Evolution of the stochastic Airy eigenvalues under a changing boundary

Angelica Gonzalez[*] Diane Holcomb[†]

October 9, 2018

Abstract

The Airy_β point process, originally introduced by Ramírez, Rider, and Virág [6], is defined as the spectrum of the stochastic Airy operator $\mathcal{H}_\beta$ acting on a subspace of $L^2[0,\infty)$ with Dirichlet boundary condition. In this paper we study the coupled family of point processes defined as the eigenvalues of $\mathcal{H}_\beta$ acting on a subspace of $L^2[t,\infty)$. These point processes are coupled through the Brownian term of $\mathcal{H}_\beta$. We show that these point processes as a function of t are differentiable with explicitly computable derivative. Moreover when recentered by t the resulting point process is stationary.

1 Introduction

Recall the β-Hermite ensemble which generalizes the eigenvalue distributions for the Gaussian Orthogonal, Unitary, and Symplectic ensembles. The β-Hermite ensemble is a set of n points on the line whose joint density is given by

$$p_H(\lambda_1,\ldots,\lambda_n) = \frac{1}{Z_{n,\beta}} \prod_{1\le i<j\le n} |\lambda_i-\lambda_j|^\beta \prod_{i=1}^n e^{-\frac{\beta}{4}\lambda_i^2} \tag{1}$$

This point process is no longer related to a full matrix model, but it does have an associated tridiagonal matrix model. The model originally introduced by Dumitriu and Edelman [1] is as follows: Let

$$A_\beta \sim \frac{1}{\sqrt{\beta}} \begin{bmatrix} \mathcal{N}(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & \mathcal{N}(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \mathcal{N}(0,2) & \chi_\beta \\ & & & \chi_\beta & \mathcal{N}(0,2) \end{bmatrix}, \tag{2}$$

[*]University of Arizona, agonzalez@math.arizona.edu
[†]KTH, holcomb@kth.se

with all of the entries independent. The χ random variables are subscripted by their parameter. A χ_k random variable has the same distribution as the norm of a vector in $\mathbb{R}^k$ with independent $\mathcal{N}(0,1)$ entries.

There are close connections between the theory of Jacobi matrices and Sturm-Liouville operators. Edelman and Sutton observed that this matrix model may be seen as an operator on step functions. The conjectured that in the limit the upper edge of the spectrum will converge to a certain differential operator [2]. Indeed, in this setting at the upper and lower edge of the spectrum Ramírez, Rider, and Virág showed that the centered and scaled matrix model converges in some sense to the "stochastic Airy operator" (SAO$_\beta$) which in turn implies convergence of the eigenvalues. Let

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x) \tag{3}$$

where we take b' to be a white noise. A precise definition and many properties of this operator can be found in [6]. We review the necessary ones below.

For our purposes it is sufficient to define an eigenfunction/eigenvalue pair in the following way: Let

$$L^*[t,\infty) = \left\{ f \in L^2[t,\infty) |\ f(t) = 0, f' \text{ exists a.e. and } \int_t^\infty (f')^2 + (1+x)f^2 dx < \infty \right\},$$

then (φ, λ) is an eigenvalue/eigenfunction pair for $\mathcal{H}_\beta$ if $\|\varphi\|_2 = 1, \varphi \in L^*[0,\infty)$ and

$$\varphi''(x) = \frac{2}{\sqrt{\beta}} \varphi(x) b'(x) + (x-\lambda)\varphi(x) \tag{4}$$

holds in the sense of distributions. This may be written as

$$\varphi'(x) - \varphi'(0) = \frac{2}{\sqrt{\beta}} \varphi(x) b(x) - \frac{2}{\sqrt{\beta}} \int_0^x \varphi'(t) b(t) dt + \int_0^x (t-\lambda)\varphi(t) dt. \tag{5}$$

In this sense, the set of eigenvalues is a deterministic function of the Brownian path b. Moreover the eigenvalues are "nice" in the following sense:

Theorem 1. *[6] With probability one, the eigenvalues of $\mathcal{H}_\beta$ are distinct (of multiplicity 1) with no accumulation point, and for each $k \geq 0$ the set of eigenvalues of $\mathcal{H}_\beta$ has a well defined $(k+1)$st lowest element $\Lambda_k(\beta)$.*

Here we consider the evolution of the eigenvalues of $\mathcal{H}_\beta$ under a smooth restriction of the domain. That is we consider the operator $\mathcal{H}_\beta$ acting on $L^*[t,\infty)$ and study the evolution of the eigenvalues as t varies. We will denote the operator acting on the particular domain by

$$\mathcal{H}_\beta^{(t)} = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'_x, \qquad \mathcal{H}_\beta^{(t)} : L^*[t,\infty) \to L^2[t,\infty), \tag{6}$$

and define $\Lambda_1(t) < \Lambda_2(t) < \Lambda_3(t) < \cdots$ to be the ordered eigenvalues of $\mathcal{H}_\beta^{(t)}$. We observe that the eigenvalue/ eigenfunction condition may be written in the same way as before, but also has an interpretation in terms of a shifted Brownian motion. That is we say that (φ,λ) is an eigenvalue/eigenfunction pair of $\mathcal{H}_\beta^{(t)}$ if for $x \ge t$

$$\begin{aligned}\varphi'(x)-\varphi'(t) &= \frac{2}{\sqrt{\beta}}\varphi(x)b(x) - \frac{2}{\sqrt{\beta}}\int_t^x \varphi'(s)b(s)ds + \int_t^x (s-\lambda)\varphi(s)ds \\ &= \frac{2}{\sqrt{\beta}}\varphi(x)(b(x)-b(t)) - \frac{2}{\sqrt{\beta}}\int_t^x \varphi'(s)(b(s)-b(t))ds + \int_t^x (s-\lambda)\varphi(s)ds.\end{aligned} \tag{7}$$

Theorem 2. *Let k be any fixed positive integer and let $\mathcal{G}_t^{(k)} = \{\Lambda_1(t),...,\Lambda_k(t)\}$. The process $\mathcal{G}_t^{(k)}$ is differentiable in time and for every fixed t we have that*

$$\frac{d}{dt}\mathcal{G}_t^{(k)} \stackrel{d}{=} \{Y_{1,t},...,Y_{k,t}\}, \qquad \text{with i.i.d.} \quad Y_{i,t} \sim \Gamma(\tfrac{\beta}{2},\tfrac{2}{\beta}). \tag{8}$$

Moreover, the process $\mathcal{G}_t^{(k)} - t$ is stationary.

