Introduction to Model Order Reduction

Lecture 9: Summary

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ACCESS Specialized Course
Graduate level
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Overview of today’s lecture

- Check the “course goal list”
- What did we not cover?
- Model reduction with structure constraints
- Project
- Exam
What you will learn in the course (from lecture 1)

1. Norms of signals and systems, some Hilbert space theory.
2. Principal Component Analysis (PCA)/Proper Orthogonal Decomposition (POD)/ Singular Value Decomposition (SVD).
3. Realization theory: Observability and controllability from optimal control/estimation perspective.
4. Balanced truncation for linear systems, with extension to nonlinear systems.
5. Hankel norm approximation.
6. Uncertainty and robustness analysis of models (small-gain theorem), controller reduction.
1. Norms of signals and systems, some Hilbert space theory

Signals
- Signals in time domain: $L_2[0,1)$, $L_2(-1,1)$
- Signals in frequency domain: $H_2$
- Hilbert spaces. $H_2$ isomorphic to $L_2[0,1)$

Systems/models
- Stable systems are in $H_1$
- Anti-stable systems are in $H_1^-$
- “Bounded” systems (on $L_2(-1,1)$) are in $L_1$
- “Bounded” systems with at most $r$ poles in $C$ are in $H_1^-(r)$
- Not Hilbert spaces

Tip: Take a course in functional analysis! It’s useful.
1. Norms of signals and systems, some Hilbert space theory

**Theorem 3.** Suppose $A : G \to H$ is a bounded linear operator, where $G$ and $H$ are Hilbert spaces.

i) For fixed $y \in H$, the vector $u \in G$ that minimizes $\|y - Au\|_H$ satisfies the normal equations

$$A^*Au = A^*y.$$ 

If $(A^*A) : G \to G$ is invertible, the unique optimal solution is $u = (A^*A)^{-1}A^*y$.

ii) Suppose $A$ has closed range in $H$ ($\overline{R(A)} = R(A)$). For example, $R(A)$ is finite dimensional. Then the vector $u \in G$ with the smallest norm $\|u\|_G$ satisfying $y = Au$, is given by

$$u = A^*z,$$

for any $z \in H$ that satisfies $AA^*z = y$.

If $(AA^*) : H \to H$ is invertible, the unique optimal solution is $u = A^*(AA^*)^{-1}y$.

The operator $A^* : H \to G$, is the Hilbert adjoint of $A$. By definition, it satisfies

$$(Au, y)_H = (u, A^*y)_G.$$
2. Principal Component Analysis (PCA)/Proper Orthogonal Decomposition (POD)/Singular Value Decomposition (SVD).

• Most model reduction methods rely on SVD, in one way or another!
• SVD gives optimal expansion of a matrix
• PCA = SVD for continuous functions
• POD = PCA

• Matrix norms:

\[ \| u \| := \| u \|_2 = \sqrt{u^* u}, \quad u \in \mathbb{C}^m \]
\[ \| A \| := \sup_x \frac{\| Au \|}{\| u \|} = \sqrt{\lambda_{\text{max}}(A^* A)}, \quad A \in \mathbb{C}^{n \times m} \]
\[ \| A \|_F := \left( \sum_{i=1}^n \sum_{j=1}^m |A_{ij}|^2 \right)^{1/2} = \sqrt{\text{Trace}(A^* A)}, \]
The SVD of $A$ is defined as follows. For all matrices $A \in \mathbb{C}^{n \times m}$ there exist unitary matrices

$$U = [u_1 \ldots u_n] \in \mathbb{C}^{n \times n},$$
$$V = [v_1 \ldots v_m] \in \mathbb{C}^{m \times m},$$

($U^*U = UU^* = I$ and $V^*V = VV^* = I$) such that

$$A = U \Sigma V^*,$$

where

$$\Sigma_1 = \begin{pmatrix} \sigma_1 & 0 \\ \vdots & \ddots \\ 0 & & \sigma_p \end{pmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times m},$$

