The Observer Effect in Estimation with Physical Communication Constraints

Henrik Sandberg
ACCESS Linnaeus Centre, KTH Royal Institute of Technology

Joint work with
Jean-Charles Delvenne, Univ. catholique de Louvain

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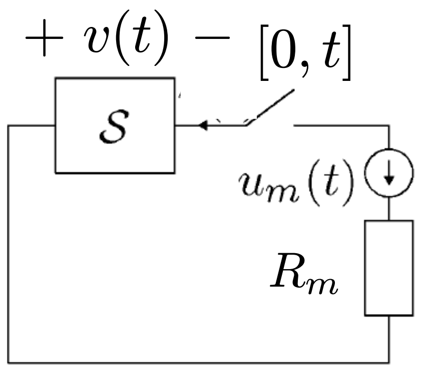
Observer Effect - Example

Voltage perturbation in passive circuit *S* — $\Delta v(t)$

Uncertainty in optimal voltage estimate — $\Delta \hat{v}(t)$



With or without active compensation, a trade-off exists:

$$|\Delta v(t)||\Delta \widehat{v}(t)| \geq 2k_B T_m / C \quad \text{small } t$$

Sandberg, Delvenne, Doyle, IEEE TAC 2010

How general is this trade-off?

Observer Effect - Example

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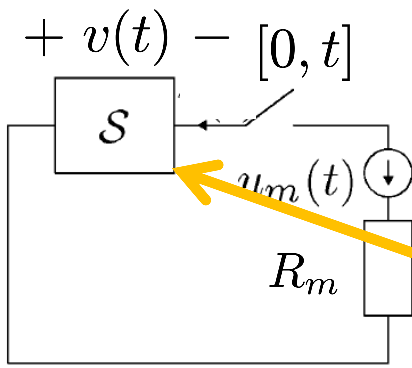

Independent of R_m

Capacitance
$C^{-1} = \lim_{s \to \infty} s[Z(s)]_+$

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How general is this trade-off?

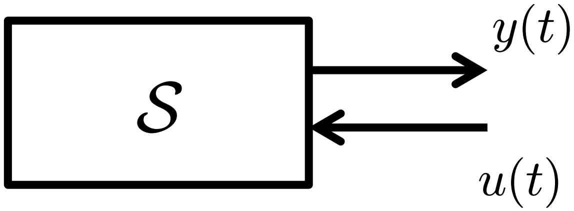
Motivation

- The observer effect has been extensively studied in quantum mechanics, but seems overlooked in a classical mechanics setting
- We want to know the **performance limits** of devices when **resources,** such as DOFs, energy, temperature, and time, are **finite** or **limited**

- How well — and what — can we actually implement from a small collection of physical building blocks? Related to circuit synthesis (*M. Smith*, *B.D.O. Anderson*, *et al*.)
- Similar questions currently being asked in synthetic biology (*R. Murray*, *D. Del Vecchio*, *et al*.)

Measurement Model

- A **physical system** *S* with a property ***y*(*t*)** we want to measure. Together with conjugate variable $u(t)$ it forms a **port** [external work rate = $u(t)^{\mathsf{T}} y(t)$]



Example: $y(t)$ = *velocity*, voltage,…

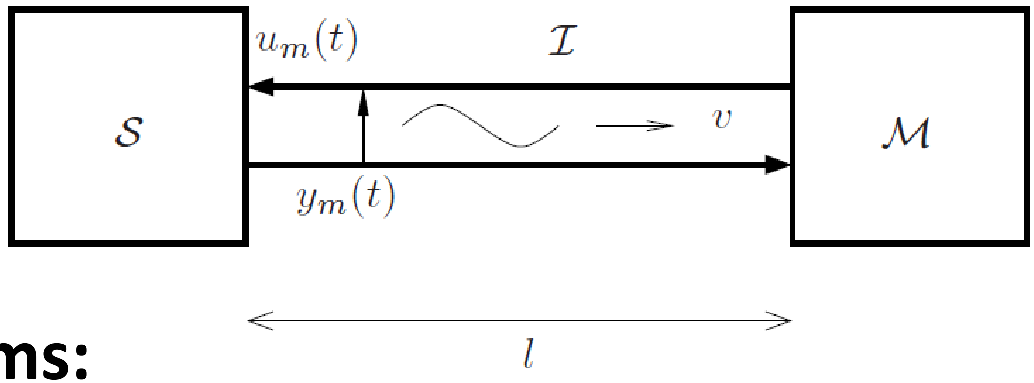
$u(t)$ = *force*, current,…

- Treat u as a *small perturbation* u_m from measurement

$$\mathcal{S}\begin{cases} \dot{x}(t) = f(x(t), u_m(t)) \\ y_m(t) = g(x(t)) \\ x(0) = x_0 \end{cases} \approx \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u_m(t) \\ y_m(t) = C(t)x(t) \\ x(0) = x_0 \end{cases}$$

