Parameterized Model Order Reduction Using Extended Balanced Truncation

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Outline

• The Parameterized Model Order Reduction (PMOR) problem

• Extended Gramians and a related dissipation inequality

• Computing parameterized Gramians

• New methods for parameterized model order reduction
The Considered Problem

**Given:**

\[
G_\xi \quad \left\{ \begin{array}{l}
x(k+1) = A_\xi x(k) + B_\xi u(k) \\
y(k) = C_\xi x(k) + D_\xi u(k)
\end{array} \right.,
\quad \begin{bmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{bmatrix} := \sum_{i=1}^{N} \xi_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}
\]

**Find:**

\[
\hat{G}_\xi \quad \left\{ \begin{array}{l}
\hat{x}(k+1) = \hat{A}_\xi \hat{x}(k) + \hat{B}_\xi u(k) \\
\hat{y}(k) = \hat{C}_\xi \hat{x}(k) + \hat{D}_\xi u(k)
\end{array} \right.,
\quad \begin{bmatrix} \hat{A}_\xi & \hat{B}_\xi \\ \hat{C}_\xi & \hat{D}_\xi \end{bmatrix} := \sum_{i=1}^{N} \xi_i \begin{bmatrix} \hat{A}_i & \hat{B}_i \\ \hat{C}_i & \hat{D}_i \end{bmatrix}
\]

such that:

(i): model order(\( \hat{G}_\xi \)) \( \ll \) model order(\( G_\xi \)); and

(ii): \( ||G_\xi - \hat{G}_\xi||_\infty \) small for all fixed \( \xi \in \Xi \).

Convex bounded polyhedron \( \Xi := \{ \xi : \xi \in \mathbb{R}^N, \sum_{i=1}^{N} \xi_i = 1, \xi_i \geq 0 \} \)
Motivation

- Model reduction of models where an operating point ($\xi$) is varying or unknown at time of reduction

- A desire to reduce all models $G_{\xi}$, $\xi \in \Xi$ in one shot

- Applications: Fluid dynamics, circuits, power systems, control systems,...
The Considered Problem (cont’d)

• A seemingly direct approach: Solve for all \( \xi \in \Xi \)
\[
P(\xi) - A_\xi P(\xi) A_\xi^T - B_\xi B_\xi^T = 0
\]
\[
Q(\xi) - A_\xi^T Q(\xi) A_\xi - C_\xi^T C_\xi = 0
\]
and apply balanced truncation or Hankel-norm approximation

• But generally **no closed-form solution** \( P(\xi), Q(\xi) \) and
reduced model \( \hat{G}_\xi \) **not easily parameterized**

• Instead we will look for **generalized Gramians**. Solve for
all \( \xi \in \Xi \)
\[
P(\xi) - A_\xi P(\xi) A_\xi^T - B_\xi B_\xi^T > 0, \quad P(\xi) > 0
\]
\[
Q(\xi) - A_\xi^T Q(\xi) A_\xi - C_\xi^T C_\xi > 0, \quad Q(\xi) > 0
\]
where \( P(\xi) \) and \( Q(\xi) \) are easily parameterized (generalized) Gramians
Contributions

• A new method to compute easily parameterized generalized Gramians based on extended balanced truncation [1]

• Two new LMI-based algorithms to solve the parameterized model order reduction problem, with \textit{a priori} $\mathcal{H}_\infty$-approximation error bounds

Related Work

• Model reduction of uncertain and linear parameter-varying models
  - Beck, Doyle, Glover (1996)
  - Li, Petersen (2010)

• Parameterized model order reduction
  - Sou, Megretski, Daniel (2008)
  - Baur, Beattie, Benner, Gugercin (2011)

• Extended LMIs
  - De Oliveira, Bernussou, Geromel (1999)
A Useful Lemma

\[ P(\xi) - A_\xi P(\xi) A_\xi^T - B_\xi B_\xi^T > 0, \quad P(\xi) > 0, \quad (1) \]

\[
\begin{bmatrix}
P(\xi) & A_\xi R(\xi) \\
R(\xi)^T A_\xi^T & R(\xi) + R(\xi)^T - P(\xi) & B_\xi \\
B_\xi^T & 0 & I
\end{bmatrix} > 0, \quad (2)
\]

Lemma ([1],[2]): Suppose \( \xi \in \mathbb{R}^n \) is fixed.