$p = \min\{n, m\}$, with the singular values

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0.$$
SVD

- $\|A\| = \sigma_1 =: \sigma$;
- $\|A\|_F = \left(\sum_{i=1}^{p} \sigma_i^2\right)^{1/2}$;
- if $\sigma_1 \geq \ldots \geq \sigma_k > \sigma_{k+1} = \ldots = \sigma_p = 0$, then $\text{Rank}(A) = k$;
- $N(A) = \text{Span}\{v_{k+1}, \ldots, v_m\}$ (orthonormal basis of the nullspace of $A$);
- $R(A) = \text{Span}\{u_1, \ldots, u_k\}$ (orthonormal basis of the range space of $A$);
- $AA^* = U\Sigma^2U^*$, and $A^*A = V\Sigma^2V^*$.

**Dyadic expansion:**

$$A = \sum_{i=1}^{k} \sigma_i u_i v_i^*.$$  

$$A_r := \sum_{i=1}^{r} \sigma_i u_i v_i^*.$$  

**Schmidt-Mirsky:**

$$\min_{\text{Rank}(B) \leq r} \|A - B\| = \|A - A_r\| = \sigma_{r+1},$$

$$\min_{\text{Rank}(B) \leq r} \|A - B\|_F = \|A - A_r\|_F = \left(\sum_{i=r+1}^{p} \sigma_i^2\right)^{1/2}.$$
Consider a signal $x \in L_2^n[0, T]$, where

$$L_2^n[0, T] = \{ x : x(t) \in \mathbb{C}^n, \| x \| < \infty \},$$

$$(x, y) = \int_0^T x(t)\ast y(t) dt, \quad \| x \| = \sqrt{(x, x)} = \left( \int_0^T x(t)\ast x(t) dt \right)^{1/2}.$$  

We define the Gramian of $x$ by

$$W = \int_0^T x(t)\ast x(t) dt \in \mathbb{C}^{n \times n}.$$  

The Gramian is a Hermitian positive semidefinite matrix ($W = W^\ast$). We define the $n$ singular values of $x$ by

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0, \quad \sigma_i = \sqrt{\lambda_i(W)},$$

with $\lambda_i(W)$ being the eigenvalues of $W$. We can now expand $x$ as

$$x(t) = \sum_{i=1}^n \sigma_i u_i v_i^\ast(t),$$
PCA

- $v_i \in L^1_2[0,T], (v_i, v_j) = 0$ if $i \neq j$, and $\|v_i\| = 1$;
- $u_i \in \mathbb{C}^n, u_i^* u_j = 0$, if $i \neq j$, and $\|u_i\| = 1$;
- $v_i^*(t) = u_i^* x(t)/\sigma_i$, and $\sigma_i^2 u_i = W u_i$.

Dyadic expansion: $x_{r_i}(t) := \sum_{i=r+1}^{n} \sigma_i u_i v_i^*(t)$.

Schmidt-Mirsky: $\min_{\dim S_{y \leq r}} \|x - y\| = \|x - x_r\| = \left( \sum_{i=r+1}^{n} \sigma_i^2 \right)^{1/2}$
3. Realization theory: Observability and controllability from optimal control/estimation perspective

- Reachability Gramian $P(t)$

$$P(t) = \int_0^t e^{As} BB^T e^{ATs} ds.$$  

$$\dot{P} = AP + PA^T + BB^T, \quad P(0) = 0.$$  

- Three interpretations:
  - PCA of impulse-to-state map $x(t) = e^{At} B$,
  - Optimal control from $x(0) = 0$ to $x(T) = X_T$

$$u(t) = R_T^* (R_T R_T^*)^{-1} x_T = B^T e^{AT(T-t)} P(T)^{-1} x_T, \quad \|u\| = \sqrt{x_T^T P(T)^{-1} x_T}.$$  

- Reachability ellipsoid, states reachable with $\|u\| \cdot 1$

$$\mathcal{R} = \{x : x = U_P \Sigma_P z, \|z\| = 1\}, \quad P(T) = U_P \Sigma_P^2 U_P^T,$$
3. Realization theory: Observability and controllability from optimal control/estimation perspective

