3 Gramians and Balanced Realizations

In this lecture, we use an optimization approach to find suitable realizations for truncation and singular perturbation of \( G \). It turns out that the recommended realizations coincide with the ones obtained through PCA of the impulse-to-state maps \( e^{At}B \) (see last lecture) and \( e^{AT}C^T \) (see Moore’s paper). The realizations are called balanced realizations.

It is good to learn both the PCA approach and the optimization approach to balanced realizations, since they yield different reduced models in the nonlinear case (a later lecture). First, we state a useful theorem, of independent interest.

3.1 Interlude: Optimization in Hilbert Space

Consider the bounded linear operator \( A : U \to Y \), where \( U \) and \( Y \) are Hilbert spaces. If the inverse \( A^{-1} \) exists, then there is a unique solution \( u \) to the equation \( y = Au \), for given \( y \). Here we are instead interested in the case when the equation is overdetermined or underdetermined.

The following results are derived in the book *Optimization by Vector Space Methods* by Luenberger.

**Theorem 3.** Suppose \( A : U \to Y \) is a bounded linear operator, where \( U \) and \( Y \) are Hilbert spaces.

i) For fixed \( y \in Y \), the vector \( u \in U \) that minimizes \( \|y - Au\|_Y \) satisfies the normal equations

\[
A^*Au = A^*y.
\]

If \( (A^*A) : U \to U \) is invertible, the unique optimal solution is \( u = (A^*A)^{-1}A^*y \).

ii) Suppose \( A \) has closed range in \( Y \) (\( R(A) = R(A^*) \)). For example, \( R(A) \) is finite dimensional. Then the vector \( u \in U \) with the smallest norm \( \|u\|_U \) satisfying \( y = Au \), is given by

\[
u = A^*z,
\]

for any \( z \in Y \) that satisfies \( AA^*z = y \).

If \( (AA^*) : Y \to Y \) is invertible, the unique optimal solution is \( u = A^*(AA^*)^{-1}y \).

The operator \( A^* : Y \to U \), is the *Hilbert adjoint* of \( A \). By definition, it satisfies

\[
(Au,y)_Y = (u,A^*y)_U.
\]

For example, if \( A \in \mathbb{R}^{n \times m} \), then \( A^* = A^T \in \mathbb{R}^{m \times n} \) using the standard scalar product \( (x,y) = x^Ty \).

Theorem 3 i) can be used to solve optimal estimation problems, and Theorem 3 ii) can be used to solve optimal control problems. Here we use them to define two Gramians.

3.2 The Reachability Gramian

Let us define the reachability operator \( R_T : L_2[0,T] \to \mathbb{R}^n \) for the linear system \( G \) by

\[
x(T) = R_T u : \ x(T) = \int_0^T e^{A(T-t)}Bu(t)dt.
\]

If the realization is controllable, there exists control signals \( u \in L_2[0,T] \) that brings the state from \( x(0) = 0 \) to any \( x(T) = x_T \). That is, there are solutions to the equation, \( x_T = R_T u \) for all \( x_T \in \mathbb{R}^n \). The question is which control uses the least amount of energy, i.e., has the smallest possible \( \|u\| ? We can use Theorem 3 ii) to solve this.
To do that, we need the Hilbert adjoint \( R_T^* \), and \( R_T R_T^* \). Simple calculations yield that
\[
R_T^* : \mathbb{R}^n \to L_2[0, T]; \quad R_T^* = B^T e^{A(T-t)}
\]
\[
R_T R_T^* \in \mathbb{R}^{n \times n}; \quad R_T R_T^* = \int_0^T e^{A(T-t)} B B^T e^{A(T-t)} dt.
\]
If we make a change of variables in the last integral, \( \tau := T - t \), we see that it coincides with the reachability Gramian (2.5),
\[
R_T R_T^* = \int_0^T e^{A\tau} B B^T e^{A\tau} d\tau \equiv P(T).
\]
The system is controllable if, and only if, \( P(T) \) is invertible for all \( T > 0 \). The optimal control and its energy is given by
\[
u(t) = R_T^* (R_T R_T^*)^{-1} x_T = B^T e^{A(T-t)} P(T)^{-1} x_T, \quad \|\nu\| = \sqrt{x_T^T P(T)^{-1} x_T}.
\]
It follows that for a fixed amount of input energy, say \( \|\nu\| \leq 1 \), the reachable states \( x_T \) in the state space are contained in the ellipsoid \( \mathcal{R} \),
\[
\mathcal{R} = \{ x : x = U_P \Sigma_P z, \|z\| = 1 \}, \quad P(T) = U_P \Sigma_P^2 U_P^T,
\]
where \( U_P \Sigma_P^2 U_P^T \) is a singular value decomposition of \( P(T) \) (\( U_P = (u_1, \ldots, u_n) \) is unitary). Note that \( U_P \) and \( \Sigma_P \) are the component vectors and singular values of \( e^{A\tau} B, 0 \leq \tau \leq T \). The vector \( u_1 \) points in the direction where it is easiest to control, and \( \sigma_1 \) quantifies how far one can reach in that direction using energy \( \|\nu\| \leq 1 \).