- If \( P(\xi) \) satisfies (1), then there exists a slack variable \( R(\xi) \) such that \( P(\xi) \) and \( R(\xi) \) satisfy (2)

- Conversely, if \( P(\xi) \) and \( R(\xi) \) satisfy (2), then \( P(\xi) \) satisfies (1)

A Useful Lemma

\[ P(\xi) - A_\xi P(\xi) A_\xi^T - B_\xi B_\xi^T > 0, \quad P(\xi) > 0, \quad (1) \]

\[
\begin{bmatrix}
P(\xi) \\
R(\xi)^T A_\xi \\
R(\xi) + R(\xi)^T - P(\xi) \\
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\end{bmatrix} > 0, \quad (2)
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Generalized and Extended Gramians

- Solutions to (1) are
  - Generalized controllability Gramians $P(\xi)$
- Solutions to (2) are
  - Extended controllability Gramians $(P(\xi), R(\xi))$
  - $P(\xi) = R(\xi)$ is always possible solutions

Feasibility set

$P(\xi)$

$R(\xi)$
The Dual

- Generalized observability Gramian $Q(\xi)$

$$Q(\xi) - A_\xi^T Q(\xi) A_\xi - C_\xi^T C_\xi > 0, \quad Q(\xi) > 0,$$

- Extended observability Gramian $(Q(\xi), S(\xi))$

$$\begin{bmatrix} S(\xi) + S(\xi)^T - Q(\xi) & S(\xi) A_\xi & 0 \\ A_\xi^T S(\xi)^T & Q(\xi) & C_\xi^T I \\ 0 & C_\xi & I \end{bmatrix} > 0,$$
Dissipation-Inequality Interpretation

Storage function parameterized by generalized Gramian $Q(\xi)$:

$$V_\xi(x(k+1), x_r(k+1)) \cdot V_\xi(x(0), x_r(0)) + \sum_{i=0}^{k} w_\xi(i),$$

Supply rate parameterized by the slack variable $S(\xi)$:

$$w_\xi(i) = 2[(x - x_r)(i + 1)]^\top S(\xi) v(i) - |(y - y_r)(i)|^2$$

(A dual inequality exists for $P(\xi), R(\xi)$. For details, see [1])
Methods for Finding Parameterized Gramians

1. Fix $P = P(\xi)$ and

$$\forall \xi \in \Xi : \quad P(\xi) - A_\xi P(\xi) A^\top_\xi - B_\xi B^\top_\xi > 0, \quad P(\xi) > 0,$$

is a finite-dimensional LMI in the unknown $P$ (similar to Wood et al.) Solve in corners of convex polyhedron. Feasibility requires *quadratic stability*

2. Fix $R = R(\xi)$ and **affine** $P(\xi) = P_\xi := \sum_{i=1}^{N} \xi_i P_i$ and

$$\forall \xi \in \Xi : \quad \begin{bmatrix} P(\xi) & A_\xi R(\xi) & B_\xi \\ R(\xi)^\top A^\top_\xi & R(\xi) + R(\xi)^\top - P(\xi) & 0 \\ B^\top_\xi & 0 & I \end{bmatrix} > 0,$$

is an LMI in the unknowns $R$ and $P_i$, $i = 1, ..., N$
New Methods and Error Bounds for PMOR

**Method 1:** Balance/truncate **constant** generalized Gramians $P$ and $Q$ (“traditional” method, similar to Wood *et al.*):

$$\forall \xi \in \Xi : \|G_\xi - \hat{G}_\xi\|_\infty \cdot 2 \sum_{i>r} \sqrt{\lambda_i(PQ)} =: \epsilon_r$$

**Method 2:** Balance/truncate **constant** slack variables $R$ and $S$ (first new method):

$$\forall \xi \in \Xi : \|G_\xi - \hat{G}_\xi\|_\infty \cdot 2 \sum_{i>r} \sqrt{\lambda_i(RS)} \cdot \epsilon_r$$

**Method 3:** Balance/truncate **affine** generalized Gramians $P_\xi$ and $Q_\xi$ (second new method):

$$\forall \xi \in \Xi : \|G_\xi - \hat{G}_\xi\|_\infty \cdot 2 \sum_{i>r} \sqrt{\lambda_i(P_\xi Q_\xi)} \cdot \epsilon_r$$
Balanced Truncation (Method 2)