- Observability Gramian $Q_T(0)$:

$$Q_T(0) = \int_0^T e^{AT}C^TCe^{AT}dt.$$  

$$-\dot{Q}_T = A^TQ_T + Q_TA + C^TC, \quad Q_T(T) = 0.$$  

- Three interpretations:
  - Initial state to output energy
  $$\|y\|^2 = \int_0^T y(t)^Ty(t)dt = \int_0^T x_0^Te^{AT}C^TCe^{AT}x_0dt = x_0Q_T(0)x_0.$$  
  - Optimal estimation of $x(0)$ from $y(t), t \in [0,T]$ (Ex.3.2)
  $$\hat{x}_0 = Q_T(0)^{-1}O^*_Ty, \quad E[\hat{x}_0 - x_0][\hat{x}_0 - x_0]^T = Q_T(0)^{-1}$$  
  - Observability ellipsoid, $||y|| \cdot 1$ for states in
  $$\mathcal{O} = \{x: x = U_Q\Sigma_Q^{-1}z, \|z\| = 1\}, \quad Q_T(0) = U_Q\Sigma_Q^2U_Q^T$$
3. Realization theory: Observability and controllability from optimal control/estimation perspective

**Theorem 4.** A realization \((A, B, C, D)\) of \(G\) is controllable if, and only if, \(P(T)\) is invertible for all \(T > 0\). The reachable subspace is spanned by the component vectors \(u_1, \ldots, u_r\) in \(U_P\) that correspond to the \(r \leq n\) strictly positive singular values in \(\Sigma_P\). The singular value \(\sigma_i\) quantifies how far one can reach in the direction \(u_i\) using the input energy \(\|u\| \leq 1\).

**Theorem 5.** A realization \((A, B, C, D)\) of \(G\) is observable if, and only if, \(Q_T(0)\) is invertible for all \(T > 0\). The unobservable subspace is spanned by the component vectors \(u_{r+1}, \ldots, u_n\) in \(U_Q\) that correspond to the possible zero singular values in \(\Sigma_Q\). The singular value \(\sigma_i\) quantifies how much energy \(\|y\|\) there is if \(x_0 = u_i\).

The Gramians contain more information than the controllability/observability matrices

\[
C = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}, \quad O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}
\]
4. Balanced truncation for linear systems, with extension to nonlinear systems.

- Model reduction in two steps:

  Step 1: Change the coordinates $x(t)$. That is, find a suitable invertible matrix $T \in \mathbb{R}^{n \times n}$ and transform the state-space model according to

  $$
  \tilde{A} = T^{-1}AT = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \quad \tilde{A}_{11} \in \mathbb{R}^{r \times r},
  \quad \tilde{B} = T^{-1}B = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}, \quad \tilde{B}_1 \in \mathbb{R}^{r \times m},
  $$

  $$
  \tilde{C} = CT = \begin{pmatrix} \tilde{C}_1 & \tilde{C}_2 \end{pmatrix}, \quad \tilde{C}_1 \in \mathbb{R}^{p \times r},
  \quad \tilde{D} = D.
  $$

- Choose balanced coordinates and then

  Step 2: Depending on method, define $G_r$ by:

  - Truncation:

  $$
  A_r = \tilde{A}_{11}, \quad B_r = \tilde{B}_1
  $$

  $$
  C_r = \tilde{C}_1, \quad D_r = \tilde{D}.
  $$
4. Balanced truncation for linear systems, with extension to nonlinear systems.

- Or

  - Singular perturbation:

    \[ A_r = \bar{A}_{11} - \bar{A}_{12} \bar{A}_{22}^{-1} \bar{A}_{21}, \quad B_r = \bar{B}_1 - \bar{A}_{12} \bar{A}_{22}^{-1} \bar{B}_2 \]
    \[ C_r = \bar{C}_1 - \bar{C}_2 \bar{A}_{22}^{-1} \bar{A}_{21}, \quad D_r = \bar{D} - \bar{C}_2 \bar{A}_{22}^{-1} \bar{B}_2. \]

  Note the following:

  (P1) using truncation, we have \( G(\infty) = G_r(\infty) \); and

  (P2) using singular perturbation, we have \( G(0) = G_r(0) \).