Remark 3. Note that in the above characterization the eigenvalue/eigenfunction pairs are defined in a path–wise sense. In this paper all calculations unless otherwise noted should be understood in this sense. Because of this it is sufficient to prove various estimates and limits for an arbitrary value of t and any Brownian path in a set of full measure.

Corollary 4. *The process $\mathcal{G}_k^{(t)} - t$ is not reversible.*

Proof. To see that the process cannot be reversible it is enough to observe that the distribution derivative of the forward process is $\Gamma(\frac{\beta}{2},\frac{2}{\beta})-1$. On the other hand the distribution of derivative of the reversed process is $1-\Gamma(\frac{\beta}{2},\frac{2}{\beta})$, which is not the same distribution. □

While the study of the eigenvalues of $\mathcal{H}_\beta$ on a changing domain is itself interesting it also has a connection to the original tridiagonal model in (2). Moreover this connection may be used to derive properties of the limiting process including the distribution of the derivatives. We are interested in the behavior of the spectrum at the upper and so begin by centering at $2\sqrt{n}$. We denote the centered, truncated matrix obtained by removing the first $k-1$ rows and columns by

$$H_{n,\beta}^{(k)} = 2\sqrt{n}I - \frac{1}{\sqrt{\beta}}\begin{bmatrix} N(0,2) & \chi_{(n-k)\beta} & & & \\ \chi_{(n-k)\beta} & N(0,2) & \chi_{(n-k-1)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2)\end{bmatrix}, \tag{9}$$

and denote its ordered eigenvalues by $\lambda_1(n,k) < \lambda_2(n,k) < \cdots < \lambda_{n-k+1}(n,k)$ with associated eigenvectors $v(1,k), v(2,k), ..., v(n-k+1,k)$

Theorem 5. *Suppose that $\lambda_1(n,k) < \lambda_2(n,k) < \cdots$ be defined as above then*

$$\left(\{n^{1/6}\lambda_i(n,n^{1/3}t)\}_{i=1}^k, t \ge 0\right) \Rightarrow \mathcal{G}_t^{(k)} \tag{10}$$

where $\mathcal{G}_k^{(t)}$ eigenvalue process of $\mathcal{H}_\beta$ defined above.

The reader might notice at this point that we are essentially considering the 'minor process' associated to the tridiagonal matrix model. This turns out to define a very different process than the classical 'minor process' when $\beta = 2$ that derives from the submatrices of the full matrix model. For more details on this classical process see [4]. In particular that process has Brownian type paths followed by the eigenvalues. The same process may be realized by considering appropriate limits of Dyson Brownian Motions [5]. The fact that two different process are obtained is particularly interesting in light of the fact that for both models when one considers the sub-matrix obtained by removing the first k rows and columns they again have the same eigenvalue distributions, and in both cases eigenvalues of successive sub-matrices satisfy interlacing.

The paper will be organized as follows: We begin recalling properties of $\mathcal{H}_\beta$ and showing that the process $\mathcal{G}_t^{(k)}$ is stationary and differentiable. In the next section we show the convergence statement in 5. Finally, in the last section we use the convergence statement to determine the distribution of the derivative vector.

Acknowledgements: The authors would like to that Bálint Virág for the problem suggestion.

2 On the eigenvalues of the restricted operator

Proposition 6. *For any fixed k the process $\mathcal{G}_t^{(k)} - t$ is stationary as a process in t.*

Proof. We use definition that (λ, φ) is an eigenvalue/eigenfunction pair for $\mathcal{H}_\beta^{(t)}$ if (7) is satisfied, we define the time sifted function $\psi(x-t) = \varphi(x)$ and shifted Brownian motion $w(x-t) = b(x) - b(t)$ then ψ satisfies the equation

$$\psi'(x-t)-\psi'(0) = \frac{2}{\sqrt{\beta}}\psi(x-t)w(x-t)-\frac{2}{\sqrt{\beta}}\int_0^{x-t}\psi'(s)w(s)ds+\int_0^{x-t}(s-(\lambda-t))\psi(s)ds. \quad (11)$$

This is equivalent in distribution to $\lambda - t$ being an eigenvalue of

$$\mathcal{H} = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}}w'_x \overset{d}{=} \mathcal{H}_\beta^{(0)}.$$

Therefore the lowest k eigenvalues of $\mathcal{H}_\beta^{(t)}$ shifted by t have the same distribution as the lowest k eigenvalue of $\mathcal{H}_\beta^{(0)}$ for all t and so $\mathcal{G}_t^{(k)} - t$ is stationary. □

Before moving on to showing that $\mathcal{G}_t^{(k)}$ is a differentiable process we must first prove several results on the eigenfunctions of the operators $\mathcal{H}_\beta^{(t)}$. We let $f_{k,t} \in L^*[t,\infty)$ denote the eigenfunction associated to $\Lambda_k(t)$ the kth lowest eigenvalue of $\mathcal{H}_\beta^{(t)}$. For $t > 0$ we can extend $f_{k,t}$ to a function on $L^*[0,\infty)$ by taking $f_{k,t}(x) = 0$ for $x < t$. This extension should be implicitly understood where necessary in the following computations.

The idea here will be to make use of the variational characterization of the eigenvalues:

$$\Lambda_k(t) = \inf_{B\subset L_t^*, \dim B = k} \sup_{g\in B} \frac{\prec g, \mathcal{H}_\beta^{(t)} g \succ}{\prec g, g \succ} = \prec f_{k,t}, \mathcal{H}_\beta^{(t)} f_{k,t} \succ . \quad (12)$$

Here the infimum is attained at the eigenfunction $f_{k,t}(x)$ associated to $\Lambda_k(t)$. We will approximate $f_{k,t}(x)$ by using $f_{k,s}(x)$ where s is close to t and replacing the start of the function with a linear approximation. In order for this to be a good approximation we need that the eigenfunctions are 'close' to linear near their boundary.

