Introduction to Model Order Reduction

Lecture 1: Introduction and overview

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Automatic Control Lab, KTH

ACCESS Specialized Course

Graduate level

Ht 2010, period 1

Overview of Today's Lecture

- What is model (order) reduction? Why is it important?
- What is included in the course? What is **not** included?
- Preliminary program
- What is expected from you? How to pass?
- Sign up for course

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Model (Order) Reduction

- **~1 000 000** hits in Google ...
- www.modelreduction.com , web.mit.edu/mor/ , ...
- Many different research communities use different forms of model reduction:

  Fluid dynamics

  Mechanics

  Computational biology

  Circuit design

  Control theory

  ...
- Many heuristics available. More or less well-motivated.
- In early 1980's some optimal approaches were developed (using AAK-lemma) in control theory.
- Not much known for nonlinear systems.

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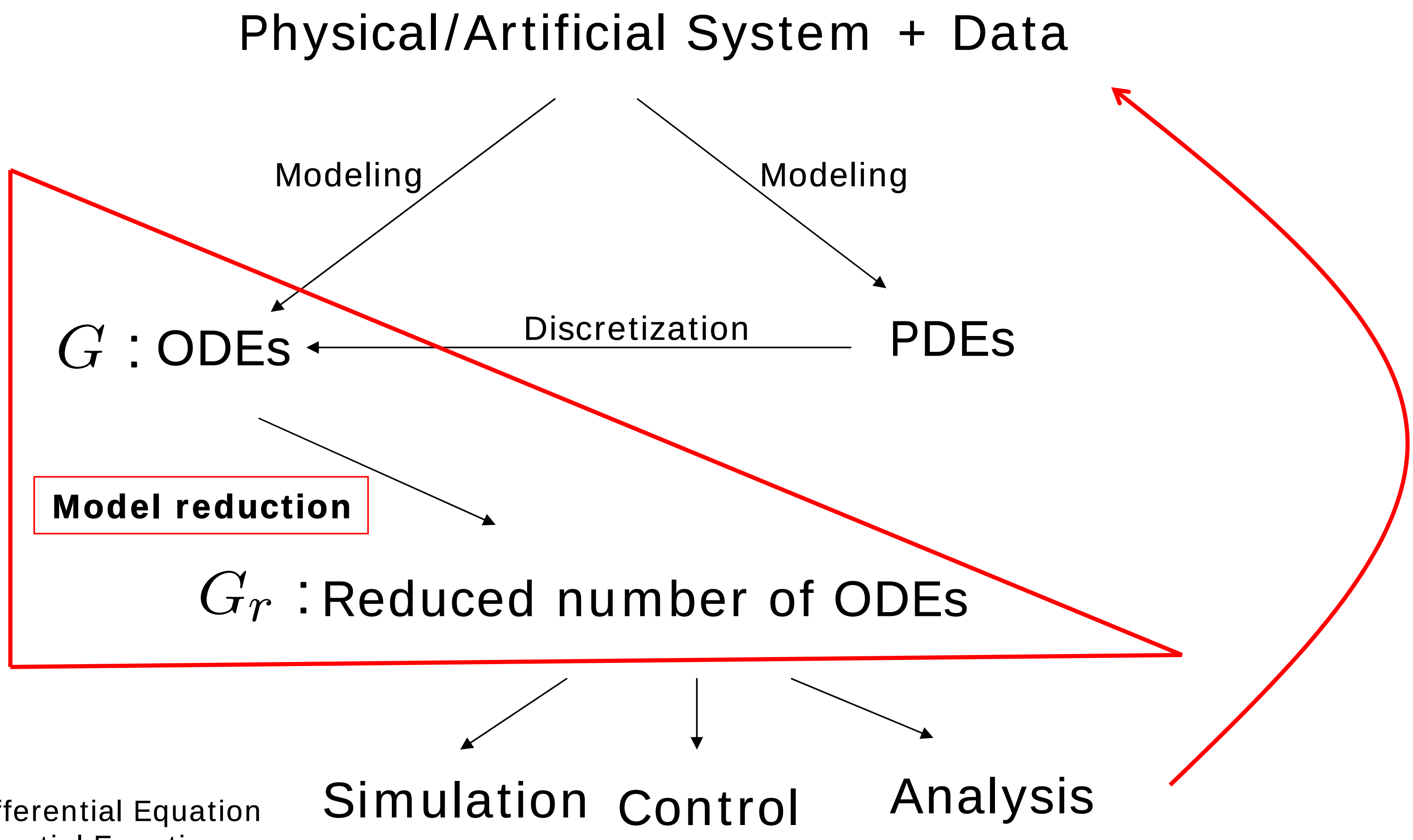
The Big Picture



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ODE = Ordinary Differential Equation
PDE = Partial Differential Equation

An Incomplete Problem Formulation

Given an ODE of order n

$$G: \quad \dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n.$$

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Find another ODE of order r

$$G_r: \quad \dot{z}(t) = f_r(z(t)), \quad z(t) \in \mathbb{R}^r, r \ll n$$

with “essentially” the same “properties”.

Not enough information for problem to make complete sense, although this captures the essence of the model-order-reduction problem.

Problem 1: "The standard problem"

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Given:

$$G : \begin{cases} \dot{x}(t) = f(x(t), u(t)), & x(t) \in \mathbb{R}^n, \ u \in \mathcal{U} \\ y(t) = g(x(t), u(t)) \end{cases}$$

Find:

$$G_r : \begin{cases} \dot{z}(t) = f_r(z(t), u(t)), & z(t) \in \mathbb{R}^r, \ u \in \mathcal{U} \\ y_r(t) = g_r(z(t), u(t)) \end{cases}$$