- Balanced coordinates make the Gramians equal and diagonal

  \[ \Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}, \quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0 \]

  \[ \bar{P} = \bar{Q} = \Sigma. \]

- Hankel singular values \( \sigma_i := \sqrt{\lambda_i(PQ)} \)
Error bounds for balanced model reduction (prime motivation)

**Theorem 6.** Suppose \((A, B, C, D)\) is a balanced realization and that \((A_{11}, B_1, C_1, D)\) is a balanced truncation. Then \(A_{11}\) is Hurwitz, and \((A_{11}, B_1, C_1, D)\) is a minimal and balanced realization of \(G_r\) with Gramian \(\Sigma_1\). Furthermore,

\[
\|G - G_r\|_\infty \leq 2 \sum_{i=l+1}^{m} \sigma_i.
\]

When \(l = m - 1\) equality holds, and \(\|G(0) - G_r(0)\| = 2\sigma_m\) if \(r_m\) is odd.

Note that for \(A_{11}\) to be guaranteed Hurwitz it is important that \(\sigma_l \neq \sigma_{l+1}\).

**Theorem 7.** Suppose \((A, B, C, D)\) is a balanced realization and that

\[
(A_r, B_r, C_r, D_r) := (A_{11} - A_{12}A_{22}^{-1}A_{21}, B_1 - A_{12}A_{22}^{-1}B_2, C_1 - C_2A_{22}^{-1}A_{21}, D - C_2A_{22}^{-1}B_2)
\]

is a singularly perturbed realization. Then \(A_r\) is Hurwitz, and \((A_r, B_r, C_r, D_r)\) is a minimal and balanced realization of \(G_r\) with Gramian \(\Sigma_1\). Furthermore,

\[
\|G - G_r\|_\infty \leq 2 \sum_{i=l+1}^{m} \sigma_i,
\]

with equality if \(l = m - 1\).
Nonlinear extension

- **POD:**
  \[ \dot{x} = f(x), \quad x(t) \in \mathbb{R}^n \]
  \[ X = [x(t_1) \ x(t_1) \ \ldots \ x(t_N)] \in \mathbb{R}^{n \times N}. \]
  \[ X = U \Sigma V^T = [u_1 \ u_2 \ \ldots \ u_n] \Sigma V^T \]
  \[ \approx U_r \Sigma_r V_r^T = [u_1 \ u_2 \ \ldots \ u_r] \Sigma_r V_r^T, \quad r < n. \]
  \[ \dot{z} = f_r(z) = U_r^T f(U_r z), \quad z(t) \in \mathbb{R}^r. \]

- **Empirical Gramians:**
  \[ \dot{x} = f(x, u), \quad x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^m \]
  \[ y = h(x, u), \quad y(t) \in \mathbb{R}^p \]
  \[ P(T) = \sum_{i=1}^m \sum_{j=1}^r \sum_{k=1}^s \frac{1}{r s c_k^2} \int_0^T \Phi_{ijk}(t) dt, \quad Q_T(0) = \sum_{j=1}^r \sum_{k=1}^s \frac{1}{r s c_k^2} \int_0^T T_j \Psi_{jk}(t) T_j^T dt, \]
Nonlinear extension

\[ \tilde{P}(T) = \tilde{Q}_T(0) = \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ \vdots & \ddots \\ 0 & \sigma_n \end{pmatrix}, \]

\[ T = (t_1 \ t_2 \ \ldots \ t_n) \in \mathbb{R}^{n \times n}, \]
\[ T^{-1} = (s_1 \ s_2 \ \ldots \ s_n)^T \in \mathbb{R}^{n \times n}, \]
\[ V = (t_1 \ t_2 \ \ldots \ t_r) \in \mathbb{R}^{n \times r}, \]
\[ W = (s_1 \ s_2 \ \ldots \ s_r) \in \mathbb{R}^{n \times r}. \]

\[ \dot{z} = f_r(z, u) = W^T f(Vz, u), \quad z(t) \in \mathbb{R}^r, \ u(t) \in \mathbb{R}^m \]
\[ y_r = h_r(z, u) = h(Vz, u), \quad y(t) \in \mathbb{R}^p. \]

