Parameterized Model Order Reduction Using Extended Balanced Truncation

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Henrik Sandberg

Automatic Control Lab, ACCESS Linnaeus Centre,
KTH, Stockholm, Sweden

Outline

- The Parameterized Model Order Reduction (**PMOR**) problem
- Extended Gramians and a related dissipation inequality
- Computing parameterized Gramians
- New methods for parameterized model order reduction

The Considered Problem

Given:

$$G_\xi \begin{cases} x(k+1) = A_\xi x(k) + B_\xi u(k) \\ y(k) = C_\xi x(k) + D_\xi u(k) \end{cases}, \quad \begin{bmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{bmatrix} := \sum_{i=1}^{N} \xi_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$$

Find:

$$\widehat{G}_\xi \begin{cases} \widehat{x}(k+1) = \widehat{A}_\xi \widehat{x}(k) + \widehat{B}_\xi u(k) \\ \widehat{y}(k) = \widehat{C}_\xi \widehat{x}(k) + \widehat{D}_\xi u(k) \end{cases}, \quad \begin{bmatrix} \widehat{A}_\xi & \widehat{B}_\xi \\ \widehat{C}_\xi & \widehat{D}_\xi \end{bmatrix} := \sum_{i=1}^{N} \xi_i \begin{bmatrix} \widehat{A}_i & \widehat{B}_i \\ \widehat{C}_i & \widehat{D}_i \end{bmatrix}$$

such that:

(i): model order$(\hat{G}_\xi) \ll$ model order(G_ξ); and

(ii): $\|G_\xi - \hat{G}_\xi\|_\infty$ small for all *fixed* $\xi \in \Xi$.

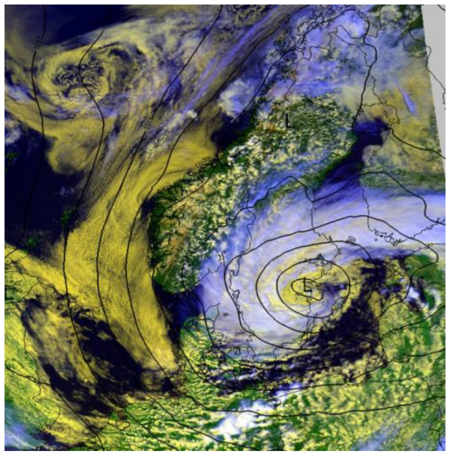
Convex bounded polyhedron $\Xi := \left\{\xi : \xi \in \mathbb{R}^N, \sum_{i=1}^{N} \xi_i = 1, \xi_i \geq 0\right\}$

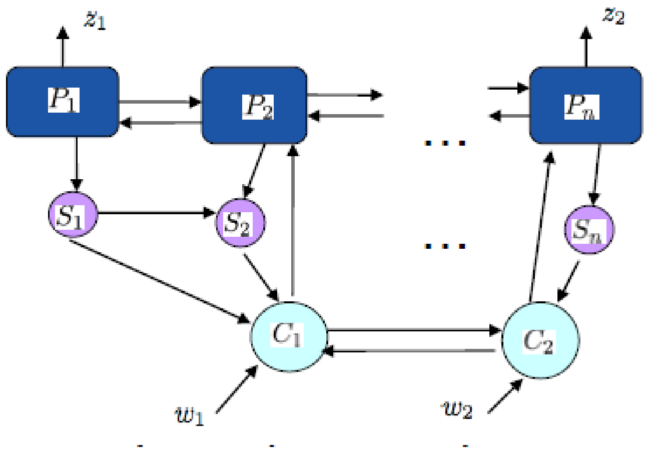
Motivation

- Model reduction of models where an operating point (ξ) is varying or unknown at time of reduction

- A desire to reduce **all** models G_ξ, $\xi \in \Xi$ in **one shot**

- Applications: Fluid dynamics, circuits, power systems, control systems,…





The Considered Problem (cont'd)

- A seemingly direct approach: Solve for all $\xi \in \Xi$

$$P(\xi) - A_\xi P(\xi) A_\xi^\top - B_\xi B_\xi^\top = 0$$
$$Q(\xi) - A_\xi^\top Q(\xi) A_\xi - C_\xi^\top C_\xi = 0$$

and apply balanced truncation or Hankel-norm approximation

- But generally **no closed-form solution $\boldsymbol{P(\xi)}, \boldsymbol{Q(\xi)}$** and reduced model $\hat{G}_\xi$ **not easily parameterized**