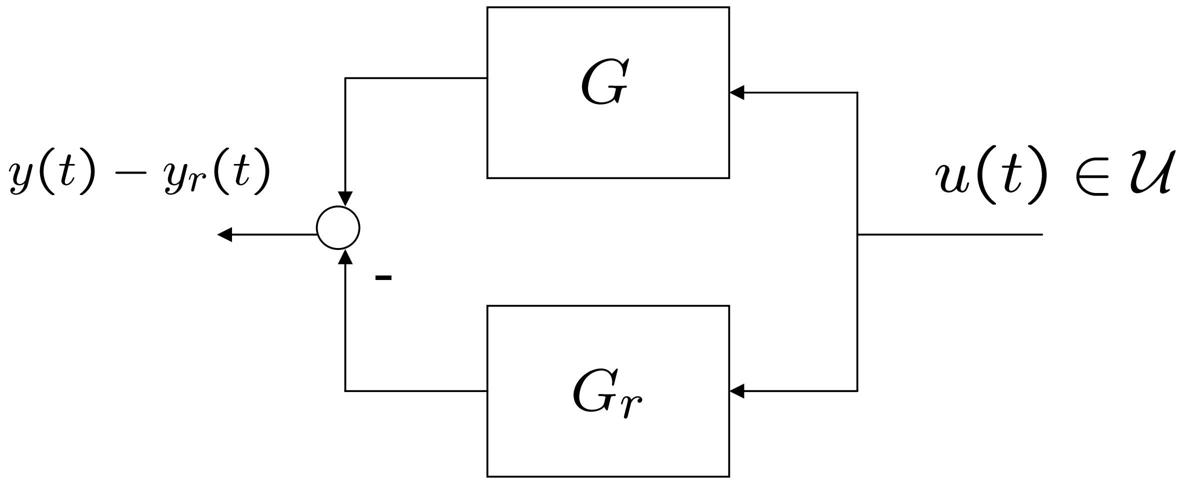
Such that

$$\|y - y_r\| \leq \text{bound}(r) \cdot \|u\|, \quad \forall u \in \mathcal{U}$$

$$\text{bound}(r) \downarrow 0 \ (\text{monotonic}), \quad r \to n.$$

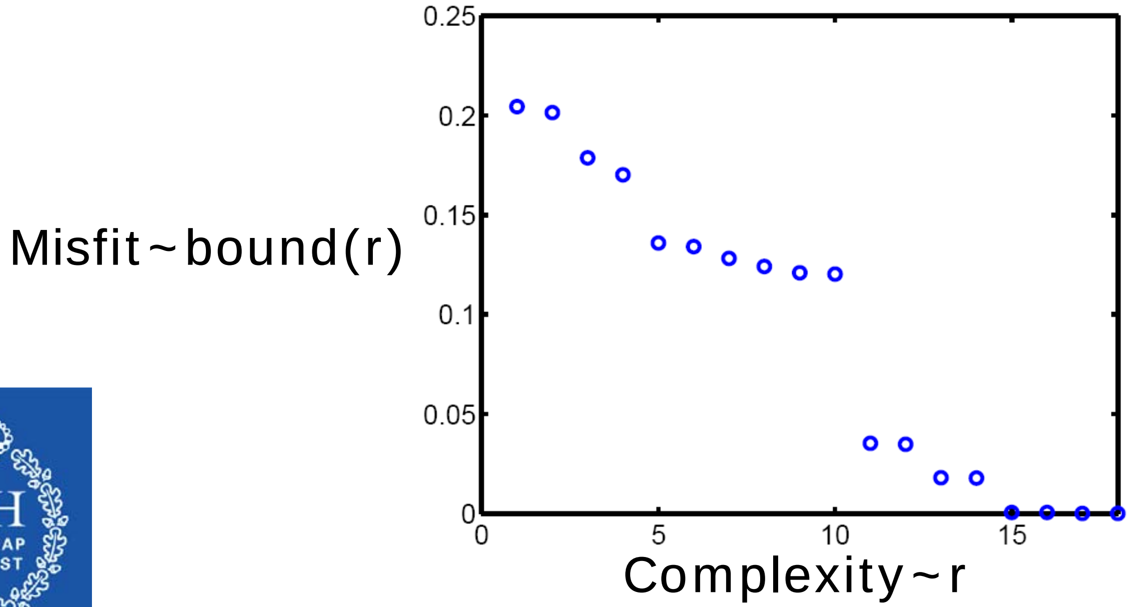
Problem 1 (cont'd)

- Choice of input u(t) determines what states are excited. Could also reflect initial conditions x(0).
- Choice of output y(t) determines what property of the states we want to preserve.



- Choice of inputs and outputs essential!

Problem 1 (cont'd)



- The function bound(r) quantifies (or bounds) the *complexity-misfit trade-off*

$$\|y - y_r\| \leq \text{bound}(r) \cdot \|u\|, \quad \forall u \in \mathcal{U}$$

$$\text{misfit}(G, G_r) = \sup_{u \in \mathcal{U}} \frac{\|y - y_r\|}{\|u\|} \leq \text{bound}(r)$$

$$\text{complexity}(G) = n, \quad \text{complexity}(G_r) = r.$$

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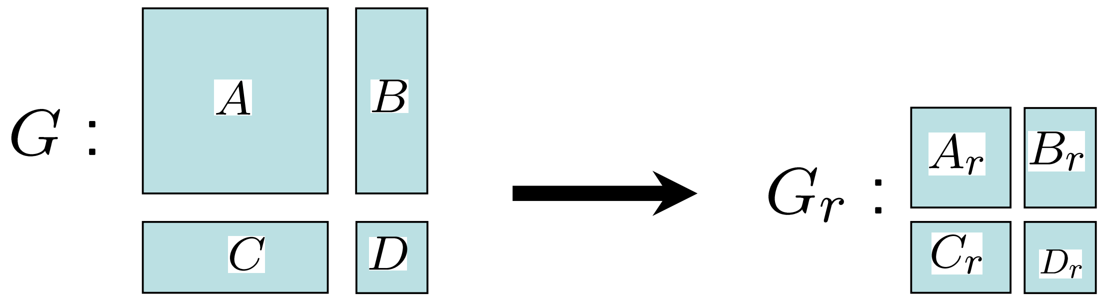
Problem 1 (cont'd)

- Often the linear problem will be treated:

$$G: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(t) \in \mathbb{R}^n,\ u \in L_2[0,\infty) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$$G_r: \begin{cases} \dot{z}(t) = A_r z(t) + B_r u(t), & z(t) \in \mathbb{R}^r,\ u \in L_2[0,\infty) \\ y_r(t) = C_r z(t) + D_r u(t) \end{cases}$$



Problem 1 (cont'd)

A good model-reduction method gives us:

1. bound(r) – To help us choose a suitable approximation order r; and
2. a reduced-order model (f_r,g_r) alt. (A_r,B_r,C_r,D_r).