- The nonlinear methods do not guarantee anything! Extensive simulation necessary!
5. Hankel norm approximation

- "SVD of linear systems"
- To hard for regular operator $G$

\[ G : L_2(-\infty, \infty) \to L_2(-\infty, \infty) : \quad (Gu)(t) = \int_{-\infty}^{t} Ce^{A(t-s)}Bu(s)ds + Du(t) \]

- Works for the Hankel operator $\Gamma_G$

\[ \Gamma_G : L_2(-\infty, 0] \to L_2[0, \infty) : \quad (\Gamma_Gu)(t) = \int_{-\infty}^{0} Ce^{A(t-s)}Bu(s)ds, \quad t > 0. \]

- Dyadic expansion of $\Gamma_G$

\[ (\Gamma_Gu)(t) = \sum_{i=1}^{n} \sigma_i u_i(t)(v_i, u)_{L_2(-\infty, 0]}, \]

- Simple truncation does not work in this case, however. Truncation does generally not result in a Hankel operator $\Gamma_{Gr}$. Use AAK-theorem.
5. Hankel norm approximation

**Theorem 12** (Nehari). Suppose that $G \in H_\infty$ and $F \in H_\infty^-$. Then $G - F \in L_\infty$, and

$$\min_{F \in H_\infty^-} \|G - F\|_\infty = \|G\|_H (= \sigma_1).$$

**Theorem 13** (Adamjan-Arov-Krein). Suppose that $G \in H_\infty$ and $Q \in H_\infty^-(r)$. Then $G - Q \in L_\infty$, and

$$\min_{Q \in H_\infty^-(r)} \|G - Q\|_\infty = \sigma_{r+1},$$

where $\sigma_{r+1}$ is the $(r + 1)$-th largest Hankel singular value of $G$.

- $\|\|G\|_H := \|\|\Gamma_G\| \cdot \|G\|_1$
- $\sigma_{r+1} \cdot \|G - G_r\| \cdot \|G - G_r\|_1$
- $G_r = [Q^*]_+$ where $[.]_+$ is the stable projection/truncation of $[.]$
- $\|G - G_r\|_H = \|\Gamma_G - \Gamma_{G_r}\| = \sigma_{r+1}$
- There are always direct terms $D$ such that $\|G - G_r - D\|_1 \cdot \sigma_{r+1} + \ldots + \sigma_n$
6. Uncertainty and robustness analysis of models (small-gain theorem), controller reduction

**Theorem 9** (Small-gain theorem). Suppose that $\tilde{G}, \Delta \in H_\infty$. The feedback interconnection of $\tilde{G}$ and $\Delta$ in Figure 5.2 is stable if $\|\tilde{G}\Delta\|_\infty \leq \|\tilde{G}\|_\infty \|\Delta\|_\infty < 1$. 

![Diagram of feedback interconnection](image-url)
Controller reduction

- Find a reduced-order controller $K_r = K + \Delta$ of $K$, that preserve closed-loop properties

- Robustness:
  \[ \| (I + GK)^{-1} G \Delta \|_\infty < 1 \quad \text{or} \quad \| \Delta G (I + KG)^{-1} \|_\infty < 1 \]

- Performance $r$ to $y$ (first order Taylor approximation):
  \[ (I + GK)^{-1} G (K_r - K) (I + GK)^{-1} \]
Plant reduction in closed-loop

\[ \| \Delta G_r K (I + G_r K)^{-1} \|_\infty < 1. \]

- For frequencies up to bandwidth: \( \tilde{G}_r K (I + \tilde{G}_r K)^{-1} \approx I \)
- \( \| (G - G_r) G_r^{-1} \|_\infty = \| G_r^{-T} (G^T - G_r^T) \|_\infty \)
- Frequency-weighted model reduction!
Weighted balanced truncation

- Controller reduction criterion: $||W_o(G - G_r)W_i||_\infty$

- Realization:

$$
\tilde{A} = \begin{bmatrix} A & 0 & BC_i \\ B_oC & A_o & 0 \\ 0 & 0 & A_i \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} BD_i \\ 0 \\ B_i \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} D_oC & C_o & 0 \end{bmatrix}
$$