- Instead we will look for **generalized Gramians**. Solve for all $\xi \in \Xi$

$$P(\xi) - A_\xi P(\xi) A_\xi^\top - B_\xi B_\xi^\top > 0, \quad P(\xi) > 0$$
$$Q(\xi) - A_\xi^\top Q(\xi) A_\xi - C_\xi^\top C_\xi > 0, \quad Q(\xi) > 0$$

where $P(\xi)$ and $Q(\xi)$ are easily parameterized (generalized) Gramians

Contributions

- A new method to compute easily parameterized generalized Gramians based on extended balanced truncation [1]

- Two new LMI-based algorithms to solve the parameterized model order reduction problem, with *a priori* $\mathcal{H}_\infty$-approximation error bounds

[1] H. Sandberg, "An extension to balanced truncation with application to structured model reduction," *IEEE Transactions on Automatic Control*, 2010

Related Work

- Model reduction of uncertain and linear parameter-varying models
  - Wood, Goddard, Glover (1996)
  - Beck, Doyle, Glover (1996)
  - Li, Petersen (2010)

- Parameterized model order reduction
  - Sou, Megretski, Daniel (2008)
  - Baur, Beattie, Benner, Gugercin (2011)

- Extended LMIs
  - De Oliviera, Bernussou, Geromel (1999)

A Useful Lemma

$$P(\xi) - A_\xi P(\xi) A_\xi^\top - B_\xi B_\xi^\top > 0, \quad P(\xi) > 0, \qquad (1)$$

$$\begin{bmatrix} P(\xi) & A_\xi R(\xi) & B_\xi \\ R(\xi)^\top A_\xi^\top & R(\xi) + R(\xi)^\top - P(\xi) & 0 \\ B_\xi^\top & 0 & I \end{bmatrix} > 0, \qquad (2)$$

Lemma ([1],[2]): Suppose $\xi \in \mathbb{R}^n$ is fixed.

- If $P(\xi)$ satisfies (1), then there exists a slack variable $R(\xi)$ such that $P(\xi)$ and $R(\xi)$ satisfy (2)
- Conversely, if $P(\xi)$ and $R(\xi)$ satisfy (2), then $P(\xi)$ satisfies (1)

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[2] M. de Oliveira et al., "A new discrete-time robust stability condition, Systems & Control Letters, 1999

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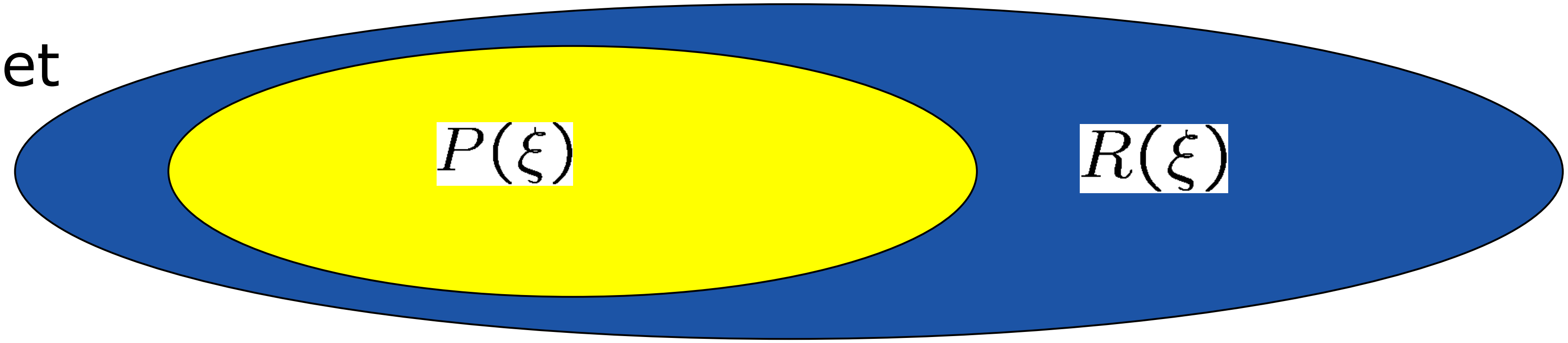
Generalized and Extended Gramians

