1 Linear Systems, Model Truncation, and Singular Perturbation

1.1 Linear State-Space Systems

We consider linear state-space systems

\[ \begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = x_0 \\
y &= Cx + Du
\end{align*} \tag{1.1} \]

with state \( x(t) \in \mathbb{R}^n \), input \( u(t) \in \mathbb{R}^m \), and output \( y(t) \in \mathbb{R}^p \). Knowledge of \( x_0 \) and \( u(t) \) in the time interval \([0, T]\) determines \( x(T) \) and \( y(T) \) uniquely by

\[ x(T) = e^{AT}x_0 + \int_0^T e^{A(T-t)}Bu(t)dt, \quad y(T) = Cx(T) + Du(T). \]

Often we assume \( A \) is a Hurwitz matrix, i.e., all eigenvalues of \( A \) are in the open left complex half plane and the system is (asymptotically) stable.

Large classes of models can be written in the form (1.1). Examples include

- discretized partial differential equations (finite difference/finite element etc.), such as diffusion and wave equations; and
- linearized nonlinear ordinary differential equations.

If the input-output mapping \( u \mapsto y \) is the main interest, and not the state \( x \), the system \( G \) can also be represented by its transfer function

\[ G(s) = C(sI - A)^{-1}B + D \in \mathbb{C}^{m \times p} \]

for complex frequencies \( s \in \mathbb{C} \). As a measure of system size, and to measure the distance between two different systems we regularly use the \( H_\infty \)-norm:

\[ \|G\|_\infty := \sup_{s \in \mathbb{C}_+} |G(s)| \quad (\mathbb{C}_+ \text{ is the open complex right-half plane}) \]

\[ = \sup_\omega |G(j\omega)| \quad (G(s) \text{ has no poles in } \mathbb{C}_+) \]

which is finite if, and only if, \( G(s) \) is stable (has no poles in the closed right complex half plane). \( |G(s)| \) denotes the largest singular value of the matrix \( G(s) \) in the MIMO (Multi-Input–Multi-Output) case. In the SISO (Single-Input–Single-Output) case this is equal to the magnitude of the complex number \( G(s) \).

1.2 Reduced Order Systems and Approximation Criteria

We identify the complexity of the system \( G \) with its order \( n \). Some motivation for this definition are

- optimal controllers (LQG/\( H_2 \)/\( H_\infty \)) for \( G \) tend to have order of at least \( n \), and
- the simulation time of the system (1.1) is strongly correlated to the number \( n \) of differential equations.

A reduced order system ("an approximation") of \( G \) is a state-space system \( G_r \)

\[ G_r : \begin{cases}
\dot{z} = A_r z + B_r u, \quad z(0) = z_0 \\
y_r = C_r z + D_r u
\end{cases} \tag{1.2} \]
such that $z(t) \in \mathbb{R}^r$ where $r < n$.

Not only should $G_r$ be of lower order than $G$, its trajectories should say something about the trajectories of $G$. Otherwise, we can hardly speak of an approximation. The main approximation criterium we will be interested in is to make $\|G - G_r\|_\infty$ small. Motivation for this choice will be given throughout the course. One simple motivation is that it is a measure of the worst-case error. Other criteria, such as the relative criterium $\|G^{-1}(G - G_r)\|_\infty$ and the frequency-weighted criterium $\|W_1(G - G_r)W_2\|_\infty$ will also be discussed later in the course.

Since we mainly look at approximating input-output behavior $u \mapsto y$, choices of inputs and outputs are essential to get an approximation $G_r$ that captures what you want.

1.3 Truncation and Singular Perturbation

Good approximations $G_r$ can often be obtained by means of truncation or singular perturbation (residualization). Both methods are done in two steps:

Step 1: Change the coordinates $x(t)$. That is, find a suitable invertible matrix $T \in \mathbb{R}^{n \times n}$ and transform the state-space model according to

$$
\tilde{A} = T^{-1}AT = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \quad \tilde{A}_{11} \in \mathbb{R}^{r \times r},

\tilde{B} = T^{-1}B = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}, \quad \tilde{B}_1 \in \mathbb{R}^{r \times m},

\tilde{C} = CT = \begin{pmatrix} \tilde{C}_1 & \tilde{C}_2 \end{pmatrix}, \quad \tilde{C}_1 \in \mathbb{R}^{p \times r},

\tilde{D} = D.
$$

Step 2: Depending on method, define $G_r$ by:

- Truncation:
  $$
  A_r = \tilde{A}_{11}, \\
  B_r = \tilde{B}_1, \\
  C_r = \tilde{C}_1, \\
  D_r = D.
  $$

- Singular perturbation:
  $$
  A_r = \tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}, \\
  B_r = \tilde{B}_1 - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{B}_2, \\
  C_r = \tilde{C}_1 - \tilde{C}_2\tilde{A}_{22}^{-1}\tilde{A}_{21}, \\
  D_r = D - \tilde{C}_2\tilde{A}_{22}^{-1}\tilde{B}_2.
  $$

Note the following:

(P1) using truncation, we have $G(\infty) = G_r(\infty)$; and

(P2) using singular perturbation, we have $G(0) = G_r(0)$.