We sum up the results of this section in the following theorem.

**Theorem 4.** A realization \( (A, B, C, D) \) of \( G \) is controllable if, and only if, \( P(T) \) is invertible for all \( T > 0 \). The reachable subspace is spanned by the component vectors \( u_1, \ldots, u_r \) in \( U_P \) that correspond to the \( r \leq n \) strictly positive singular values in \( \Sigma_P \). The singular value \( \sigma_i \) quantifies how far one can reach in the direction \( u_i \) using the input energy \( \|\nu\| \leq 1 \).

### 3.3 The Observability Gramian

The observability Gramian is dual to the reachability Gramian. It can be derived in many ways, for example using PCA (see paper by Moore) or by using Theorem 3 (see Exercise 3.2). Here we use the perhaps simplest approach.

We want to study how much output energy there is in a particular initial state \( x(0) = x_0 \). That is, what is \( \|y\|^2 = \int_0^T y(t)^T y(t) dt \), when \( x(0) = x_0 \) and \( u(t) = 0 \), \( 0 \leq t \leq T \)? Using that \( y(t) = Ce^{At} x_0 \), straightforward calculations yield
\[
\|y\|^2 = \int_0^T y(t)^T y(t) dt = \int_0^T x_0^T e^{AT} C^T C e^{AT} x_0 dt = x_0 Q_T(0) x_0 \text{,}
\]
where we have defined the **observability Gramian** by
\[
Q_T(0) = \int_0^T e^{AT} C^T C e^{AT} dt.
\]

In practice, it is often convenient to compute the reachability Gramian through the Lyapunov differential equation
\[
-\dot{Q}_T = A^T Q_T + Q_T A + C^T C, \quad Q_T(T) = 0.
\]
When \( T \to \infty \) and \( A \) is Hurwitz, \( Q(= Q_\infty(0)) \) can be computed from the algebraic Lyapunov equation

\[
A^T Q + QA + C^T C = 0. \tag{3.2}
\]

The Gramian \( Q_T(0) \) contains a lot of information. For example, the system is observable if, only if, \( Q_T(0) \) is nonsingular for all \( T > 0 \).

Just as for the reachability Gramian, it is useful to make a singular value decomposition of \( Q_T(0) \),

\[
Q_T(0) = U_Q \Sigma_Q U_Q^T, \quad \text{where} \quad U_Q = (u_1 \ldots u_n) \quad \text{is unitary.}
\]

The vector \( u_1 \) points in the direction of the state space that generates the largest amount of energy in the output. The initial states \( x_0 \) that lie on the ellipsoid \( O \),

\[
O = \{ x : x = U_Q \Sigma_Q^{-1} z, \| z \| = 1 \}, \quad Q_T(0) = U_Q \Sigma_Q U_Q^T,
\]

all result in an output \( \| y \| = 1 \). Hence, points very close to the origin are very observable in the output.

We sum up the results of this section in the following theorem.

**Theorem 5.** A realization \((A, B, C, D)\) of \( G \) is observable if, and only if, \( Q_T(0) \) is invertible for all \( T > 0 \).

The unobservable subspace is spanned by the component vectors \( u_{r+1}, \ldots, u_n \) in \( U_Q \) that correspond to the possible zero singular values in \( \Sigma_Q \). The singular value \( \sigma_i \) quantifies how much energy \( \| y \| \) there is if \( x_0 = u_i \).

### 3.4 Balanced Realizations

The goal of the model-reduction problem we set out to study, is to obtain a model \( G_r \) from \( G \) such that

\[
\| G - G_r \|_\infty = \sup_{u \in L_2} \frac{\| y - y_r \|}{\| u \|} \tag{3.3}
\]

is small, where \( y = Gu \) and \( y_r = G_r u \). We have argued that truncation and singular perturbation of certain state coordinates in \( G \) can be a good way of obtaining \( G_r \). Since the norm \( \| G \|_\infty \) measures how much energy that is transferred from the input into the output, it seems like a good idea to remove states in \( G \) that are not very involved in this energy transfer. This can be quantified using the analysis in Sections 3.2 and 3.3. We next try to find state coordinate directions that are very hard to excite with finite amounts of input energy, and also do not result in much energy in the output.

If we perform the coordinate transformation \( x = T \bar{x} \), the reachability and observability Gramians transform as

\[
P = T^{-1} P T^{-T}, \quad Q = T^T Q T. \tag{3.4}
\]

One can note that the eigenvalues of the product \( PQ \) are invariant under coordinate transformations, and we define

\[
\sigma_i := \sqrt{\lambda_i(PQ)} \tag{3.3}
\]

as the *Hankel singular values* of \( G \). These turn out to be fundamental in model reduction, and are the most important of all singular values in the course.