Proposition 7. *Let $\varphi_t(x)$ be an eigenfunction of $\mathcal{H}_\beta(t)$. Suppose that $\sup_{x\in[t,x_0]} |\varphi_t'(x) - \varphi_t'(t)| < \eta_{x_0}$ for some $\eta_{x_0} > 0$. Then for every $0 < \delta < 1/2$ and $x \in [t, x_0]$ there exists $C_{x_0,\delta}$,*

$$|\varphi_t(x) - (x-t)\varphi_t'(t)| \leq C_{b,x_0,\delta}(\eta_{x_0} + \varphi_t'(t))|x-t|^{2+\delta}.$$

Proof. We begin with two bounds.
Bound on Brownian Motion: Using that Brownian Motion α-Holder continuous for $\alpha < 1/2$ and for all $t > 0$ we have that almost surely for all BM paths and $x, t < x_0$

$$|b(x) - b(t)| \leq C_{x_0,\delta}|x-t|^\delta, \text{ for } 0 < \delta < 1/2. \tag{13}$$

Bound for $\varphi_t(x)$: We apply the Mean Value Theorem to $\varphi_t(x)$ to get that for some $r \in (t, x)$

$$\varphi_t(x) = \varphi_t'(r) \cdot (x-t) \leq (\eta_{x_0} + \varphi_t'(t))(x-t). \tag{14}$$

Now suppose that $\varphi_t(x)$ is the eigenfunction corresponding to eigenvalue λ. Then

$$\varphi_t'(x) - \varphi_t'(t) = \frac{2}{\sqrt{\beta}}\varphi_t(x)(b_x - b_t) - \frac{2}{\sqrt{\beta}}\int_t^x (b_y - b_t)\varphi_t'(y)dy + \int_t^x (y-\lambda)\varphi_t(y)dy.$$

Applying the two bounds given at the beginning of this proof ((13) and (14)) we obtain that

$$|\varphi_t'(x) - \varphi_t'(t)| \leq C_{x_0,\delta}(\eta_{x_0} + \varphi_t'(t))|x-t|^{1+\delta}$$

We use the mean value theorem again to get $\varphi_t(x) - (x-t)\varphi_t'(t) = (\varphi_t'(r) - \varphi_t'(t))(x-t)$ which gives us the final bound

$$|\varphi_t(x) - (x-t)\varphi_t'(t)| \leq C_{x_0,\delta}(\eta_{x_0} + \varphi_t'(t))|x-t|^{2+\delta}.$$

□

Proposition 8. *Let $[c, d]$ be any interval, then for any $\gamma < 1/2$ and $0 < \varepsilon < \frac{\rho}{t-s}$ there exists a constant $C_{\rho,\gamma}$ depending only on γ and ρ such that for any $s < t \in [c, d]$ and $i = 1, ..., k$ we have*

$$\frac{\varepsilon}{1+\varepsilon}(f_{i,s}'(s))^2 - C_{\rho,\gamma}\varepsilon^{1+\gamma}(t-s)^\gamma \leq \frac{\Lambda_i(t) - \Lambda_i(s)}{t-s} \leq \frac{1+\varepsilon}{\varepsilon}(f_{i,t}'(t))^2 + C_{\rho,\gamma}\varepsilon^{1+\gamma}(t-s)^\gamma. \tag{15}$$

Remark 9. Observe that the choise $\varepsilon = (t-s)^{-\delta}$ meets the conditions of the theorem and converges to ∞ as $t \to s$, therefore if we can show that $\lim_{t\to s} f_{i,t}'(t) = f_{i,s}'(s)$ this will be enough to show that the process is differentiable with the derivative at t being given by $(f_{i,t}'(t))^2$.

Corollary 10. *The process $\mathcal{G}_k^{(t)}$ is continuous as a function of t.*

This follows immediately from the inequality in the previous Proposition 8, simply multiply through by $(t-s)$.

Proof of Proposition 8. For every pair $s<t$ we define two new families of functions

$$\phi^{(a)}_{k,s,t}(x)=\begin{cases}(x-s)\frac{f_{k,t}(a)}{a-s} & s\le x<a\\ f_{k,t}(x) & x\ge a\end{cases},\qquad \psi^{(a)}_{k,s,t}(x)=\begin{cases}(x-t)\frac{f_{k,s}(a)}{a-t} & t\le x<a\\ f_{k,s}(x) & x\ge a\end{cases}. \tag{16}$$

The function $\varphi^{(a)}_{k,s,t}$ approximates the kth eigenfunction for $\mathcal{H}^{(s)}_\beta$ by building a function from the kth eigenfunction of $\mathcal{H}^{(t)}_\beta$. The function $\psi^{(a)}_{k,s,t}$ does something similar, but instead approximates the kth eigenfunction of $\mathcal{H}^{(t)}_\beta$ by looking at the kth eigenfunction of $\mathcal{H}^{(s)}_\beta$. We will begin by showing the necessary estimates for $k=1$ and then show how to expand the result to arbitrary k.

The idea is to use the variational characterization of our eigenvalues to get upper bounds using $\psi^{(a)}_{1,s,t}$ and $\varphi^{(a)}_{1,s,t}$. In particular we have that

$$\Lambda_1(t)\le\frac{\prec\psi^{(a)}_{1,s,t},\mathcal{H}^{(t)}_\beta\psi^{(a)}_{1,s,t}\succ}{\|\psi^{(a)}_{1,s,t}\|_2^2},\qquad\text{and}\qquad\Lambda_1(s)\le\frac{\prec\varphi^{(a)}_{1,s,t},\mathcal{H}^{(t)}_\beta\varphi^{(a)}_{1,s,t}\succ}{\|\psi^{(a)}_{1,s,t}\|_2^2}.$$

Before continuing we show that $\|\psi^{(a)}_{k,s,t}\|_2^2$ is close enough to 1 that it may be neglected for the remainder of the calculations. In particular we have

$$\|\psi^{(a)}_{k,s,t}\|_2^2=\|f_{k,s}\|_2^2+\int_t^a(x-t)^2\frac{f^2_{k,s}(a)}{(a-t)^2}dx-\int_s^a f^2_{k,s}(x)dx$$

Applying Proposition 7 we obtain that

$$|\|\psi^{(a)}_{k,s,t}\|_2^2-1|\le\left((a-t)(a-s)^2+(a-s)^3\right)\frac{(f'_{k,s})^2(a)}{3}+\tilde{C}_{a,\gamma}(a-s)^{3+\gamma} \tag{17}$$

Taking $a=t+\varepsilon(t-s)$ we obtain that $|\||\psi^{(a)}_{k,s,t}\|_2^{-2}-1|\le C(t-s)^3$. These errors may be bounded using the constant $C_{\rho,\gamma}\varepsilon^{1+\gamma}(t-s)^\gamma$ term in equation (15). Because of this we will neglect the normalization for the remainder of the argument.