Hence, these methods always achieve perfect approximation either at steady state or at infinite frequency.

For successful application of these methods (usually it is not enough to have a good approximation only at one frequency), we need to find good coordinate transformations $T$, and a suitable approximation order $r$. The following lectures will deal with this. Mostly we will use the truncation method.

1.4 Truncation = Projection*

The truncation method can also be seen as a (generally non-orthogonal) projection from the original state-space in $\mathbb{R}^n$ to the reduced state-space in $\mathbb{R}^k$. We have the transformations

$$
W^T = \begin{pmatrix} I_r & 0_{r \times (n-r)} \end{pmatrix} T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^r \quad ("x \mapsto z")
$$

$$
V = \begin{pmatrix} I_r \\ 0_{(n-r) \times r} \end{pmatrix} : \mathbb{R}^r \rightarrow \mathbb{R}^n \quad ("z \mapsto x")
$$
Notice that $W$ and $V$ satisfy
\[ W^TV = I, \quad V^TW = (VW^T)(VW^T). \]
Such a projection is called a Petrov Galerkin projection. If $W^T = V^T$ the projection is called a Galerkin projection. For Petrov Galerkin projections we have that
\[
\begin{align*}
A_r &= W^T AV, \\
B_r &= W^T B, \\
C_r &= CV, \\
D_r &= D.
\end{align*}
\]
To understand the Petrov Galerkin projection we can do the following analysis. Assume we want to try to express the solution $x(t) \in \mathbb{R}^n$ to the model $G$ only in $r$ variables. Such a solution can be written as $x(t) = Vz(t)$, where $z(t) \in \mathbb{R}^r$ and $V \in \mathbb{R}^{n \times r}$. $V$ should be chosen such that its columns span a subspace where we think the solution $x(t)$ will lie in. If the potential solution is put into the original state-space model $G$, we obtain
\[
\dot{x} = V\dot{z} = AVz + Bu + E,
\]
where $E$ is the residual. $x(t) = Vz(t)$ is a solution to the original problem if, and only if, $E(t) = 0$ for all $t$. There are $n$ equations in (1.3), but only $r$ unknowns $z$. The system is generally over determined. To find a unique solution, we can require that the projection of the residual $E(t)$ onto the subspace spanned by $V$ is zero. This projection is given by $W^T$. Hence, we add the condition
\[ W^TE(t) = 0, \quad \forall t \]
to (1.3). We then obtain the equation
\[ \dot{z} = W^T AVz + W^T Bu, \]
which exactly is the Petrov Galerkin projection of (1.1). From this, $z(t)$ can be computed, and the projection of the resulting residual $E$ onto $V$ is zero.

1.5 Recommended Reading

Sections 3.2–3.2.3 in *Linear Robust Control* discuss linear systems and the $H_\infty$-norm. We will discuss signals, systems, and function spaces more during Exercise 1.

Sections 9.1–9.2.2 in *Linear Robust Control* discuss model truncation and singular perturbation in more detail. The controllability and observability Gramians mentioned on page 315 will be introduced at a later stage in this course.

1.6 Exercises

**EXERCISE 1.1** (Modal representation and truncation)

Assume that $A$ is Hurwitz, and has $n$ distinct eigenvalues $\lambda_i$. Then there exists a coordinate transformation $T$ such that
\[
\bar{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad \bar{C} = (c_1 \ldots c_n).
The state corresponding to $\lambda_i$ is often called the $i$-th mode of the system. Prove that using truncation on the diagonalized system we obtain

$$G(s) - G_r(s) = \sum_{i=r+1}^{n} \frac{c_i b_i}{s - \lambda_i},$$

(1.4)

$$\|G - G_r\|_\infty \leq \sum_{i=r+1}^{n} \frac{|c_i b_i|}{|\text{Re} \lambda_i|},$$

(1.5)

Discuss how an error bound like (1.5) can be used. What modes should be truncated?

**EXERCISE 1.2 (Modal truncation)**

What modes should be truncated in the following systems if $\|G - G_r\|_\infty$ should be small?

a) 
\[
\bar{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 \\ 0.1 \\ 0.1 \end{pmatrix}, \quad \bar{C} = (1 \ 1 \ 0.1).
\]

b) 
\[
\bar{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.101 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{C} = (1 \ 1 \ 1).
\]

c) 
\[
\bar{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.101 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{C} = (1 \ 1 \ 1).
\]

d) 
\[
G(s) = \frac{(s + 2)(s + 4)(s + 6)(s + 8)}{(s + 1)(s + 3)(s + 5)(s + 7)}.
\]

(1.6)

Relate your choices of truncated states to the error bound (1.5). Plot Bode diagrams of the systems $G$ and the approximations $G_r$ you construct.

**EXERCISE 1.3**

What are the conditions for controllability and observability of the modes in a state-space system in modal representation?

**EXERCISE 1.4**

Prove properties P1 and P2.