Measurement Model

- Interconnect *S* to a measurement device *M* through a **communication medium *I***, during a short time $[0,t]$

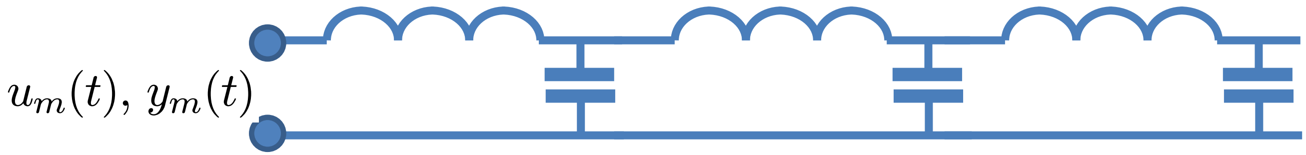


- **Problems:**
  - What is the best estimate of $y(t)$ we can get?
  - How much must *S* simultaneously be perturbed? (=back action, retroactivity,…)
- **Assumption:** Medium *I* is well modeled by a *lossless wave equation*, in thermal equilibrium at time t=0

Why Lossless Wave Equation?

- Mechanics:

- Circuits:



- Electromagnetic fields: $\left(\nabla^2 - \epsilon\mu \frac{\partial^2}{\partial t^2}\right) E(x,t) = 0$

- At the terminals for times $0 < t < l/v$:

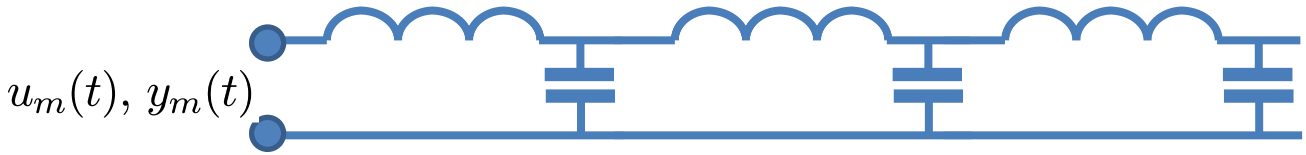
$$u_m(t) = -Z y_m(t) - \sqrt{2 k_B T_Z Z} w_Z(t)$$

(Z>0 [acoustic/characteristic] impedance, T_Z temperature, v wave velocity, l medium length, and white noise $w_Z(t)$ due to FD-theorem. Transmission line $Z = \sqrt{\frac{R+i\omega L}{G+i\omega C}} \to \sqrt{\frac{L}{C}}, \quad \omega \sim 1/t \to \infty$.)

Where Does the Noise Come From?

- Mechanics:

- Circuits:

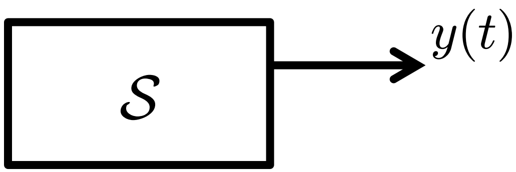


- Electromagnetic fields: $\left(\nabla^2 - \epsilon\mu \frac{\partial^2}{\partial t^2}\right) E(x,t) = 0$

- Assume exact initial state of wave equation unknown
- **In thermal equilibrium**: Assign the expected energy ½$k_B T_Z$ to each mode
- Total effect at the terminal sums up to white noise, see *Nyquist, 1928* (or *Sandberg, Delvenne, Doyle, 2010*)