We can then compute the following:

$$\prec\psi^{(a)}_{1,s,t},\mathcal{H}^{(t)}_\beta\psi^{(a)}_{1,s,t}\succ-\Lambda_1(s)=\int_t^a\psi^{(a)}_{1,s,t}(x)\mathcal{H}^{(t)}_\beta\psi^{(a)}_{1,s,t}(x)dx-\int_s^a f_{1,s}(x)\mathcal{H}^{(s)}_\beta f_{1,s}(x)dx.$$

We show that for $a=t+\varepsilon(t-s)$

$$0\le\prec\psi^{(a)}_{1,s,t},\mathcal{H}^{(t)}_\beta\psi^{(a)}_{1,s,t}\succ-\Lambda_1(s)\le(f'_{1,s})^2(s)(t-s)\frac{1+\varepsilon}{\varepsilon}+M(1+\varepsilon)^{1+\gamma}(t-s)^{1+\gamma} \tag{18}$$

In order to do this we must bound $\int_t^a\psi^{(a)}_{1,s,t}(x)\mathcal{H}^{(t)}_\beta\psi^{(a)}_{1,s,t}(x)dx$ above and $\int_s^a f_{1,s}(x)\mathcal{H}^{(s)}_\beta f_{1,s}(x)dx$ below. We use Holder continuity of Brownian motion to say that for $c<x<y<a$ and $\gamma<1/2$ fixed with probability 1 there exists a constant C such that

$$|b_x-b_y|\le C_{a,\gamma}|x-y|^\gamma.$$

This gives us

$$\int_t^a \psi^{(a)}_{1,s,t}(x)\mathcal{H}^{(t)}_\beta \psi^{(a)}_{1,s,t}(x)dx = \int_t^a \left(\frac{f_{k,s}(a)}{a-t}\right)^2 \left[1 + x(x-t)^2 + (b_x - b_t)(x-t)\right] dx$$
$$\leq f_{1,s}^2(a)\left(\frac{1}{a-t} + \frac{(a-t)^2}{4} + t\frac{(a-t)^3}{3} + \frac{C_{a,\gamma}(a-t)^\gamma}{1+\gamma}\right).$$

An application of proposition 7 allows us to write $f_{1,s}(x) \leq (x-s)f'_{1,s}(s) + C_{a,\gamma}(x-s)^{1+\gamma}$, which leads us to

$$\int_t^a \psi^{(a)}_{1,s,t}(x)\mathcal{H}^{(t)}_\beta \psi^{(a)}_{1,s,t}(x)dx \leq (f'_{1,s})^2(s)\frac{(a-s)^2}{a-t} + M_{a,\gamma}(a-t)^\gamma(a-s)^2. \tag{19}$$

Before continuing we make the following observation

$$\left|\int_s^a (x-s)^{1+\gamma}\mathcal{H}^{(s)}_\beta(x-s)ds\right| \leq C_{a,\gamma}(a-s)^{\gamma+1},$$

where C is a random constant depending on the interval $[c,d]$ and the choice of a and γ. From this we can check that Proposition 7 implies that

$$\left|\int_s^a f_{1,s}(x)\mathcal{H}^{(s)}_\beta f_{1,s}(x)dx - \int_s^a (x-s)f'_{1,s}(s)\mathcal{H}^{(s)}_\beta(x-s)f'_{1,s}(s)dx\right| \leq C_{a,\gamma}(a-s)^{1+\gamma}. \tag{20}$$

We finish the lower bound on $\int_s^a f_{1,s}(x)\mathcal{H}^{(s)}_\beta f_{1,s}(x)dx$ by computing

$$\int_s^a (x-s)f'_{1,s}(s)\mathcal{H}^{(s)}_\beta(x-s)f'_{1,s}(s)dx \geq (f'_{1,s})^2(s)(a-s) - M_{a,\gamma}(a-s)^{2+\gamma}. \tag{21}$$

Putting together (19), (20), and (21) we are led to the conclusion that for all $a = t + \varepsilon(t-s)$ we have

$$\Lambda_1(t) - \Lambda_1(s) \leq (f'_{1,s})^2(s)(t-s)\frac{1+\varepsilon}{\varepsilon} + M\varepsilon^{1+\gamma}(t-s)^{1+\gamma}. \tag{22}$$

This leads us one of the inequalities in Propostion 8 for $i = 1$. Similar techniques may be used to study $\varphi^{(a)}_{1,s,t}$. These lead to the inequality

$$\Lambda_1(t) - \Lambda_1(s) \geq (f'_{1,t})^2(t)(t-s)\frac{\varepsilon}{1+\varepsilon} - M\varepsilon^{1+\gamma}(t-s)^{1+\gamma}. \tag{23}$$

We now consider the problem for the higher eigenvalues. For these we turn to the Courant-Fisher characterization which is given by

$$\Lambda_k(t) = \inf_{B\subset L^*_t, \dim B = k}\ \sup_{g\in B} \frac{\prec g, \mathcal{H}^{(t)}_\beta g \succ}{\prec g, g\succ}. \tag{24}$$

From this characterization we have

$$\Lambda_k(t) \leq \sup_{g\in B_s} \frac{\prec g, \mathcal{H}^{(t)}_\beta g \succ}{\prec g, g\succ}, \qquad \text{where} \qquad B_s = \mathrm{Span}\,\{\psi^{(a)}_{1,s,t}, ..., \psi^{(a)}_{k,s,t}\}.$$

We make the following observations: For $i \neq j \leq k$

$$\prec \psi^{(a)}_{i,s,t}, \psi^{(a)}_{j,s,t} \succ = \int_t^a \frac{(x-t)^2}{(a-t)^2} f_{i,s}(a) f_{j,s}(a) dx - \int_s^a f_{i,s}(x) f_{j,s}(x) dx$$

And so an application of Proposition 7 gives us that

$$| \prec \psi^{(a)}_{i,s,t}, \psi^{(a)}_{j,s,t} \succ | \leq \frac{1}{3}(t-s)(a-s)^2 f'_{i,s}(s) f'_{j,s}(s) + C(a-s)^{3+\gamma}. \tag{25}$$

First observe that for all j using bounds identical to those used to prove (18) we can show that

$$0 \leq \prec \psi^{(a)}_{j,s,t}, \mathcal{H}^{(t)}_\beta \psi^{(a)}_{j,s,t} \succ - \Lambda_j(s) \leq (f'_{j,s})^2(s) \frac{(a-s)^2}{a-t} + M_{a,\gamma}(a-s)^{1+\gamma}. \tag{26}$$