1. Solve

\[
\begin{align*}
\text{minimize} & \quad \text{trace}(R) \\
\text{subject to} & \quad \begin{bmatrix} P_i & A_i R & B_i \\
R^T A_i & R + R^T - P_i & 0 \\
B_i^T & 0 & I \end{bmatrix} > 0 \quad R = R^T \leq P^*, \quad i = 1, \ldots, N
\end{align*}
\]

2. Extended Hankel values [1]: \( \sigma_{e,i} := \sqrt{\lambda_i(R^*S^*)} \)

3. Balance the optimal solutions \( R^* \) and \( S^* \):
\[
TR^*T^T = \bar{R}^* = T^{-T} S^*T^{-1} = \bar{S}^* = \text{diag}\{\sigma_{e,1}, \ldots, \sigma_{e,N}\}
\]

4. Truncate the balanced \( G_\xi \)
\[
\hat{G}_\xi \begin{cases}
\hat{x}(k+1) = \hat{A}_\xi \hat{x}(k) + \hat{B}_\xi u(k) \\
\hat{y}(k) = \hat{C}_\xi \hat{x}(k) + \hat{D}_\xi u(k)
\end{cases}, \quad \begin{bmatrix} \hat{A}_\xi & \hat{B}_\xi \\
\hat{C}_\xi & \hat{D}_\xi \end{bmatrix} := \sum_{i=1}^{N} \xi_i \begin{bmatrix} \bar{A}_i^{11} & \bar{B}_i^{1} \\
\bar{C}_i & \bar{D}_i \end{bmatrix}
\]

(For details on Methods 1 and 3, see the paper.)
Numerical Example

\[
G_\xi \left\{ \begin{array}{l}
x(k+1) = A_\xi x(k) + B_\xi u(k) \\
y(k) = C_\xi x(k) + D_\xi u(k)
\end{array} \right. , \quad \begin{bmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{bmatrix} := \sum_{i=1}^{N} \xi_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} ,
\]

where

\[
A(\alpha) = \begin{pmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.8\alpha & -0.5\alpha & 0.2 & 0.03 + \alpha \\ 0 & 0 & 1 & 0 \end{pmatrix} , \quad |\alpha| < \gamma
\]

\[
B(\beta) = \beta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + (1 - \beta) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad 0 \cdot \beta \cdot 1
\]

\[
C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} , \quad D = 0
\]

Reduce to second order!

(Model from De Oliveira et al.)
Numerical Example (Typical Path in $\Xi$)

Path 1 ($|\alpha| \leq 0.4$)

- $\square$ = Regular balanced truncation (non-PMOR, lower bound)
- $\circ$ = Method 1 ($P, Q$)
- $\times$ = Method 2 ($R, S$)
- $\oplus$ = Method 3 ($P_\xi, Q_\xi$)

- Method 2-3 close to lower bound
- Method 3 best since $\xi$-dependent coordinates
Numerical Example (Not A Typical Path in $\Xi$)

- Method 1 cannot handle uncontrollable mode at $\xi_1 = 0.5$.

- $\square$ = Regular balanced truncation (non-PMOR, lower bound)
- $\circ$ = Method 1 ($P, Q$)
- $\times$ = Method 2 ($R, S$)
- $\mathbf{+}$ = Method 3 ($P_\xi, Q_\xi$)

- Method 1-3 cannot handle uncontrollable mode at $\xi_1 = 0.5$!
A Priori $\mathcal{H}_\infty$ Error Bounds

Path 3 ($|\alpha| \leq 0.04$)

A priori error bounds:

Method 1 ($P, Q$):

Method 2 ($R, S$):

Method 3 ($P_\xi, Q_\xi$): \n
\[ \xi_4 = 1 - \xi_1 \ (\xi_2 = \xi_3 = 0) \]
Summary

- Extended balanced truncation can be applied to the PMOR problem
  - Methods 1-2: $\hat{G}_\xi$ has simple affine parameterization
  - Method 3: $\hat{G}_\xi$ has no closed-form parameterization (but Gramians have simple affine parameterization)
  - In general: Method 3 better than Method 2 better than Method 1

- Methods only work for relatively low-order models due to required solution of LMIs

- Interesting to use non-convex methods to compute non-affinely parameterized Gramians and reduced models $\hat{G}_\xi$