We show next how the coordinates of \( G \) can be balanced. The procedure has four steps:

1. Compute the Gramians \( P \) and \( Q \) for any stable and minimal realization \((A, B, C, D)\) of \( G \).
2. Compute the Cholesky factor \( R \) of \( P \), that is, \( P = RR^T \).
3. Compute the singular value decomposition \( R^TQR = U \Sigma^2 U^T \) of \( R^TQR \).
4. Use the coordinate transformation \( x = T \bar{x}, \quad T = RU \Sigma^{-1/2} \).
Using these coordinates, it holds that
\[ \Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}, \quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0 \]
\[ \bar{P} = \bar{Q} = \Sigma. \]

The reachability and observability Gramian become identical, and diagonal. This is called a balanced realization. It means that in the new coordinate vector, \( \bar{x} = (\bar{x}^1 \ldots \bar{x}^n)^T \), \( \bar{x}^1 \) is the most controllable and observable coordinate, and \( \bar{x}^n \) is the least controllable and observable direction.

The balanced coordinates are very good for both truncation and singular perturbation. Among the highlights that we will discuss more later, we mention that:

- if \( A \) is Hurwitz and \( \sigma_{r+1} < \sigma_r \), then \( A_{11} \) and \( A_{11} - A_{12}A_{22}^{-1}A_{21} \) are Hurwitz; and
- \( \sigma_{r+1} \leq \|G - \bar{G}_r\|_\infty \leq 2 \sum_{i=r+1}^n \sigma_i. \)

The last bound can be compared to the bound in the Schmidt-Mirsky theorem.

### 3.5 Recommended Reading

Read the proof of Theorem 3 in *Optimization by Vector Space Methods* by Luenberger.

Balanced realizations were introduced for model reduction by Bruce Moore in *Principal Component Analysis in Linear Systems*. Here a slightly different motivation for balanced realizations is used, but Moore’s paper is still very readable.

Read Sections 9.3-9.4 and 3.1 in *Linear Robust Control*.

### 3.6 Exercises

**Exercise 3.1** (Capacitor control)

Consider the electrical circuit in Figure 3.1, modelled by

\[ C \frac{dv(t)}{dt} = -\frac{v(t)}{R_1} + i(t), \]

where \( v(t) \) is the voltage over the capacitor. We can control the current source \( i(t) \), but there is an internal resistance \( R_2 \) in the actuator, and an amount \( D(T) = \int_0^T R_2 i(t)^2 dt \) of energy is dissipated there.

What voltages \( v(T) \) are reachable from \( v(0) = 0 \), if the upper bound on the dissipated energy is

\[ D(T) = \int_0^T R_2 i(t)^2 dt \leq 1? \]

Assign the parameters \( R_1, R_2, C, \) and \( T \) (non-trivial) numerical values if it simplifies the problem for you.
**EXERCISE 3.2** (Optimal estimation)

Suppose that we can measure the corrupted output $y_m(t)$ from a linear system $G$,

$$\dot{x} = Ax, \quad x(0) = x_0$$

$$y_m = Cx + e,$$

where $e(t)$ is measurement noise.

Define the observability operator for $G$ by $O_T : \mathbb{R}^n \rightarrow L_2[0, T]$,

$$y(t) = O_T x_0 : \quad y(t) = C e^{At} x_0.$$

Use Theorem 3 to find an optimal estimate $\hat{x}_0$ of the initial state $x_0$, given a measurement $y_m \in L_2[0, T]$. That is, solve $\min_x \|y_m - O_T x\|$. What is the covariance of the estimation error $\mathbb{E}(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T$? You can assume that the measurement noise is approximately white, that is, $\mathbb{E}e(t) = 0$ and $\mathbb{E}e(t_1)e(t_2) = \delta(t_1 - t_2)$.

**EXERCISE 3.3** (Balanced realizations)

Find balanced realizations for the following systems $G$ and truncate them to find approximations $G_r$ which make $\|G - G_r\|_\infty$ small. Compare the result to what you obtained in Exercise 1.2.

a)

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0.1 \\ 0.1 \end{pmatrix}, \quad C = (1 \ 1 \ 0.1).$$

b)

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.101 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad C = (1 \ 1 \ 1).$$

c)

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.101 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad C = (1 \ 1 \ 1).$$

d)

$$G(s) = \frac{(s + 2)(s + 4)(s + 6)(s + 8)}{(s + 1)(s + 3)(s + 5)(s + 7)}.$$  

How can you use the Hankel singular values to select approximation order $r$?

**EXERCISE 3.4**

Let $P$ and $Q$ be reachability and observability Gramians for a linear system $G$.

a) Verify that (3.4) balances $P$ and $Q$, i.e., that $\bar{P} = \bar{Q} = \Sigma$, and that $\Sigma$ in Step 3 contains the Hankel singular values $\sigma_i$ in (3.3).
b) Express

\[
\max_x \frac{x^T Q x}{x^T P^{-1} x},
\]  

(3.5)

using the Hankel singular values. Can you think of a system-theoretic interpretation of the ratio (3.5) for the system \( G \)?

c) Prove that a truncated balanced realization still is a balanced realization. You may assume that \( \bar{A}_{11} \) is Hurwitz.