve network structures very well
• LMIs are very expensive to solve [$\sim O(n^{5.5})$]

Thank You!

Sponsors

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