- Weighted Gramians:

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}^T + \tilde{B}\tilde{B}^T = 0$$
$$\tilde{Q}\tilde{A} + \tilde{A}^T\tilde{Q} + \tilde{C}^T\tilde{C} = 0.$$

$$P := [I_n \ 0] \tilde{P} \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad Q := [I_n \ 0] \tilde{Q} \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$
Balanced stochastic truncation

- Plant reduction/relative criteria:
  \[ \| G^{-1} (G - G_r) \|_\infty \]
  \[ \| (G - G_r) G_r^{-1} \|_\infty = \| G_r^{-T} (G_r^T - G_r^T) \|_\infty \]

- Bounds on error in Bode plot
  \[ \Delta(j\omega) = \frac{(G(j\omega) - G_r(j\omega))}{G(j\omega)} \]
  \[ 20 \log_{10} \left| \frac{G_r(j\omega)}{G(j\omega)} \right| \leq 8.69 |\Delta(j\omega)| \text{ dB} \]
  \[ |\text{phase } G(j\omega) - \text{phase } G_r(j\omega)| \leq |\Delta(j\omega)| \text{ rad.} \]

- Realization:
  \[ \tilde{A} = \begin{bmatrix} A & 0 \\ -BD^{-1}C & A - BD^{-1}C \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} D^{-1}C & D^{-1}C \end{bmatrix} \]
Balanced stochastic truncation

**Theorem 8.** Suppose $G, G^{-1} \in H_\infty$, and let $G_r$ be a truncated realization of $G$ that has been balanced with the weighted Gramians of (4.7). Then $G_r$ is stable and minimum phase, $G_r, G_r^{-1} \in H_\infty$, and satisfies

$$
\|G^{-1}(G - G_r)\|_\infty \leq \prod_{i=l+1}^{m} \left( 1 + 2\sigma_i \left( \sqrt{1 + \sigma_i^2} + \sigma_i \right) \right) - 1 \\
\|G_r^{-1}(G - G_r)\|_\infty \leq \prod_{i=l+1}^{m} \left( 1 + 2\sigma_i \left( \sqrt{1 + \sigma_i^2} + \sigma_i \right) \right) - 1,
$$

- If $G^{-1}$ is not in $H_1$, use stable and minimum phase spectral factor instead.
H$_2$-model reduction

- H$_2$-model reduction was introduced by Bo
- Statistical measure of model error
- Good for system identification
- Weights reflect signal-to-noise ratio
- No simple a priori error bounds
- Complexity of method depends on model structure. FIR and ARX simple.
- Generally nonconvex optimization and iterative methods
What you will learn in the course

1. Norms of signals and systems, some Hilbert space theory.
2. Principal Component Analysis (PCA)/Proper Orthogonal Decomposition (POD)/ Singular Value Decomposition (SVD).
3. Realization theory: Observability and controllability from optimal control/estimation perspective.
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You did it!
Classification of methods

- $O(n^3)$ operations
- $n \sim 100-1000$
- Provable stability, bound($r$) exist

- $O(r^2n)$ operations
- $n \sim 10^6$
- Not provable stability, bound($r$) does not exist

[Figure from Antoulas: *Approximation of Large-Scale Dynamical Systems*]
What did you not learn?

- **Krylov-methods:**
  - Iterative methods with no guarantees
  - Matches moments
  - Computationally “cheap”: $O(r^2n)$
  - For really large models, PDEs
  - Numerical linear algebra

- **LMI (Linear Matrix Inequality)-methods:**
  - For uncertain/time-varying/nonlinear/structured systems
  - Semidefinite programs/convex optimization
  - Generalized Gramians
  - Computationally “expensive”: $O(n^4)$...$O(n^6)$
  - For small systems
  - Optimization/control theory
What did you not learn?