Unmeasured vs. Measured System

- Unmeasured system

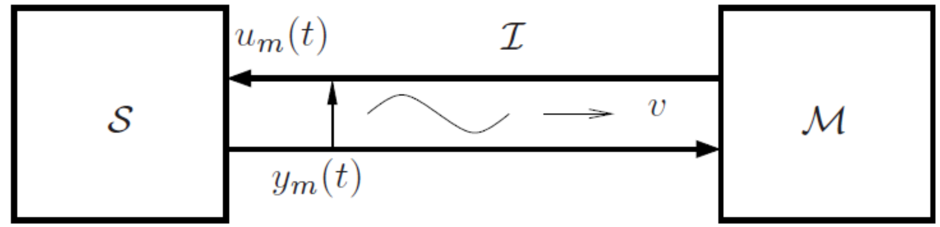


$$\dot{x}(t) = Ax(t) \quad x(0) = x_0,$$
$$y(t) = Cx(t) \qquad y_0 := Cx_0$$

- For small t:

$$y(t) = Ce^{At}x_0$$
$$= y_0 + CAx_0t + O(t^2)$$

- Measured system



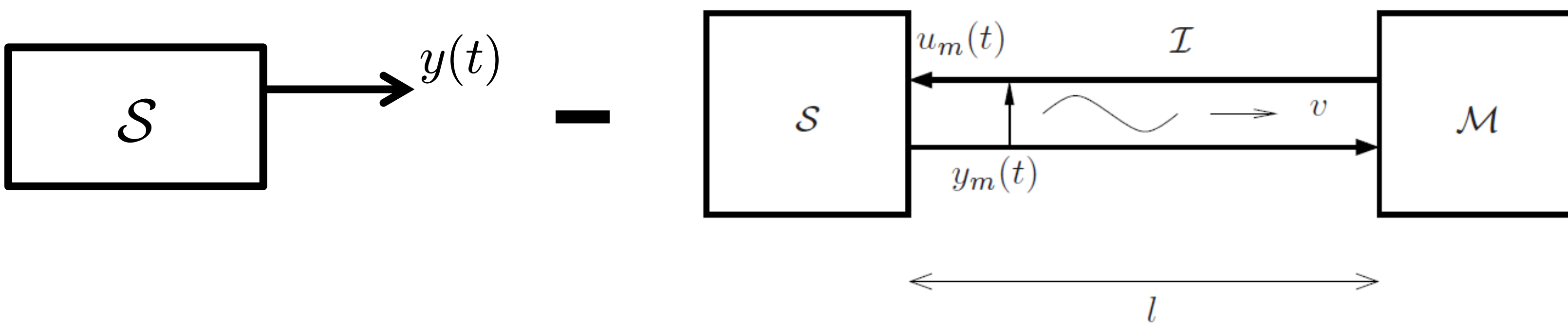
$$\dot{x}_m(t) = (A - BZC)x_m(t) - B\sqrt{2k_BT_ZZ}w_Z(t),$$
$$y_m(t) = Cx_m(t), \quad x_m(0) = x_0,$$
$$u_m(t) = -Zy_m(t) - \sqrt{2k_BT_ZZ}w_Z(t),$$

- For small t:

$$y_m(t) = Ce^{(A-BZC)t}x_0$$
$$- \int_0^t Ce^{(A-BZC)(t-\tau)}B\sqrt{2k_BT_ZZ}w_Z(\tau)d\tau$$
$$= y_0 + C(A - BZC)x_0t$$
$$- CB\sqrt{2k_BT_ZZ}\int_0^t w_Z(\tau)d\tau + O(t\sqrt{t}),$$

Measurement Back Action

- Difference between un-measured and measured output =: **back action of measurement**



- **Deterministic back action:**

$$y(t) - \mathbf{E} y_m(t) = M^{-1} Z y_0 t + O(t^2)$$

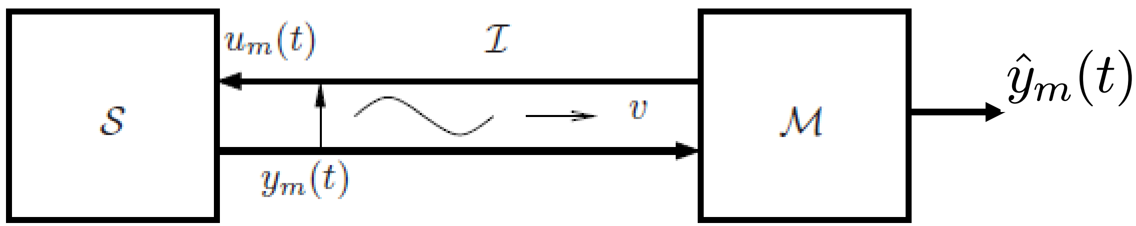
(Present even when $T_Z = 0$. $M^{-1} := CB$ is the “inverse inertia” of *S*)