Such methods exist for some classes of models (typically linear). Many heuristics fail to provide bound(r).

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Why Decrease the Order?

(Why complexity(G)=n?)

- Simulation: Each evaluation of f(x(t,u(t))) is $O(n^2)$ operations in linear case.
- Simulation: Data compression, roughly $O(n^2)$ numbers to store a linear model.

- Control: Computation time of LQG controller is $O(n^3)$ operations (solve the Riccati equation).
- Control: Optimal controller is at least of order n $\Rightarrow$ can be hard to implement.

- Analysis: Curse of dimensionality. Problem complexity often exponential in number of equations (=order).

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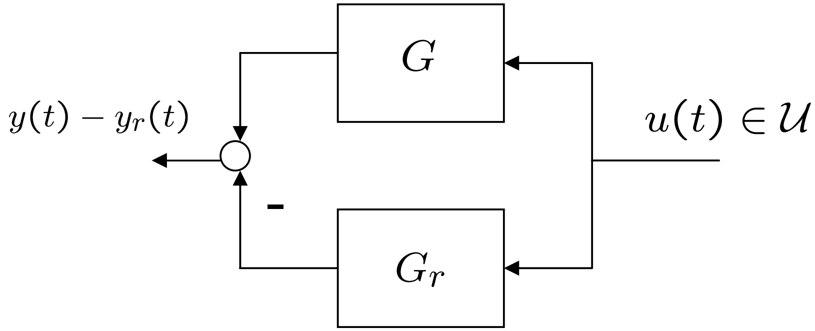
Why Define Misfit This Way?



$$\text{misfit}(G, G_r) = \sup_{u \in \mathcal{U}} \frac{\|y - y_r\|}{\|u\|} \leq \text{bound}(r)$$

- Misfit is a measure of the worst-case error of the approximation. Can be pessimistic...
- Other measures are possible, statistical for example.
- Worst-case error often good for control theory (robust control theory).
- Simple expressions of bound(r) are available for worst-case errors, but not for statistical error measures.

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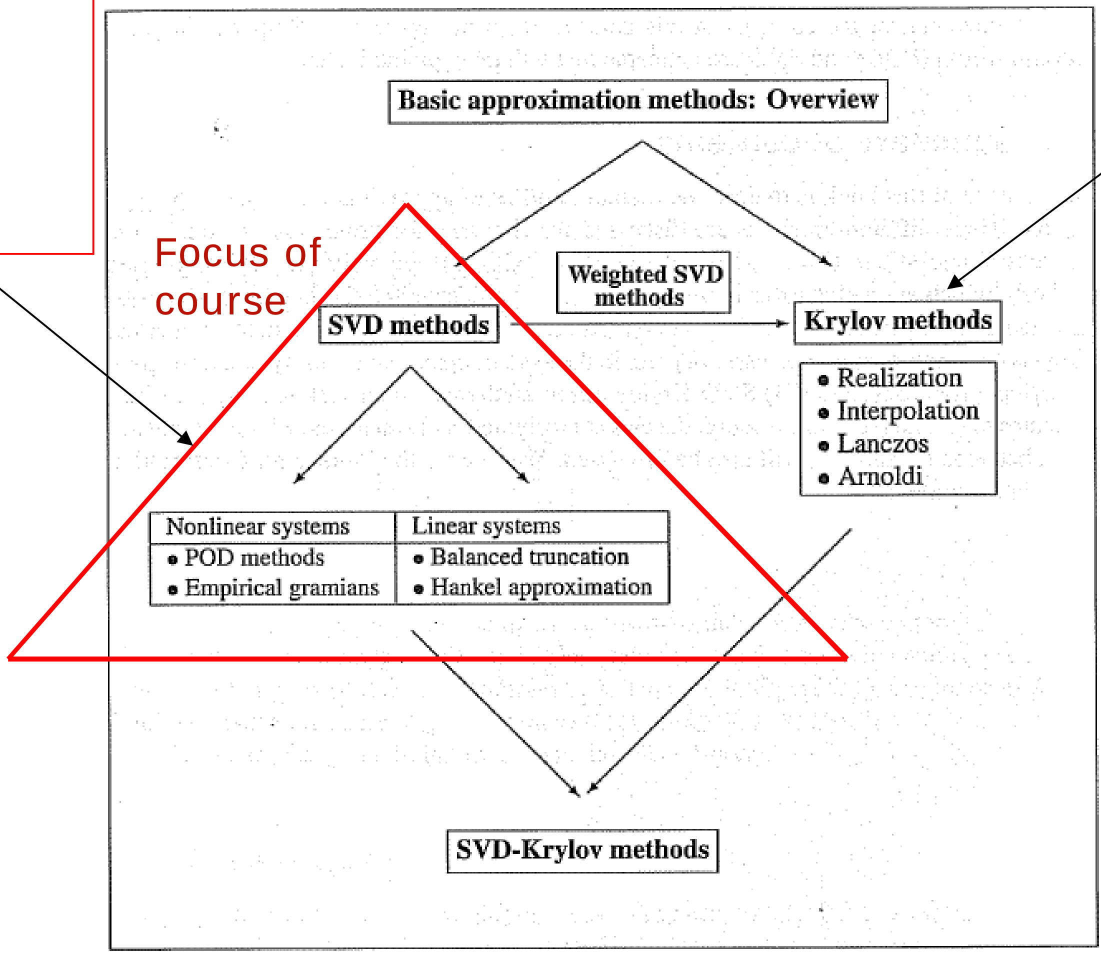
Classification of Methods

- $O(n^3)$ operations
- $n \sim 100\text{-}500$
- Provable stability, bound(r) exist



- $O(r^2 n)$ operations
- $n \sim 10^6$
- Not provable stability, bound(r) does not exist