- Solutions to (1) are
  - *Generalized controllability Gramians* $P(\xi)$
- Solutions to (2) are
  - *Extended controllability Gramians* $(P(\xi), R(\xi))$
  - $P(\xi)=R(\xi)$ is always possible solutions

Feasibility set



The Dual

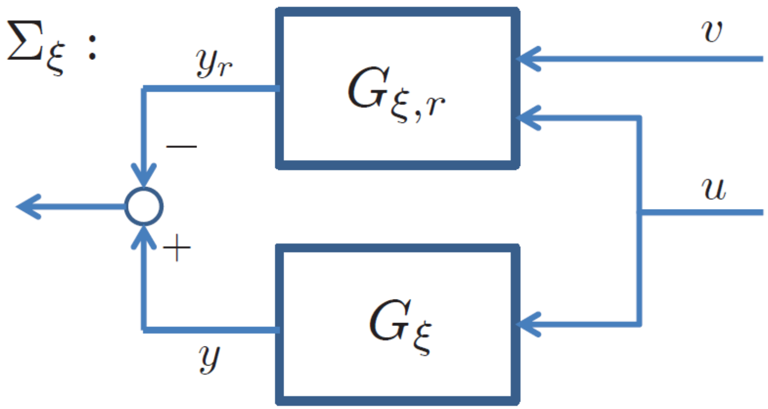
- Generalized observability Gramian $Q(\xi)$

$$Q(\xi) - A_\xi^\top Q(\xi) A_\xi - C_\xi^\top C_\xi > 0, \quad Q(\xi) > 0,$$

- Extended observability Gramian $(Q(\xi), S(\xi))$

$$\begin{bmatrix} S(\xi) + S(\xi)^\top - Q(\xi) & S(\xi)A_\xi & 0 \\ A_\xi^\top S(\xi)^\top & Q(\xi) & C_\xi^\top \\ 0 & C_\xi & I \end{bmatrix} > 0,$$

Dissipation-Inequality Interpretation



$$G_{\xi,r} \begin{cases} x_r(k+1) = A_\xi x_r(k) + B_\xi u(k) + v(k), \\ y_r(k) = C_\xi x_r(k) + D_\xi u(k). \end{cases}$$

$$G_\xi \begin{cases} x(k+1) = A_\xi x(k) + B_\xi u(k) \\ y(k) = C_\xi x(k) + D_\xi u(k) \end{cases}$$

$$\boxed{V_\xi(x(k+1), x_r(k+1)) \cdot \quad V_\xi(x(0), x_r(0)) + \sum_{i=0}^{k} w_\xi(i),}$$

Storage function parameterized by generalized Gramian $Q(\xi)$:

$$V_\xi(x, x_r) = (x - x_r)^\top Q(\xi)(x - x_r)$$

Supply rate parameterized by the slack variable $S(\xi)$:

$$w_\xi(i) = 2[(x - x_r)(i+1)]^\top S(\xi) v(i) - |(y - y_r)(i)|^2$$

(A dual inequality exists for $P(\xi), R(\xi)$. For details, see [1])

Methods for Finding Parameterized Gramians

1. Fix $P = P(\xi)$ and

$$\forall \xi \in \Xi : \quad P(\xi) - A_\xi P(\xi) A_\xi^\top - B_\xi B_\xi^\top > 0, \quad P(\xi) > 0,$$

is a finite-dimensional LMI in the unknown P (similar to Wood *et al*.) Solve in corners of convex polyhedron. Feasibility requires *quadratic stability*

2. Fix $R = R(\xi)$ and **affine** $P(\xi) = P_\xi := \sum_{i=1}^{N} \xi_i P_i$ and

$$\forall \xi \in \Xi : \quad \begin{bmatrix} P(\xi) & A_\xi R(\xi) & B_\xi \\ R(\xi)^\top A_\xi^\top & R(\xi) + R(\xi)^\top - P(\xi) & 0 \\ B_\xi^\top & 0 & I \end{bmatrix} > 0,$$

is an LMI in the unknowns R and P_i, $i = 1, \ldots, N$

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New Methods and Error Bounds for PMOR