- **Stochastic back action:**

$$\mathrm{Var}[y_m(t)] = 2 k_B T_Z M^{-1} Z M^{-T} t + O(t^2)$$

Idealized Measurement Device *M**

- The measurement device *M* produces an estimate $\hat{y}_m$ of y_m



- In best case, *M* has exact models of *S* and *I*, and knows the temperature T_Z (but not exact state of *I*) $\Rightarrow$ Kalman filter *M** optimal:

$$\dot{\hat{x}}_m(t) = (A - BZC)\hat{x}_m(t) + K(t)(u_m(t) - \hat{u}_m(t))$$
$$\hat{u}_m(t) = -ZC\hat{x}_m(t),$$
$$\hat{y}_m^*(t) = C\hat{x}_m(t),$$

- *Any M with less or equal knowledge is no better than M**

Lower Bounds on Accuracy

- Accuracy of M^* determined from differential filter Riccati equation:

$$\dot{P} = (A - BZC)P + P(A - BZC)^T - (PC^T Z - 2k_B T_Z BZ)(2k_B T_Z Z)^{-1}(PC^T Z - 2k_B T_Z BZ)^T + 2k_B T_Z BZB^T$$

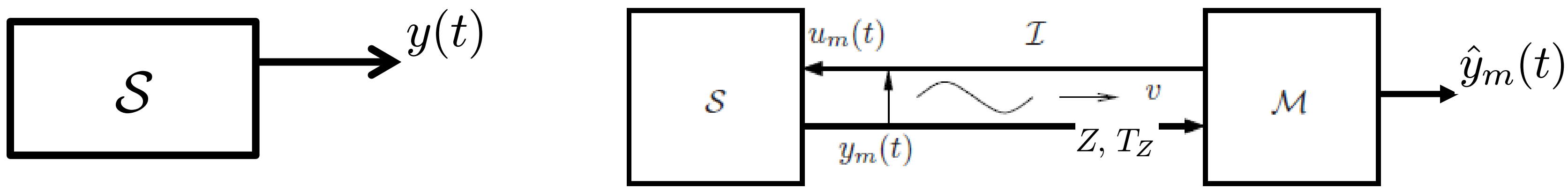
- Assume zero knowledge at t = 0: $P(0) = \mathbf{E}[\hat{x}_m(0) - x_0]^2 = +\infty$
- Series expansion of $P\,(t)$ gives

$$P(t) = P_{-1}t^{-1} + P_0 + P_1 t + O(t^2) \qquad P_{-1} = P_{-1}C^T ZCP_{-1}/2k_B T_Z$$

- Optimal measurement accuracy:

$$\mathrm{Var}[\hat{y}^*_m(t) - y_m(t)] = 2k_B T_Z Z^{-1} t^{-1} + O(1)$$

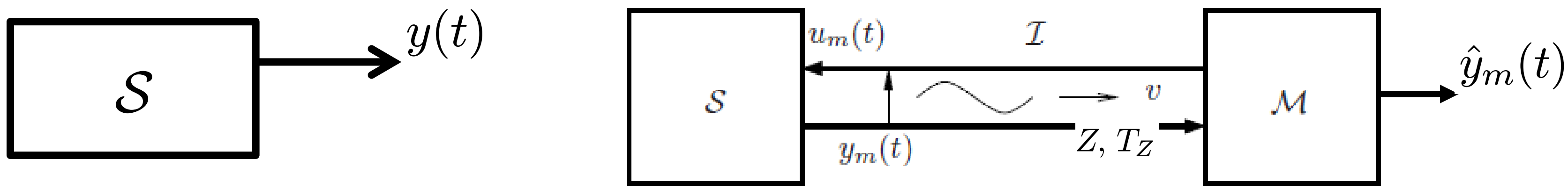
Back Action and Accuracy Trade-Off



- **Back action:** $$y(t) - \mathbf{E}y_m(t) = M^{-1}Zy_0t + O(t^2),$$ $$\mathrm{Var}[y_m(t)] = 2k_BT_ZM^{-1}ZM^{-T}t + O(t^2)$$
- **Measurement accuracy:** $$\mathrm{Var}[\hat{y}_m(t) - y_m(t)] \geq \mathrm{Var}[\hat{y}^*_m(t) - y_m(t)] = 2k_BT_ZZ^{-1}t^{-1} + O(1)$$
- **Trade-off:** $$\sqrt{\mathrm{Tr}\ \mathrm{Var}[y_m(t)]}\ \sqrt{\mathrm{Tr}\ \mathrm{Var}[\hat{y}_m(t) - y_m(t)]} \geq 2k_BT_Z\mathrm{Tr}\ M^{-1} + O(t)$$

Trade-off independent on small t and medium impedance Z!