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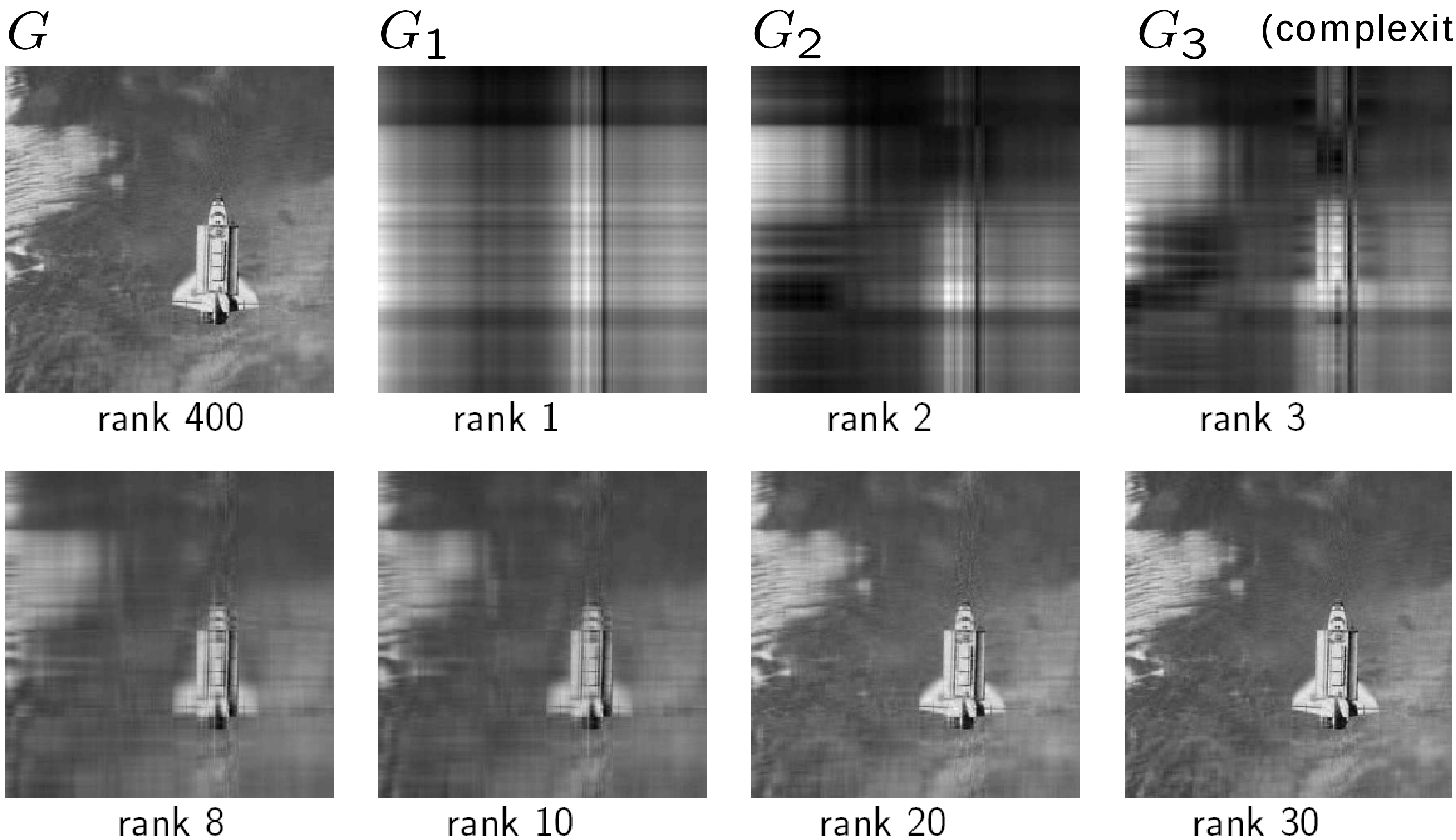
[Figure from Antoulas: *Approximation of Large-Scale Dynamical Systems*]

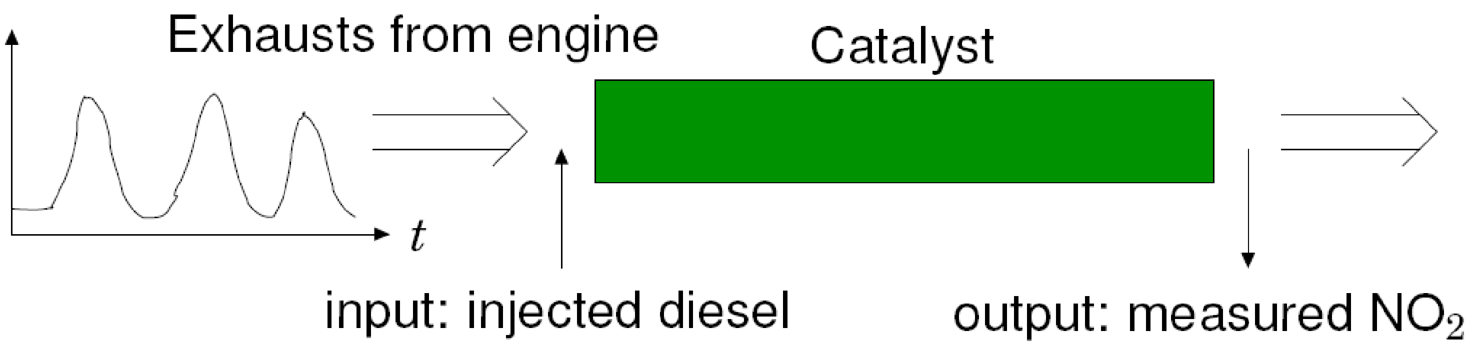
Example 1: Image compression



(complexity ~ rank)

Example 2: Chemical Reactions

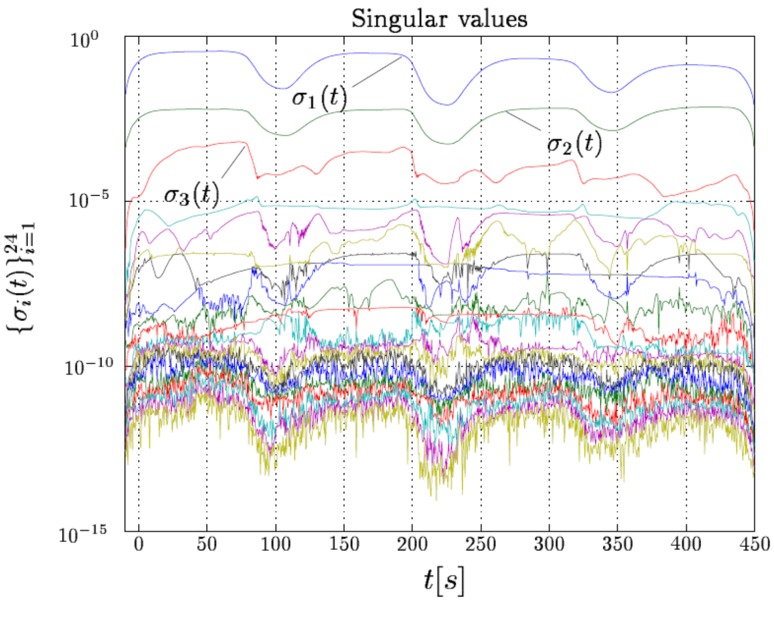
- Model reduction of a diesel exhaust catalyst from [Sandberg, 2006].