Method 1: Balance/truncate **constant** generalized Gramians P and Q ("traditional" method, similar to Wood *et al*.):

$$\forall \xi \in \Xi : \; \|G_\xi - \hat{G}_\xi\|_\infty \cdot \; 2 \sum_{i>r} \sqrt{\lambda_i(PQ)} =: \epsilon_r$$

Method 2: Balance/truncate **constant** slack variables R and S (first new method):

$$\forall \xi \in \Xi : \; \|G_\xi - \hat{G}_\xi\|_\infty \cdot \; 2 \sum_{i>r} \sqrt{\lambda_i(RS)} \cdot \; \epsilon_r$$

Method 3: Balance/truncate **affine** generalized Gramians P_ξ and Q_ξ (second new method):

$$\forall \xi \in \Xi : \; \|G_\xi - \hat{G}_\xi\|_\infty \cdot \; 2 \sum_{i>r} \sqrt{\lambda_i(P_\xi Q_\xi)} \cdot \; \epsilon_r$$

Balanced Truncation (Method 2)

1. Solve

$$\begin{aligned}
\underset{P_i,R}{\text{minimize}} \quad & \operatorname{trace}(R) \\
\text{subject to} \quad & \begin{bmatrix} P_i & A_i R & B_i \\ R^\top A_i^\top & R + R^\top - P_i & 0 \\ B_i^\top & 0 & I \end{bmatrix} > 0 \\
& R = R^\top \leq P^\star, \quad i = 1, \dots, N
\end{aligned}$$

$$\begin{aligned}
\underset{Q_i,S}{\text{minimize}} \quad & \operatorname{trace}(S) \\
\text{subject to} \quad & \begin{bmatrix} S + S - Q_i & S A_i & 0 \\ A_i^\top S^\top & Q_i & C_i^\top \\ 0 & C_i & I \end{bmatrix} > 0 \\
& S = S^\top \leq Q^\star, \quad i = 1, \dots, N.
\end{aligned}$$

2. Extended Hankel values [1]: $\sigma_{e,i} := \sqrt{\lambda_i(R^\star S^\star)}$

3. Balance the optimal solutions $R^\star$ and $S^\star$:

$$T R^\star T^\top = \bar{R}^\star = T^{-\top} S^\star T^{-1} = \bar{S}^\star = \operatorname{diag}\{\sigma_{e,1}, \dots, \sigma_{e,n}\}$$

4. Truncate the balanced G_ξ

$$\widehat{G}_\xi \begin{cases} \hat{x}(k+1) = \widehat{A}_\xi \hat{x}(k) + \widehat{B}_\xi u(k) \\ \hat{y}(k) = \widehat{C}_\xi \hat{x}(k) + \widehat{D}_\xi u(k) \end{cases}, \quad \begin{bmatrix} \widehat{A}_\xi & \widehat{B}_\xi \\ \widehat{C}_\xi & \widehat{D}_\xi \end{bmatrix} := \sum_{i=1}^{N} \xi_i \begin{bmatrix} \bar{A}_i^{11} & \bar{B}_i^1 \\ \bar{C}_i^1 & D_i \end{bmatrix}$$

(For details on Methods 1 and 3, see the paper.)

Numerical Example

$$G_\xi \begin{cases} x(k+1) = A_\xi x(k) + B_\xi u(k) \\ \quad y(k) = C_\xi x(k) + D_\xi u(k) \end{cases}, \quad \begin{bmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{bmatrix} := \sum_{i=1}^{N} \xi_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix},$$

where

$$A(\alpha) = \begin{pmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.8\alpha & -0.5\alpha & 0.2 & 0.03 + \alpha \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad |\alpha| < \gamma$$