Back Action and Accuracy Trade-Off

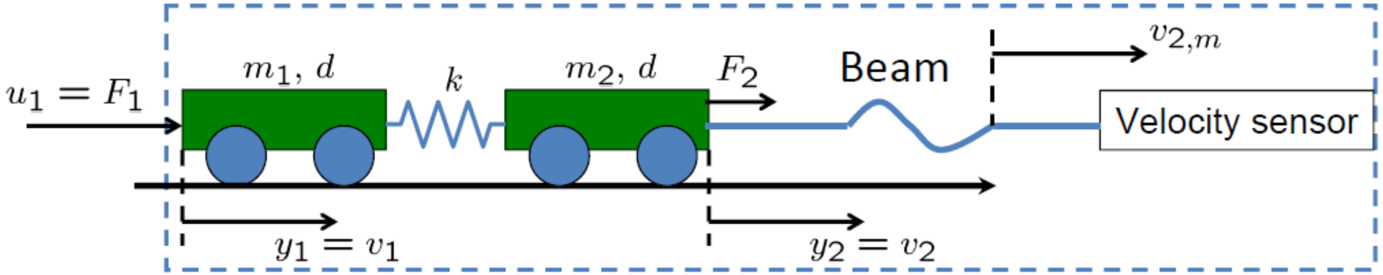


- **Back action:** $$y(t) - \mathbf{E}y_m(t) = M^{-1}Zy_0 t + O(t^2),$$ $$\mathrm{Var}[y_m(t)] = 2k_B T_Z M^{-1} Z M^{-T} t + O(t^2)$$
- **Measurement accuracy:** $$\mathrm{Var}[\hat{y}_m(t) - y_m(t)] \geq \mathrm{Var}[\hat{y}^*_m(t) - y_m(t)]$$ $$= 2k_B T_Z Z^{-1} t^{-1} + O(1)$$
- **Trade-off:**

  back action × accuracy
  $$\sqrt{\mathrm{Tr}\ \mathrm{Var}[y_m(t)]}\ \sqrt{\mathrm{Tr}\ \mathrm{Var}[\hat{y}_m(t) - y_m(t)]}$$ $$\geq 2k_B T_Z \mathrm{Tr}\ M^{-1} + O(t)$$

Trade-off independent on small t and medium impedance Z!

Mechanical Example



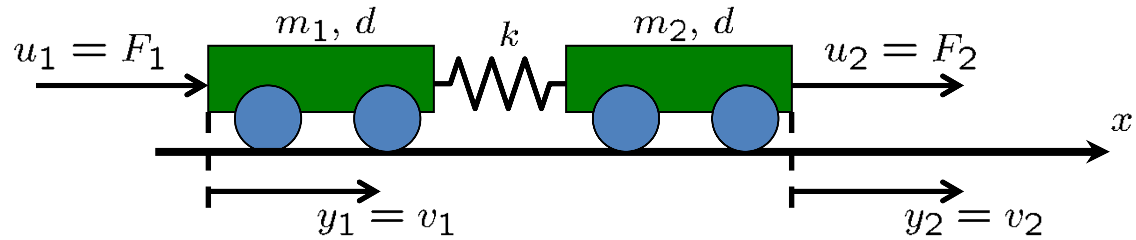
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & -d_1/m_1 & k/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k/m_2 & 0 & -k/m_2 & -d_2/m_2 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{pmatrix}$$