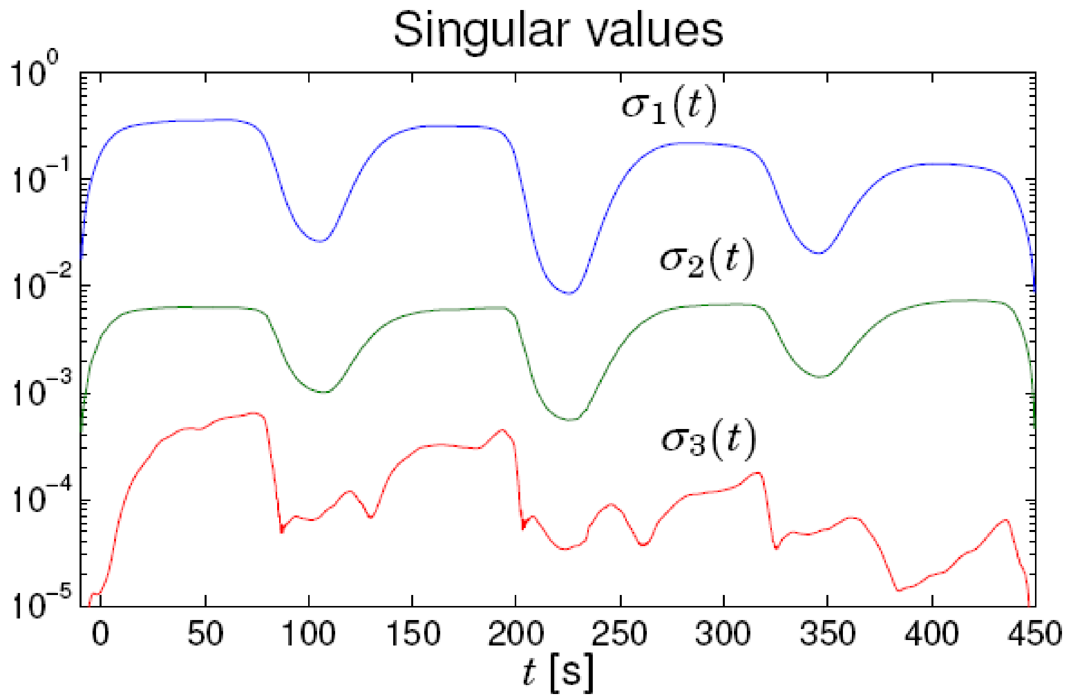


- Reduction of NO_x
- Model by Westerberg et al. ('02)
- 24 nonlinear ODEs. Linearize around pulsating trajectory $\rightarrow \{A(t), B(t), C(t)\}$

Example 2 (cont'd)

- Complexity-misfit trade-off

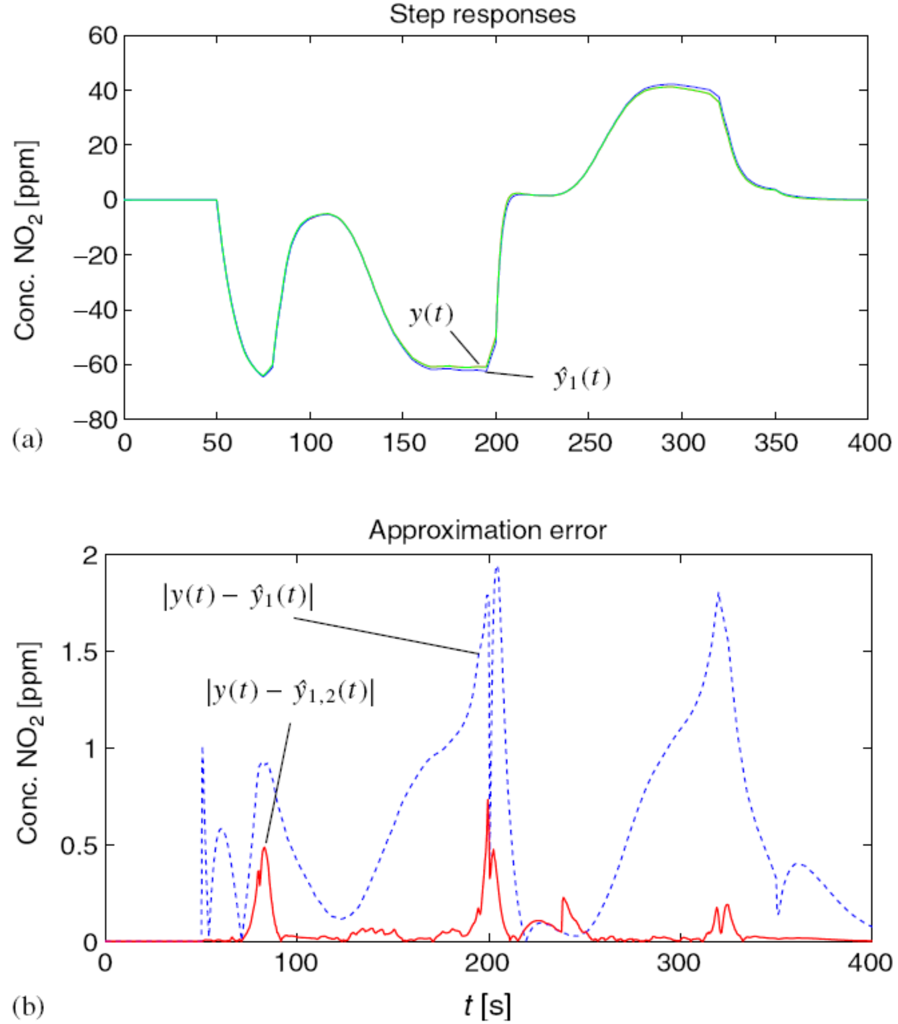




$$\text{bound}(r) = \sum_{i=r+1}^{n} f(\sigma_i(t))$$

Example 2 (Cont'd)