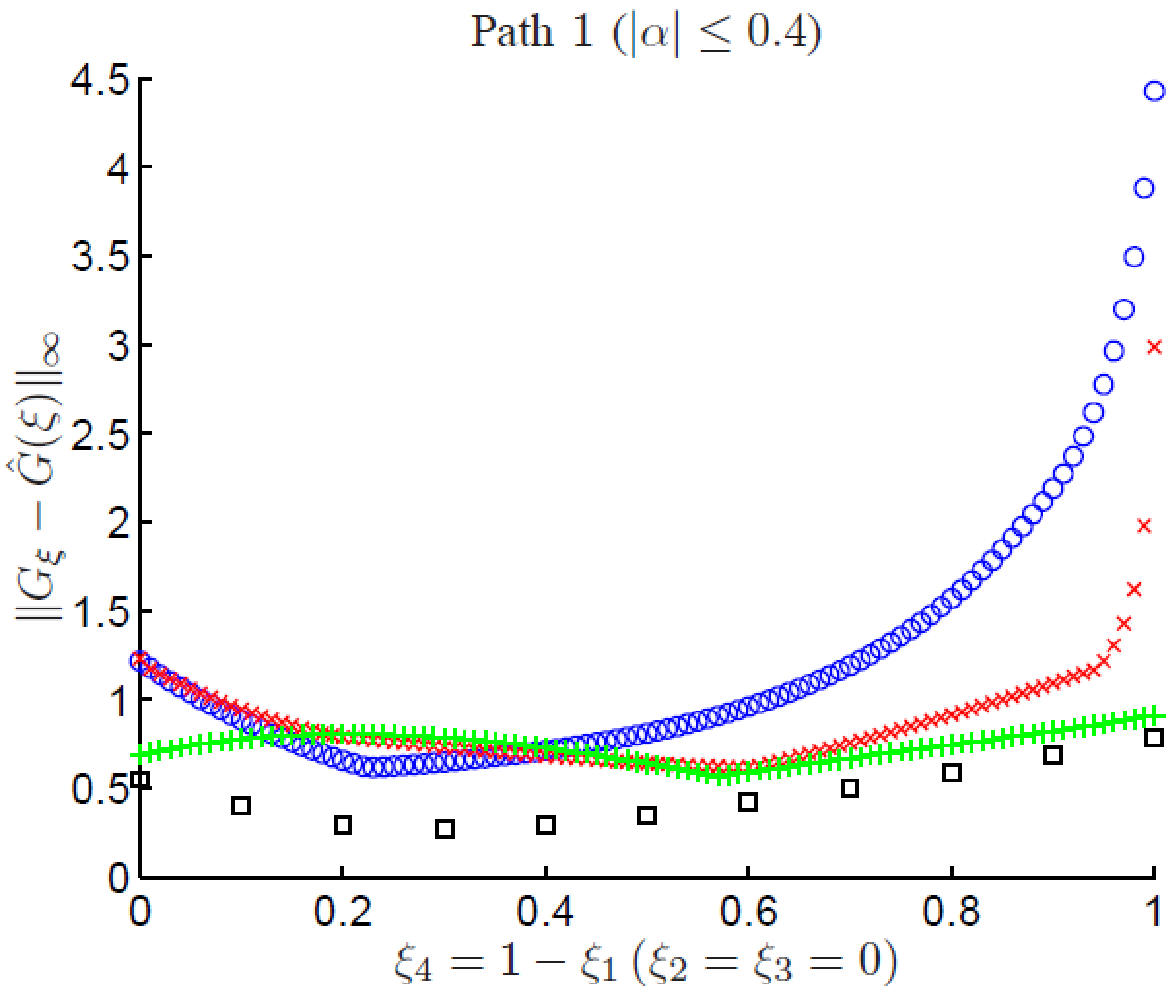
$$B(\beta) = \beta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + (1-\beta) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad 0 \cdot \;\; \beta \cdot \;\; 1$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad D = 0$$

Reduce to second order!

(Model from De Oliveira *et al*.)

Numerical Example (Typical Path in Ξ)



□ = Regular balanced truncation (non-PMOR, lower bound)