Now observe that for $g = c_1 \psi^{(a)}_{1,s,t} + \cdots + c_k \psi^{(a)}_{k,s,t} \in B_s$ we have

$$\begin{aligned} \prec g, \mathcal{H}^{(t)}_\beta g \succ \leq \sum_{j=1}^k c_j^2 \left(\Lambda_j(s) + (f'_{j,s})^2(s) \frac{(a-s)(t-s)}{a-t} + M_{a,\gamma}(a-s)^{1+\gamma} \right) \\ + 2\Lambda_k(t) \sum_{j_1 < j_2} c_{j_1} c_{j_2} | \prec \psi^{(a)}_{j_1,s,t}, \psi^{(a)}_{j_2,s,t} \succ | \end{aligned}$$

Taking $a = t + \varepsilon(t-s)$ and applying the bound in (25) we are led to

$$\begin{aligned} \Lambda_k(t) \leq \sup_{c_1,\ldots,c_n} \|g\|_2^{-2} \sum_{j=1}^k c_j^2 \left(\Lambda_j(s) + (f'_{j,s})^2(s) \frac{(1+\varepsilon)(t-s)}{\varepsilon} + M_{a,\gamma}\big((1+\varepsilon)(t-s)\big)^{1+\gamma} \right) \\ + 2\Lambda_k(t) \sum_{j_1<j_2} c_{j_1} c_{j_2} (1+\varepsilon)^2 (t-s)^3 (f'_{j_1,s}(s) f'_{j_2,s}(s) + C) \end{aligned}$$

For t sufficiently close to s this is maximal for $c_j = 0$ for $j \neq k$ and $c_k = \|\psi^{(t+\varepsilon(t-s))}_{k,s,t}\|_2^{-1}$. This is because the $\Lambda_1(s) < \Lambda_2(s) < \cdots < \Lambda_k(s)$ are fixed and distinct with probability 1, but all the remaining terms (except possibly the error term on the first line) converge to 0 as $t \to s$. The error term is identical in all terms so does not change the optimization. Therefore for some t sufficiently close to s $\Lambda_k(s) + (f'_{k,s})^2(s)\frac{(1+\varepsilon)(t-s)}{\varepsilon} + M_{a,\gamma}\big((1+\varepsilon)(t-s)\big)^{1+\gamma}$ will be the dominant term and so the right hand side is maximized when all of the c_i are 0 except for c_k. By previous argument in line (17) we have that $c_k^2 = 1 + O(t-s)^3$ and so the error we obtain by replacing c_k with 1 may be neglected. This gives us that

$$\Lambda_k(t) - \Lambda_k(s) \leq (f'_{k,s})^2(s) \frac{(1+\varepsilon)(t-s)}{\varepsilon} + M_{a,\gamma}\big((1+\varepsilon)(t-s)\big)^{1+\gamma},$$

which complete the upper bound in the proposition. To complete the lower bound we perform a similar analysis with $B_s = \text{Span}\{\varphi^{(a)}_{1,s,t}, \ldots, \varphi^{(a)}_{k,s,t}\}$. □

Lemma 11. *The eigenfunctions $f_{1,s}, \ldots, f_{k,s}$ of $\mathcal{H}^{(t)}_\beta$ converge uniformly on compact subsets to the eigenfunctions $f_{1,t}, \ldots, f_{k,t}$ of $\mathcal{H}^{(t)}_\beta$ as $s \to t$.*

Proof. We begin by reusing the notion introduced in equation (16) from the proof of proposition 8. We will show that the proposition holds for $s \searrow t$ by using the functions $\varphi^{(a)}_{k,s,t}$. One can show the identical result for $s \nearrow t$ by instead using the functions $\psi^{(a)}_{k,s,t}$. We consider families of functions of the form

$$(\varphi^{(t+\varepsilon(t-s))}_{1,s,t}, \varphi^{(t+\varepsilon(t-s))}_{2,s,t}, ..., \varphi^{(t+\varepsilon(t-s))}_{k,s,t}).$$

From the proof or 8 we get that $\prec \varphi^{(t+\varepsilon(t-s))}_{j,s,t}, \mathcal{H}^{(s)}_{\beta} \varphi^{(t+\varepsilon(t-s))}_{j,s,t} \succ \to \Lambda_j(s)$ as $s \searrow t$.

We apply fact 2.2 from [6] to get that there exists a subsequence $s_j \searrow t$ and functions $(g_1, ..., g_k)$ such that

$$(\varphi^{(t+\varepsilon(t-s_j))}_{1,s_j,t}, \varphi^{(t+\varepsilon(t-s_j))}_{2,s_j,t}, ..., \varphi^{(t+\varepsilon(t-s_j))}_{k,s_j,t}) \to (g_1, ..., g_k)$$

uniformly on compact subsets in L^2 and weakly in H^1. It remains to be shown that $(g_1, ..., g_k) = (f_{1,t}, ..., f_{k,t})$ the eigenfunctions of $\mathcal{H}^{(t)}_{\beta}$. To complete the picture we use the variational derivative characterization $\frac{d}{d\varepsilon} \prec g_j + \varepsilon h, \mathcal{H}^{(t)}_{\beta}(g_j + \varepsilon h) \succ |_{\varepsilon=0}$ to get that g_j satisfies $\mathcal{H}^{(t)}_{\beta} g_j = \tilde{\Lambda}_j g_j$ for some $\tilde{\Lambda}_j$ and so g_j is an eigenfunction of $\mathcal{H}^{(t)}_{\beta}$. The strict ordering of the eigenvalues is enough to complete the picture and give $\tilde{\Lambda}_j = \Lambda_j(t)$. It follows that $g_j = f_{j,t}$. Therefore we conclude that we in fact have

$$(\varphi^{(t+\varepsilon(t-s_j))}_{1,s_j,t}, \varphi^{(t+\varepsilon(t-s_j))}_{2,s_j,t}, ..., \varphi^{(t+\varepsilon(t-s_j))}_{k,s_j,t}) \to (f_{1,t}, ..., f_{k,t})$$

uniformly on compact subsets in L^2 and weakly in H^1.

□

This weak convergence in H^1 suggests that we should have convergence of the derivatives $f'_{j,t}(t) \to f'_{j,t_0}(t_0)$ as $t \to t_0$, and indeed by making use of the fact that the eigenfunctions are almost linear near the boundary point this can be shown. In particular if the eigenfunctions are approximately linear near their endpoint then convergence on compact subsets will imply that the derivatives converge at the end points.