- Verification using r=1 and r=2



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Explanation

- Kalman decomposition:

$G:$

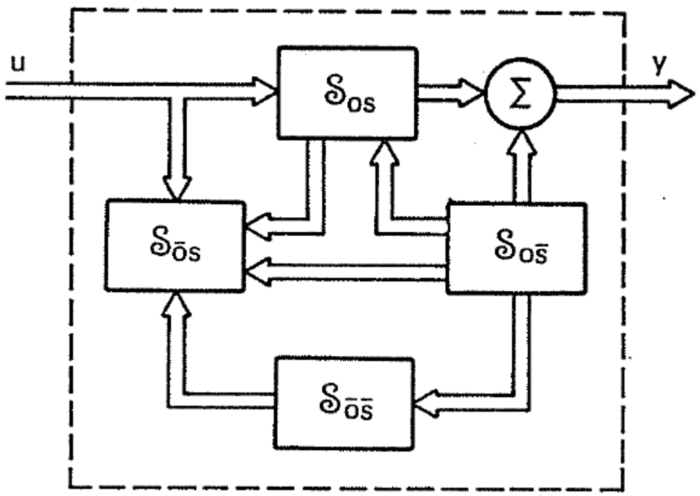

Fig. 6.2. Blockschema som illustrerar Kalmans uppdelning av ett godtyckligt system i delsystemen $\mathcal{S}_{os}$, $\mathcal{S}_{o\bar{s}}$, $\mathcal{S}_{\bar{o}s}$, $\mathcal{S}_{\bar{o}\bar{s}}$.

[Figure from Åström: *Reglerteori*]

- Only S_{os} contribute to the mapping $u(t) \rightarrow y(t)$.
- Also, states in S_{os} do not contribute equally.
- $G_r = S_{os}$ is one obvious reduced model candidate, but we can often do much better!

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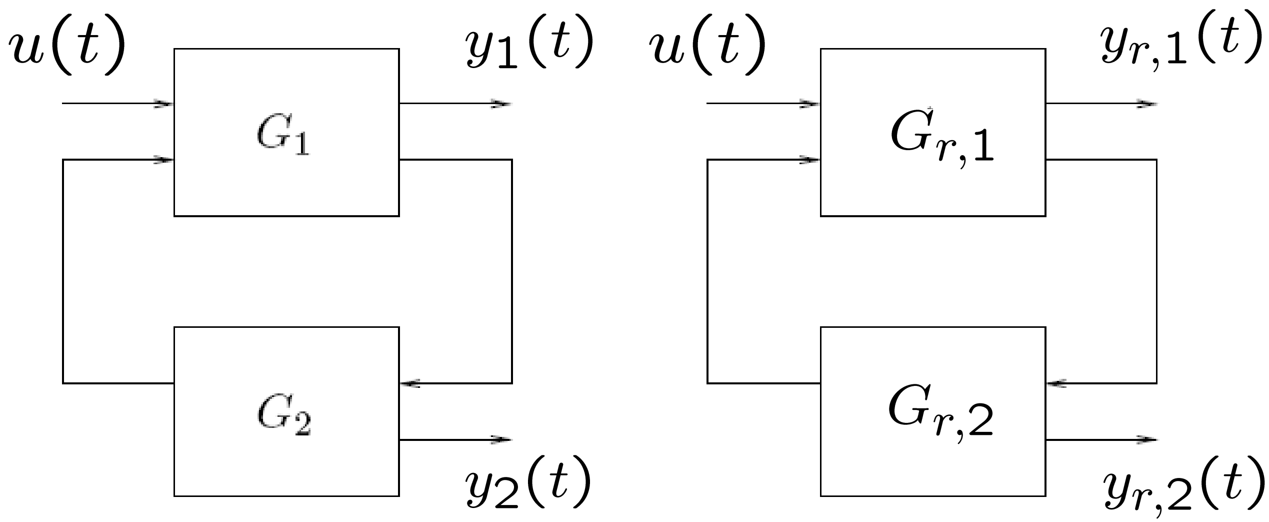
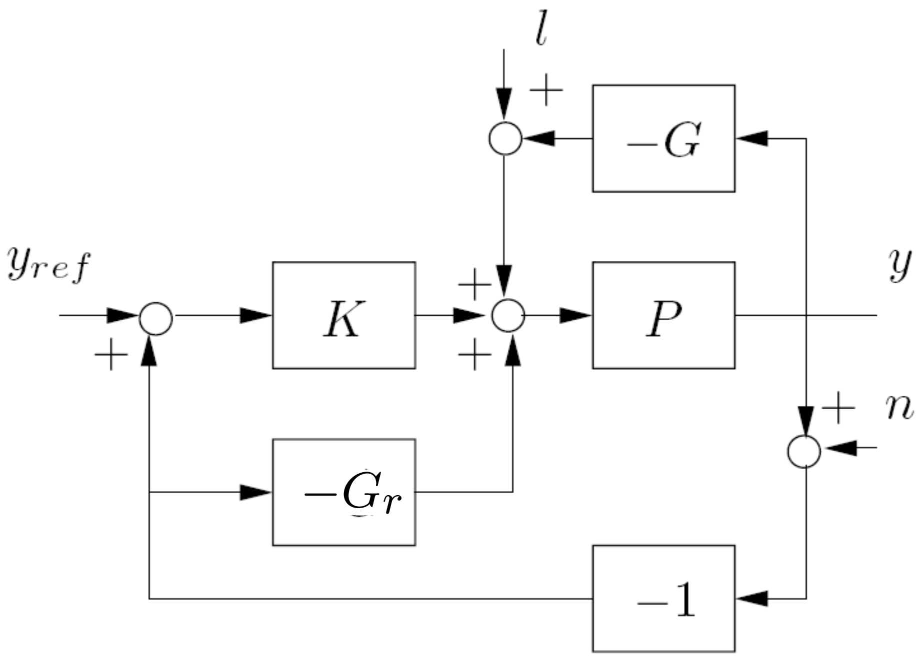
Problem 2: Model Reduction with Structure Constraints



- States in the model G are physically constrained to certain blocks, for example.
- Example: G_1 is a plant. G_2 is a controller.