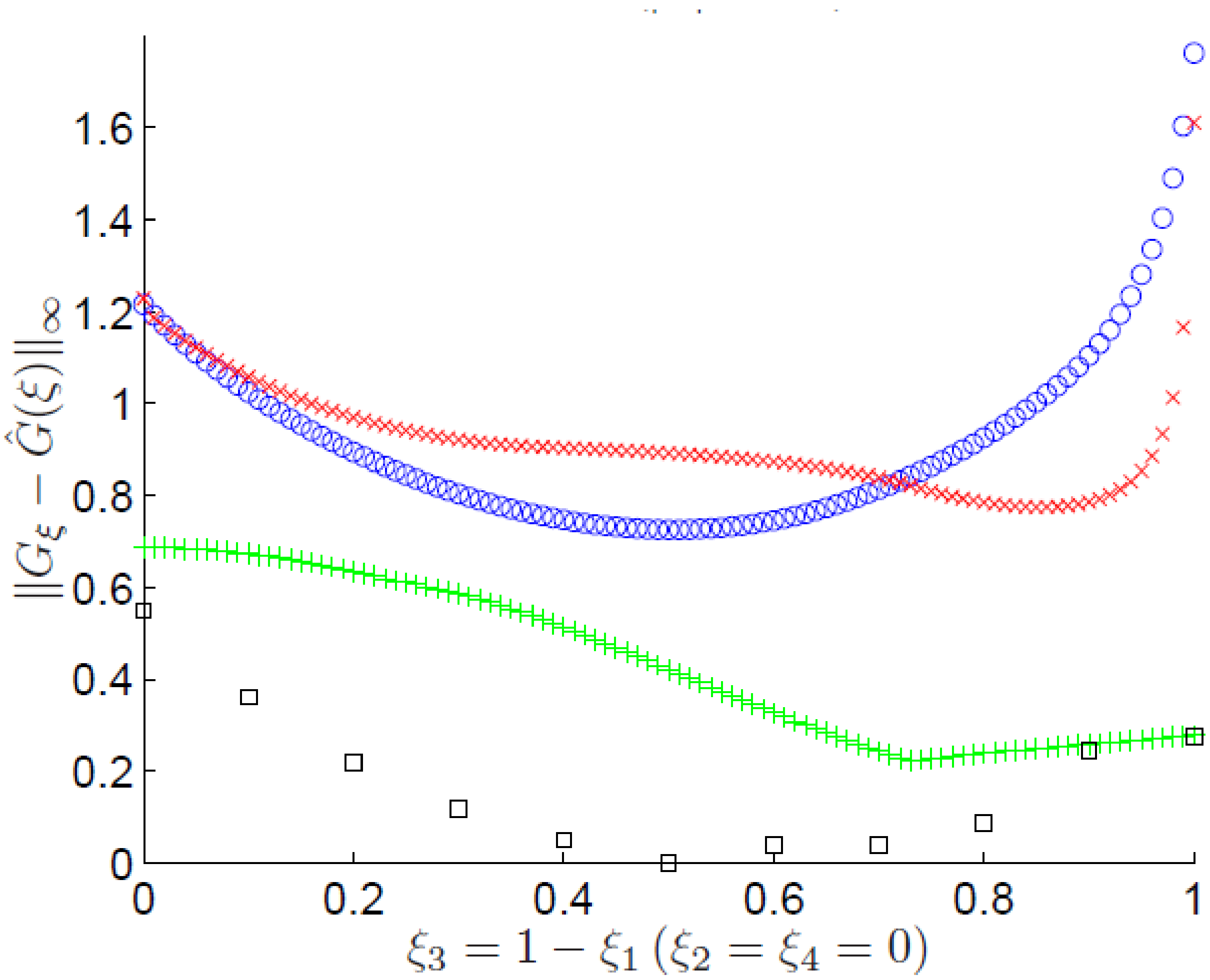
o = Method 1 (P, Q)

x = Method 2 (R, S)

+ = Method 3 (P_ξ, Q_ξ)

- **Method 2-3 close to lower bound**
- **Method 3 best since ξ-dependent coordinates**

Numerical Example (Not A Typical Path in Ξ)



□ = Regular balanced truncation (non-PMOR, lower bound)