Lemma 12. *For all t_0, and any $j = 1, ..., k$ we have $\lim_{t\to t_0} f'_{j,t}(t) = f'_{j,t_0}(t_0)$.*

Proof. Let $\varepsilon > 0$ We use the following: In a fixed neighborhood of t_0 we have the bound from Proposition 7 with a $C_{\delta,\gamma}$ depending on the neighborhood size δ and $0 < \gamma < 1/2$. We now observe that for $x > t \wedge t_0$ in a neighborhood of t_0 we have

$$\begin{aligned} |f'_{j,t}(t) - f'_{j,t_0}(t_0)| &\leq |f'_{j,t}(t) - \frac{f_{j,t}(x)}{x-t}| + |f'_{j,t_0}(t) - \frac{f_{j,t_0}(x)}{x-t_0}| + |\frac{f_{j,t}(x)}{x-t} - \frac{f_{j,t_0}(x)}{x-t_0}| \\ &\leq C_{\delta,\gamma}\big((\eta_\delta + f'_{j,t}(t))(x-t)^{1+\gamma} + (\hat{\eta}_\delta + f'_{j,t_0}(t_0))(x-t_0)^{1+\gamma}\big) \\ &\qquad + |\frac{f_{j,t}(x)}{x-t} - \frac{f_{j,t_0}(x)}{x-t_0}| \end{aligned}$$

The previous convergence result Lemma 11 give us that the final term may be made arbitrarily small as $t \to t_0$ for any fixed x. Choose t and t_0 close enough and x small enough so that $(x-t)^{1+\gamma}$ and $(x-t_0)^{1+\gamma}$ are both small enough that the first two terms are both less than $\varepsilon/3$, then by letting t go to t_0 (which won't hurt the bounds on the first two terms) we will get that the final term is also bounded by $\varepsilon/3$. Therefore

$$\lim_{t\to t_0} f'_{j,t}(t) = f'_{j,t_0}(t_0).$$

□

Proposition 13. *For any fixed k the process $\mathcal{G}_t^{(k)}$ is differentiable as a function of t. With the derivatives given by*

$$\frac{d}{dt}\Lambda_j(t) = (f'_{j,t}(t))^2.$$

See Remark 9 for the proof.

3 The discrete to continuous convergence

In this section we wish to use the machinery developed for the proof of the original soft edge limit in order to show convergence of the t dependent eigenvalue process. To do this we begin by recalling the general convergence theorem from section 5 of [6].

Theorem 14 (Theorem 5.1 [6])**.** *Suppose that H_n is a tridiagonal matrix with*

$$\begin{array}{ll} \textit{diagonal} & 2m_n + m_n y_{n,1}(1), 2m_n + m_n y_{n,1}(2), 2m_n + m_n y_{n,1}(3), \ldots \\ \textit{off-diagonal} & -m_n + \frac{1}{2}m_n y_{n,2}(1), -m_n + \frac{1}{2}m_n y_{n,2}(2), -m_n + \frac{1}{2}m_n y_{n,2}(3), \ldots \end{array}$$

and $H = -\partial_x^2 + Y'(x)$ acting on $H'_{\text{loc}} \mapsto D$ the space of distributions with boundary condition $f(0) = 0$ (see [6] for further details). Let $Y_{n,i}(x) = \sum_{j=1}^{\lfloor nx \rfloor} y_{n,i}(j)$. For any fixed k, the bottom k eigenvalues of H_n converge to the bottom k eigenvalues of H if the following two conditions are met:

1. *(Tightness/Convergence) There exists a process $x \mapsto Y(x)$ such that*

$$\begin{array}{lcl} (Y_{n,i}(x) : x \geq 0) & & i = 1, 2 \quad \textit{are tight in law,} \\ (Y_{n,1}(x) + Y_{n,2}(x) : x \geq 0) & \Rightarrow & (Y(x); x \geq 0) \quad \textit{in law,} \end{array}$$

   with respect to the Skorokhod topology of paths; see [3] for the definitions.

2. *(Growth/Oscillation bound). There is a decomposition*

$$y_{n,i}(k) = \frac{1}{m_n}(\eta_{n,i}(k) + \omega_{n,i}(k)),$$

   for $\eta_{n,i}(k) \geq 0$, deterministic, unbounded non-decreasing functions $\bar{\eta}(x) > 0, \zeta(x) \geq 1$, and random constants $\kappa_n(\omega) \geq 1$ defined on the same probability space which satisfy the following: The κ_n are tight in distribution, and, almost surely,

$$\bar{\eta}(x)/\kappa_n - \kappa_n \leq \eta_{n,1}(x) + \eta_{n,2}(x) \leq \kappa_n(1 + \bar{\eta}(x)),$$
$$\eta_{n,2}(x) \leq 2m_n^2.$$
$$|\omega_{n,1}(\xi) - \omega_{n,1}(x)|^2 + |\omega_{n,2}(\xi) - \omega_{n,2}(x)|^2 \leq \kappa_n(1 + \bar{\eta}(x)/\zeta(x))$$

   for all n and $x, \xi \in [0, n/m_n]$ with $|x - \xi| \leq 1$.

Ramírez, Rider, and Virág show in section 6 of [6], that the tridiagonal model $H^{(k)}_{n,\beta}$ defined in (9) with $k = 1$ satisfies the the conditions of the theorem with $m_n = n^{1/3}$ and $Y(x) = \frac{x^2}{2} + \frac{2}{\sqrt{\beta}} b_x$. The same arguments may be used to show that for $H^{(\lfloor tn^{1/3} \rfloor)}_{n,\beta}$ the same convergence statements hold with $m_n = n^{1/3}$ and $Y(x) = \frac{x^2}{2} - tx + \frac{2}{\sqrt{\beta}} b_x$. These are two different distributional convergence statements, but with a slight modification of the proof of Theorem 14 we may show a joint distributional convergence for any finite collection $\{t_1, t_2, ..., t_j\}$.