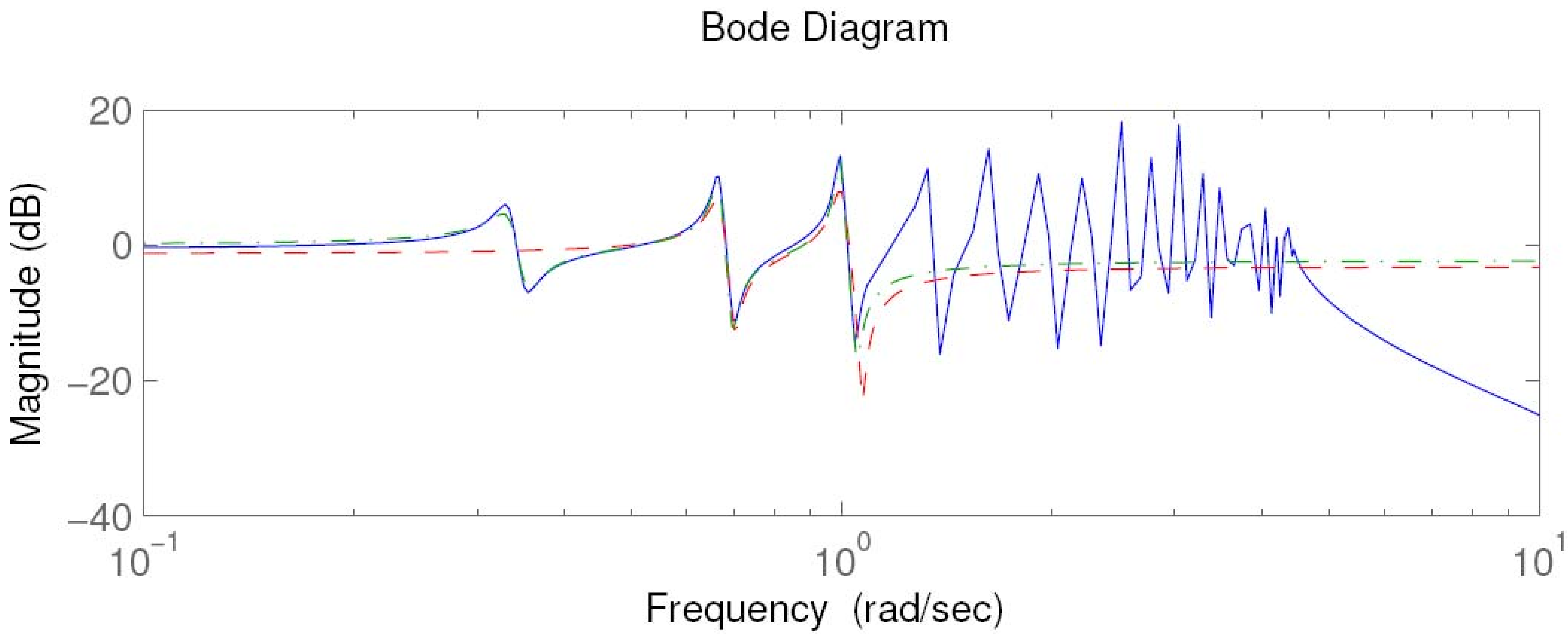
Example 3: Networked Control

- Example from [Sandberg and Murray, 2007].



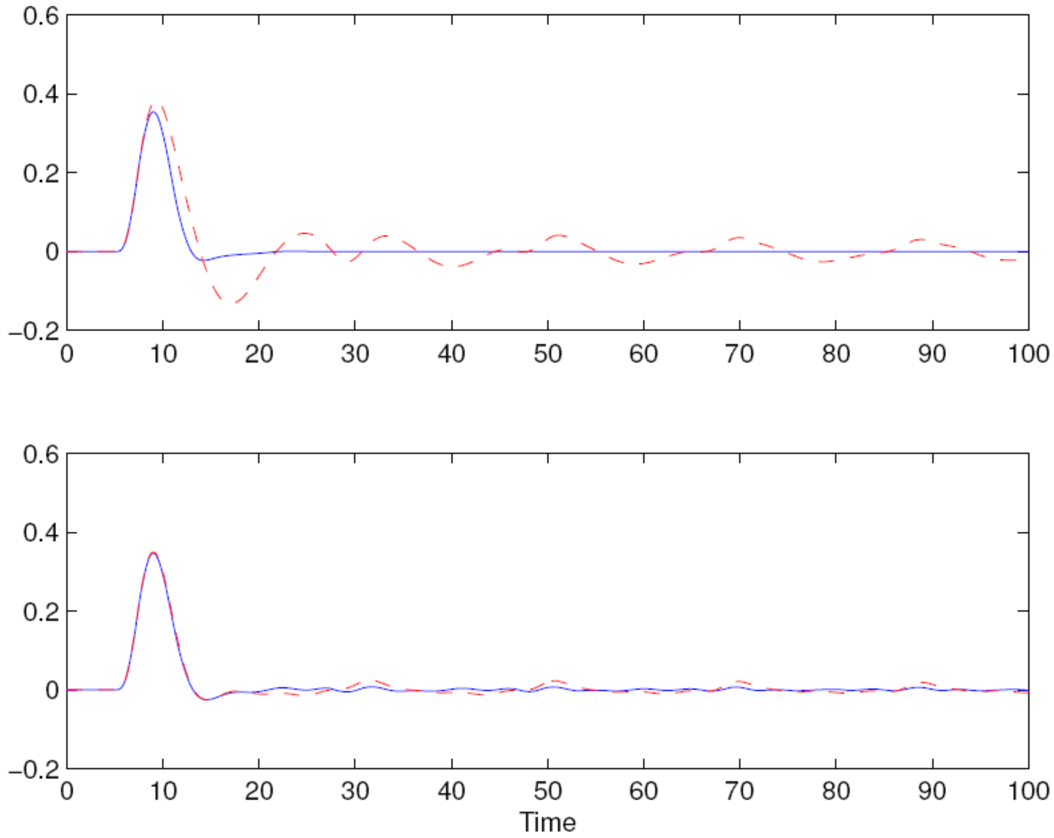
- K is a decentralized controller of P.
- G models P's interaction with the surrounding environment.
- G_r is a local environment model, to be added to controller K. How to choose G_r?