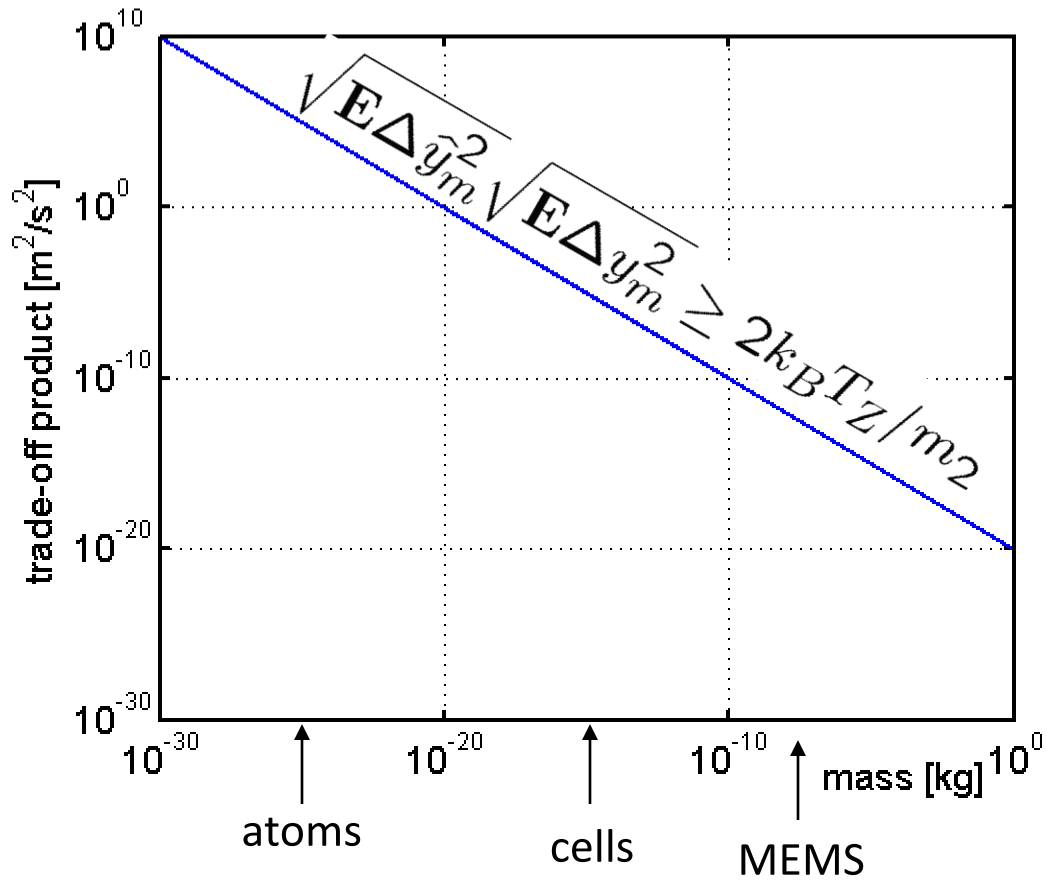
$$C = (0\ 0\ 0\ 1),$$

- Inverse inertia $M^{-1} := CB = 1/m_2$
- Det. back action $Zv_{2,0}t/m_2$, stoch. back action $2k_BT_ZZt/m_2^2$
- Measurement accuracy $2k_BT_Z/Zt$
- Trade-off: back action $\times$ accuracy $\geq 2k_BT_Z/m_2$

Mechanical Example

- $k_B = 1.4 \cdot 10^{-23}$ J/K
- $T_Z = 300$ K
- Not visible at macroscopic level
- Significant at microscopic level





Outline

- Measurement model
- Measuring deterministic systems
- Measuring port-Hamiltonian systems in thermal equilibrium (details in paper)

System *S* in Thermal Equilibrium

$$\text{back action} \geq \frac{2T_Z}{T_S} \times \Delta\text{accuracy}$$

- Only ratio T_Z/T_S determines trade-off. Is this result essentially different from before?

$$\text{back action} \times \text{accuracy} \geq 2k_B T_Z/m$$

- Not really, as system temperature $T_\text{S} \to \infty$ we obtain earlier result since rate of learning also goes to infinity
- System inertia disappeared and was replaced by system temperature T_S
- When T_S is really small, only small improvement in accuracy since we already know a lot about the system

Summary

- There are general **trade-offs** between **measurement accuracy** and **back action** for classical measurements
- Holds not only for passive system *S* and particular *M* (compare *Sandberg, Delvenne, Doyle, IEEE TAC 2010*)
- Lossless measurement medium of temperature T_Z
- System with "inverse inertia" $M^{-1} = CB$ (first Markov parameter) and completely unknown initial state

$$\text{back action} \times \text{accuracy} \geq 2k_B T_Z \operatorname{Tr} M^{-1}$$

- System of temperature T_S

$$\text{back action} \geq \frac{2T_Z}{T_S} \times \Delta\text{accuracy}$$