- “Non-standard norm”-methods
  - Real-part norm: \( \| \text{Re } G - \text{Re } G_r \|_1 \)
  - Magnitude matching: \( \| \log |G| - \log |G_r| \|_1 \)
  - Phase matching: \( \| \text{phase } G - \text{phase } G_r \|_1 \)
  - Megretski’s method:
    \[
    \min_{A_r, B_r, C_r} \left\| G - \frac{B_r}{A_r} - \frac{C_r}{A_r^\sim} \right\|_\infty, \quad G_r = \frac{B_r}{A_r}
    \]
  - Analytic interpolation (Nevanlinna-Pick)
  - Convex optimization
  - Frequency data samples
Problem 1: “The standard problem”

Given:

\[ G : \begin{cases} \dot{x}(t) = f(x(t), u(t)), & x(t) \in \mathbb{R}^n, u \in \mathcal{U} \\ y(t) = g(x(t), u(t)) \end{cases} \]

Find:

\[ G_r : \begin{cases} \dot{z}(t) = f_r(z(t), u(t)), & z(t) \in \mathbb{R}^r, u \in \mathcal{U} \\ y_r(t) = g_r(z(t), u(t)) \end{cases} \]

Such that

\[ \|y - y_r\| \leq \text{bound}(r) \cdot \|u\|, \quad \forall u \in \mathcal{U} \]

\[ \text{bound}(r) \downarrow 0 \text{ (monotonic)}, \quad r \to n. \]
Problem 2: Model reduction with structure constraints

- States in the model $G$ are physically constrained to certain blocks, for example.
- Example: $G_1$ is a plant. $G_2$ is a controller.
Explanation

- Find proper “inputs” and “outputs” to each subsystem which reflect the subsystem’s interaction with the global system.
- Then apply methods that solve Problem 1.

Motivation:
1. Low-order feedback/feedforward controllers
2. Large interconnected systems in computer science and biology
3. Modular model reduction
One heuristic:
Generalized weighted balanced truncation

\[ AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0. \]

\[ Q = \begin{bmatrix} Q_N & Q_{NG} \\ Q_{NG}^T & Q_G \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & \cdots & Q_{1q} \\ \vdots & \ddots & \vdots \\ Q_{1q}^T & \cdots & Q_q \end{bmatrix} \]

\[ P = \begin{bmatrix} P_N & P_{NG} \\ P_{NG}^T & P_G \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & \cdots & P_{1q} \\ \vdots & \ddots & \vdots \\ P_{1q}^T & \cdots & P_q \end{bmatrix} \]
Mechanical example

\[ w = u_{1,1}, \quad u_{1,2}, \quad u_2 \]

\[ z_1 = y_1, \quad z_2 = y_2 \]

Structured Hankel singular values (Method 1)

Hankel singular values
To get credits, you need to complete...

1. Exercises
   - At the end of the course, at least 75% of the exercises should have been solved and turned in on time.

2. Project
   - See upcoming slide.

3. Exam
   - A 48h hand-in exam.
2. Project

- Choose a model that preferably relates as much as possible to your own research project. If you don’t have a good model, talk to me.

- You should write a report and turn it in at the end of the course. The report should (at least) contain the following sections:
  1. Introduction. Describe or derive the model, and explain why it should be reduced. What is there to gain by model reduction? Give a clear problem formulation. Choices of inputs and outputs.
  2. Apply and analyze the result of model reduction method 1 (R1)
  3. Apply and analyze the result of model reduction method 2 (R2)
  4. Summarize what has been achieved. What worked, what did not work. Objectives achieved?

- R1 and R2 can be methods we have talked about in the course, a method you have found in the literature, or a method you come up with yourself. Method choices should be well motivated.

- **End date: 2008-12-31**
3. Exam

- **Exam period**: 2008-10-31 to 2008-12-31
- **Instructions**: You can take the exam anytime you want to during the above exam period. After collecting the exam, you have 48 hours to complete it. *No cooperation* is allowed, but you are allowed to use any other material you like. All solutions should be *well motivated*, and if you use computer tools to solve problems, you should hand in both *figures* and *source code*.

- 5 problems similar to the exercises.

- *Note that I will be away Dec. 7 – Dec. 31. I recommend that you take the exam before this, but I can send you the exam with an email if you want to take it during this period.*