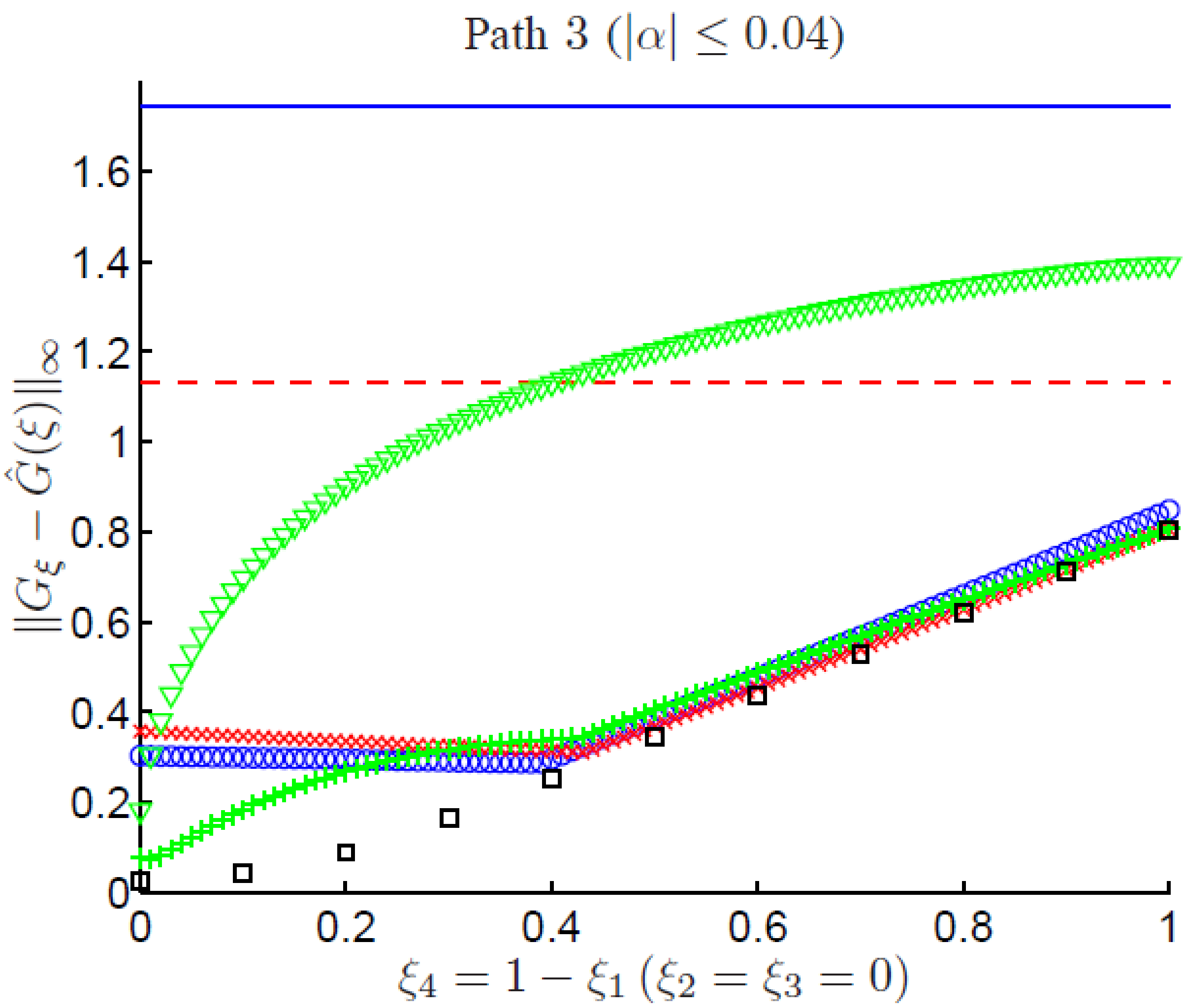
o = Method 1 (P, Q)

x = Method 2 (R, S)

+ = Method 3 (P_ξ, Q_ξ)

- **Method 1-3 cannot handle uncontrollable mode at** $\xi_1 = 0.5$!

A Priori $\mathcal{H}_\infty$ Error Bounds



A priori error bounds:

Method 1 (P, Q):

Method 2 (R, S):

Method 3 (P_ξ, Q_ξ): ∇

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Summary

- Extended balanced truncation can be applied to the PMOR problem
  - Methods 1-2: $\hat{G}_{\xi}$ has simple affine parameterization
  - Method 3: $\hat{G}_{\xi}$ has no closed-form parameterization (but Gramians have simple affine parameterization)
  - In general: Method 3 better than Method 2 better than Method 1

- Methods only work for relatively low-order models due to required solution of LMIs

- Interesting to use non-convex methods to compute non-affinely parameterized Gramians and reduced models $\hat{G}_{\xi}$