Proof of 5. Let $t_1 < \cdots < t_\ell$ be any finite collection of times (possibly negative). We observe using the work in Section 6 of [6] that the matrices $H^{(\lfloor t_1 n^{1/3} \rfloor)}_{n,\beta}, H^{(\lfloor t_2 n^{1/3} \rfloor)}_{n,\beta}, ..., H^{(\lfloor t_\ell n^{1/3} \rfloor)}_{n,\beta}$ satisfy the conditions of Theorem 14 with $m_n = n^{1/3}$ and $Y^{(t_j)}(x) = \frac{x^2}{2} - t_j x + \frac{2}{\sqrt{\beta}} b_x$. Moreover we have

$$y^{(t_j)}_{n,i}(k) = y^{(t_1)}_{n,i}(k + \lfloor t_j n^{1/3} \rfloor).$$

Because of this identity the if the conditions of Theorem 14 hold for t_1 then they also hold for $t_2, ..., t_\ell$. Therefore for any subsequence we can extract a further subsequence such that we have the following joint distributional convergence:

$$\begin{aligned} (\int_0^x \eta^{(t_j)}_{n,i}(y)dy; x \geq 0) &\Rightarrow (\int_0^x \eta^{(t_j)}_i(y)dy; x \geq 0), \\ (Y_{n,i}(x); x \geq 0) &\Rightarrow (\frac{x^2}{2} - t_j x + b_i(x + t_j); x \geq 0), \qquad j = 1, ..., \ell \\ \kappa^{(t_j)}_n &\Rightarrow \kappa^{(t_j)} \end{aligned}$$

where the first line converges uniformly on compact subsets and the second in the Skorokhod topology. Notice that the brownian motions b_i that appear are the same for all j. The Skorokhod representation theorem (see Theorem 1.8, Chapter 3, or [3]) gives us that there exists a probability space so that the necessary convergence statements hold with probability 1. This allows us to reduce to working with the deterministic case and the remainder of the proof goes through unchanged. In all at this point we have proved that

$$\{\lambda_1(\lfloor t_j n^{1/3} \rfloor), \lambda_2(\lfloor t_j n^{1/3} \rfloor), ..., \lambda_k(\lfloor t_j n^{1/3} \rfloor)\}_{j=1,...,\ell} \Rightarrow \{\Lambda_1(t_j), \Lambda_2(t_j), ..., \lambda_k(t_j)\}_{j=1,...,\ell}$$

where $\Lambda_1(t_j) < \Lambda_2(t_j) < \cdots$ are the eigenvalues of the operator

$$H^{(t_j)} = -\frac{d^2}{dx^2} + x + t_j + db(x + t_j)$$

acting on functions in $L^*[0, \infty)$. These are exactly the eigenvalues of the operator $\mathcal{H}^{(t_j)}_\beta$ defined in (6). Therefore we have convergence of finite dimensional distributions which completes the proof of Theorem 5. □

4 Distribution of the derivatives

We need to begin by showing that the eigenvalues of the discrete operator follow an approximately linear pattern where the 'slope' is determined by the first entry of the eigenvector.

Because we know the distribution of the spectral weights which are found in these first entries we can then use this property to determine the distribution of the eigenfunctions in the limit. This will in turn give us the derivative of the process as desired.

Proposition 15. *Let v be the eigenvector associated to $\lambda_i(n,0)$ the ith lowest eigenvalue of $H^{(0)}_{n,\beta}$ defined in (9), and let $1/4 < \gamma < 1/2$, then for $t \in [0, x_0]$ there exists C_{γ,x_0} depending only on γ and x_0 such that*

$$|v_{\lfloor tn^{1/3} \rfloor} - tn^{1/3} v_1| \leq C_{\gamma,x_0} t^{2+\gamma} n^{1/6-\frac{2}{3}\gamma}.$$

Before continuing on to the proof of the proposition we will need some information on the distribution of v_1.

Lemma 16 (Dumitriu-Edelman [1])**.** *The spectral weights associated to the tridiagonal model in (2) are Dirichlet with parameters $(\frac{\beta}{2}, ..., \frac{\beta}{2})$. These weights are the same as the first entry of each normalized eigenvector. The marginal distribution of a single spectral weight is*

$$q_i \sim \mathrm{Beta}\Big(\frac{\beta}{2}, (n-1)\frac{\beta}{2}\Big), \qquad Eq_i = \frac{1}{n}, \qquad Var\ q_i = \frac{\beta(n-1)}{n^2(\beta n + 2)}.$$

proof of Lemma 15. Recall that we're working with the matrix $H^{(0)}_{n,\beta} = n^{1/6}(2\sqrt{n} - A_\beta)$. To start let's scale the $n^{2/3}$ out of the leading term then the resulting matrix has the form

$$n^{-2/3} H^\beta_n = \begin{bmatrix} 2+\rho_1 & -1+r_1 & & \\ -1+r_1 & 2+\rho_2 & -1+r_2 & \\ & \ddots & \ddots & \ddots \end{bmatrix}.$$

Under this we have that $\rho_i \sim \frac{1}{\sqrt{n}}\mathcal{N}(0,2)$ and $r_i = \frac{i}{2n} + \frac{1}{\sqrt{n}}\eta_i$ where η_i is an order 1 random variable with Gaussian tails. Notice that this rescaling does not change the distribution of the eigenvectors, but the lowest eigenvalues will now be on the order of $n^{-2/3}$. Before we start we give two bounds: First suppose we have a sequence $X_1, X_2, ...$ of independent random variables with $EX_i = 0$ and $EX_i^2 = \sigma^2$ and let $0 < \gamma < 1/2$ there exists a random constant C_γ such that

$$\sum_{j=1}^k X_j \leq C_\gamma (\sigma^2 k)^\gamma \qquad \text{for all } k = 1, 2, 3, ... \tag{27}$$

To prove this we merely need to observe that the sum when normalized by dividing by $(\sigma^2 k)^\gamma$ converges to 0 in probability. This is a consequence of the Lindeberg-Feller theorem. The second bound involves sequences. Suppose that $a_1, a_2, a_3, ...$ is a sequence such that $|(a_{k+1} - a_k) - a_1| \leq \eta$ for all $k \geq 1$, then

$$|a_k| \leq k(a_1 + \eta). \tag{28}$$

Now we move on the the main part of the proof. Suppose that v solves $H^{(0)}_{n,\beta} v = \lambda_j v$ with $\|v\| = 1$. We can check that v_k satisfies the following:

$$\begin{aligned} v_{k+1} - v_k &= (v_k - v_{k-1}) + r_{k-1}v_{k-1} + (\rho_k - \lambda)v_k + r_k v_{k+1} \\ &= \sum_{\ell=1}^k r_{\ell-1}v_{\ell-1} + (\rho_\ell - \lambda)v_\ell + r_\ell v_{\ell+1} \end{aligned} \tag{29}$$

if we assume that $|v_{k+1} - v_k - v_1| \le \eta/\sqrt{n}$ for $k \le x_0 n^{1/3}$ then using (28) we get that $|v_k| \le k(v_1 + \eta/\sqrt{n})$. We rewrite the first term in the sum:

$$\sum_{\ell=1}^{k} r_\ell v_{\ell+1} = \sum_{\ell=1}^{k} (R_{\ell+1} - R_\ell) v_{\ell+1} = v_{k+1} R_{k+1} + \sum_{\ell=1}^{k} R_\ell (v_\ell - v_{\ell-1})$$