Example 3 (cont'd)



- Environment G (solid blue) is a highly resonant system.
- In open loop, G is hard to reduce. In closed loop, only certain frequencies are important.
- Reduced models: G_4 (dashed red), G_6 (dashed green).

Example 3 (cont'd)

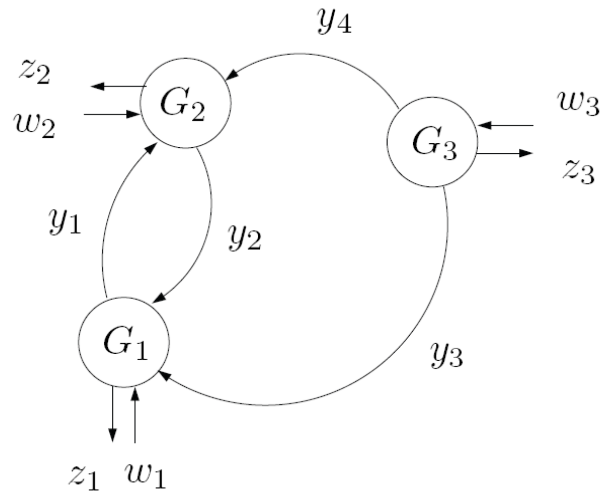


- Upper plot: Load step response with/without G.
- Lower plot: Load step response with G_4 and G_6.
- A low-order environment model can compensate for a very complex environment!

Explanation

- Find proper "inputs" and "outputs" to each subsystem which reflect the subsystem's interaction with the global system.
- Then apply methods that solve Problem 1.

- Motivation:
  1. Low-order feedback/feedforward controllers
  2. Large interconnected systems in computer science and biology
  3. Modular model reduction



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What You Will Learn in the Course

- Norms of signals and systems, some Hilbert space theory.
- Realization theory: Observability and controllability from optimal control/estimation perspective.
- Principal Component Analysis (PCA)/Proper Orthogonal Decomposition (POD)/Singular Value Decomposition (SVD).
- Balanced truncation for linear systems, with extension to nonlinear systems.
- Hankel norm approximation.
- Uncertainty and robustness analysis of models (small-gain theorem), controller reduction.
- (Quasi-)convex optimization/LMI approaches
- Applications in Fluid Dynamics (by KTH Mechanics)

Course Basics

- Graduate level
- Pass/fail
- 7 ECTS
- Course code: FEL3500
- Prerequisites:
  1. Linear algebra
  2. Basic systems theory (state-space models, controllability,observability etc.)
  3. Familiarity with MATLAB

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Preliminary Schedule

Week	Date	Topic
1	27/8	Lecture 1: Introduction. The model-order-reduction problem
2	3/9	Lecture 2: Model truncation, singular perturbation.
3	7/9	Exercise 1: Review of linear systems and Hilbert spaces etc.
	9/9	Lecture 3: Linear systems: POD/PCA/SVD-based simplification
4	13/9	Exercise 2
	17/9	Lecture 4: Linear systems: Gramians and balanced realizations
5	20/9	Exercise 3
	24/9	Lecture 5: Linear systems: Balanced truncation and weighted extensions
6	27/9	Exercise 4
	1/10	Lecture 6: Applications: Controller and nonlinear model reduction
7	4/10	Exercise 5
	8/10	Lecture 7: Optimal model reduction: Hankel norm approximation
8	11/10	Exercise 6
	15/10	Lecture 8: Applications in fluid mechanics, by Dan Henningson, KTH Mechanics.
9	18/10	Exercise 7
	22/10	Lecture 9: Quasi-convex model reduction techniques, by Kin Cheong Sou
10	28/10	Project presentations

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Course Material 1

- Two books entirely devoted to model reduction are available:
  1. Obinata and Anderson: *Model Reduction for Control Systems Design*
  2. Antoulas: *Approximation of Large-Scale Dynamical Systems*

  These books are **not** required for the course (although they are very good). Complete references on webpage.
- Parts of robust control books are used instead
  1. Green and Limebeer: *Linear Robust Control*
  2. Doyle, Francis, and Tannenbaum: *Feedback Control Theory*

  Available freely on the internet. Links on course webpage.

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Course Material 2

- Relevant research articles will be distributed.
- Generally no slides. White/black board will be used.
- Minimalistic lecture notes provided every lecture, containing:
  1. Summary of most important equations and notation (generally no proofs)
  2. Exercises
  3. Reading advice

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To Get Credits, You Need to Complete...

1. Exercises
   - At the end of the course, at least 75% of the exercises should have been solved and turned in on time.
2. Project
   - See upcoming slide.
3. Exam
   - A 24h hand-in exam.

1. Exercises

- Every set of lecture notes comes with 2-4 exercises. Handed out at each lecture (except this lecture).
- Exercises to be solved and turned in to me <u>7 days after they have been handed out</u>.
- Choose what exercises to solve yourself. At least 75% of the exercises should have been solved at the end of course.
- Cooperation allowed.
- The exercises are then discussed in the following exercise session.

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2. Project

- Choose a model that preferably relates as much as possible to your own research project. If you don't have a good model, talk to me.
- You should write a report and turn it in at the end of the course. The report should (at least) contain the following sections:
    1. Introduction. Describe or derive the model, and explain why it should be reduced. What is there to gain by model reduction? Give a clear problem formulation. Choices of inputs and outputs.
    2. Apply and analyze the result of model reduction method 1 (R1)
    3. Apply and analyze the result of model reduction method 2 (R2)
    4. Summarize what has been achieved. What worked, what did not work. Objectives achieved?
- R1 and R2 can be methods we have talked about in the course, a method you have found in the literature, or a method you come up with yourself. Method choices should be well motivated.
- Project presentation on Thursday 28/10.

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3. Exam

- 24 hours
- No cooperation allowed
- Problems similar to exercises

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Next Lecture

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- Friday September 3rd at 10-12.
- We start with the simplest methods:
  - Modal truncation
  - Singular perturbation
  - Model projection
- Model-reduction-method complexity increases with time in the course.
- First set of exercises handed out
- First exercise session on Tuesday September 7th will be devoted to repetition of basic linear systems concepts, Hilbert spaces, norms, operators,…
- Hope to see you next week!