Notice that R_k is the sum of k independent mean $1/(2n)$ random variables each with variance $2/\beta n$ (approximately). By (27) there exists some random constant C_γ such that $|R_k| \le C_\gamma (k/n)^\gamma$ for all k. Therefore we have

$$\sum_{\ell=1}^{k} r_\ell v_{\ell+1} \le 2 C_1 k^{1+\gamma} (v_1 + \eta/\sqrt{n}) n^{-\gamma}.$$

This same argument holds for the other two terms in (29). Using Lemma 16 and the observation $v_1 = q_j$ and the fact that λ_j is order $1/n^{2/3}$ we get that:

$$|v_{k+1} - (k+1)v_1| \le C \frac{q_j + \eta/\sqrt{n}}{n^\gamma} \sum_{\ell=1}^{k} \ell^{1+\gamma} = C' \frac{q_j \sqrt{n} + \eta}{n^{\frac{1}{2}+\gamma}} k^{2+\gamma}$$

for some constants C and C' uniform in k (for $k \le x_0 n^{1/3}$). For $k \sim t n^{1/3}$ this give us

$$|v_{\lfloor t n^{1/3} \rfloor} - t n^{1/3} v_1| \le C' t^{2+\gamma} n^{1/6 - \frac{2}{3}\gamma}.$$

Choose $\gamma > 1/4$, then for n sufficiently large this still fit's within our original assumption, and so for $t^{2+\gamma} \ll \eta$ the initial assumption on the increments holds.

□

Proposition 17. *At any fixed time $t > 0$ the derivatives of the process $\mathcal{G}_k^{(t)}$ in t are independent with distribution*

$$\frac{d}{dt} \Lambda_j(t) = \Gamma_j, \qquad \Gamma_j \sim \text{ Gamma } (\frac{\beta}{2}, \frac{2}{\beta}),$$

for $j = 1, 2, ..., k$.

Proof. We begin with the observation that the process $\mathcal{G}_k^{(t)} - t$ stationary in t and therefore the distribution of the derivative for all t is determined by the distribution of the derivate at $t = 0$. Let $v^{(1)}, ..., v^{(\ell)}$ be the eigenfunctions of $H_{n,\beta}^{(0)}$. These may be embedded as step functions with $v^{(i)}(x) = n^{1/6} v^{(i)}_{\lfloor x n^{1/3} \rfloor}$ in $L^2[0, n^{2/3})$. We need to check that $\|v^{(i)}\|_{L^2} = 1$. Indeed we have $\|v(x)\|_{L^2} = (n^{1/6})^2 n^{-1/3} \|v^{(i)}\|_2$. From the proof of Theorem 5 (see Lemma 5.8 [6]) we get that there exists a subsequence along which

$$n^{1/2}(v_1^{(1)}, ..., v_1^{(\ell)}) \Rightarrow (f'_{1,0}(0), f'_{2,0}(0), ..., f'_{\ell,0}(0)).$$

The continuous mapping theorem immediately gives us convergence of the squared entries to the squared derivatives. Therefore if we can compute the limiting distribution of the first entry of each of the first ℓ eigenvectors of the matrix $H^{(0)}_{n,\beta}$ this will determine the distribution of $\{(f'_{i,0}(0))^2\}_{i=1}^{\ell}$, which by Proposition 13 gives us the distribution of the derivative of the eigenvalue process in t.

From Lemma 16 we know that the square of the spectral weights $q_1^2, ..., q_n^2$ are exchangeable with distribution Dirichlet$(\frac{\beta}{2}, ..., \frac{\beta}{2})$ and $v_1^{(i)} = q_i$. We now use the following characterization of a Dirichlet distribution: Let $X_1, X_2, ..., X_n$ be independent identically distributed with $X_i \sim$Gamma $(\frac{\beta}{2}, 1)$, then for

$$q_i^2 = \frac{X_i}{X_1 + X_2 + \cdots + X_n}, \qquad \text{we get} \qquad (q_1^2, q_2^2, ..., q_n^2) \sim \text{Dirichlet}(\tfrac{\beta}{2}, ..., \tfrac{\beta}{2}).$$

We observe that by the strong law of large numbers $(X_1 + \cdots + X_n)/n \to \frac{\beta}{2}$ in probability, and note that for $\eta \sim$ Gamma (k, θ), $c\eta \sim$ Gamma $(k, c\theta)$. Therefore this characterization this is enough to give the joint convergence statement

$$n(q_1^2, q_2^2, ..., q_\ell^2) \Rightarrow (Y_{1,0}, Y_{2,0}, ..., Y_{\ell,0}), \qquad Y_{i,0} \sim \Gamma(\tfrac{\beta}{2}, \tfrac{2}{\beta}),$$

where $Y_{1,0}, ..., Y_{\ell,0}$ are independent. This is using the shape and scale convention for Gamma random variables. □

References

[1] Ioana Dumitriu and Alan Edelman, *Matrix models for beta ensembles*, J. Math. Phys. **43** (2002), no. 11, 5830–5847. MR MR1936554 (2004g:82044)

[2] Alan Edelman and Brian D. Sutton, *From random matrices to stochastic operators*, 2007, math-ph/0607038.

[3] Stewart N. Ethier and Thomas G. Kurtz, *Markov processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986, Characterization and convergence. MR 838085 (88a:60130)

[4] Peter J Forrester and Taro Nagao, *Determinantal correlations for classical projection processes*, Journal of Statistical Mechanics: Theory and Experiment **2011** (2011), no. 08, P08011.

[5] Taro Nagao and Peter J Forrester, *Multilevel dynamical correlation functions for dyson's brownian motion model of random matrices*, Physics Letters A **247** (1998), no. 1, 42 – 46.

[6] José A. Ramírez, Brian Rider, and Bálint Virág, *Beta ensembles, stochastic Airy spectrum, and a diffusion*, J. Amer. Math. Soc. **24** (2011), no. 4, 919–944. MR 2813333 (2012c:60022)