Sobolev spaces

Sobolev spaces, introduced by Sergei Sobolev, is an important class of function partly/mainly for solving PDEs, that have non-smooth solutions.
At the same time it is a natural extension of L^p-spaces.

Suppose we want to solve a PDE

$$\Delta u = f(x) \quad \text{in a domain } D$$

with bdry values

Since we take second derivatives of u we expect u to be C^2, and f continuous, at least

If we want to solve this problem then we immediatly encounter troubles [?]

The trouble comes from this example

$$u(x) = x_1 x_2 \left[-\log|x|\right]^a$$

$0 < a < 1$

$|x| < 1$

Here $\Delta u = \ldots = f(x)$

is continuous, but $u \notin C^2$

So how do we find solutions to PDEs with R.H.S. that are not "smooth" enough? say even contin. R.H.S. ?

A way of circumventing this problem is to solve it in weak space, or weakly and not pointwise.

If $\Delta u = f$ then

$$\Delta u \varphi = f \varphi \qquad \forall \varphi \in C_0^2(B_1)$$

so

$$\int \Delta u \varphi = \int f \varphi$$

integration gives

$$(*) \qquad -\int \nabla u \nabla \varphi = \int f \varphi$$

This last identity asks only for ∇u to be integrable

From L^p-theory we also know that the L.H.S. of $(*)$ is defined

If $\nabla u, \nabla \varphi \in L^2$

and that the R.H.S. is defined

if $f, \varphi \in L^2$

so all to all we seem to need

$$f \in L^2, \varphi \in L^2, \quad \nabla u, \nabla \varphi \in L^2$$

to have a meaning for $\circledast$.

We may thus define the PDE

$$\Delta u = f$$

weakly by asking

$$-\int \nabla u \nabla \varphi = \int f \varphi \qquad \begin{array}{l} \forall \varphi \in L^2(B_1) \\ \nabla \varphi \in L^2(B_1) \end{array}$$

and $\nabla u \in L^2(B_1)$

I also add $u \in L^2(B_1)$ (for fun!)

This is called weak solution

A more interesting thing is that Sobolev spaces can be used to find solutions to PDEs.

E.g. if

$$J(v) = \int_{B_1} |\nabla v|^2 + 2f(x)v$$

Then for all $v \in L^2$, with $\nabla v \in L^2$ over B_1, with $v = g$ (bdry values) we can minimize J over

$$W_g^{1,2}(B_1) = \{v \in L^2, \nabla v \in L^2, v = g \text{ on } \partial B_1\}$$

and prove that the minimizer u satisfies

$$-\int \nabla u \nabla \varphi = \int f \varphi$$

$$\forall \varphi \in W_0^{1,2}(B_1)$$

Weak derivatives

suppose f is differentiable of order $\underline{\alpha}$
$\alpha=(\alpha_1,\dots,\alpha_n)$, then for φ smooth $C_0^\infty(B_r)$

$$\int D^\alpha f\,\varphi = (-1)^{|\alpha|}\int f\,D^\alpha\varphi$$

A natural way of defining $D^\alpha f$ for non-smooth functions is to say g is the D^α derivative of f if

$$\int g\varphi = (-1)^{|\alpha|}\int f\,D^\alpha\varphi$$

$\forall\,\varphi$ smooth $C_0^\infty(B_r)$.

Reasonably f should be in $L^1_{loc}(B_r)$ at least.

obviously $D^\alpha f$ is uniquely (up to a.e.) defined

Def. $W^{k,p}(B)$ = all L^p functions f on B, s.t. $D^\alpha f \in L^p$, for $|\alpha| \le k$.

We set

$1 \le p < \infty$: $$\|u\|_{W^{k,p}(B)} = \left(\sum_{|\alpha| \le k} \int_B |D^\alpha u|^p \right)^{1/p}$$

$p = \infty$ $$\|u\|_{W^{k,\infty}(B)} = \sum_{|\alpha| \le k} \operatorname*{ess\text{-}sup}_{x \in B} |D^\alpha u|$$

Def. $W_0^{k,p}(B)$

is defined as the closure of $C_0^\infty(B)$ in $W^{k,p}(B)$; i.e. $u \in W_0^{k,p}(B)$ iff $\exists\, u_n \in C_0^\infty(B)$:

$$\|u - u_n\|_{W^{k,p}(B)} \to 0.$$

We also define Hilbert spaces

$H^k = W^{k,2}$, with norm coming

from the inner product

$$\langle u,v\rangle_{H^k} = \sum_{|\alpha|\le k} \int_B D^\alpha u D^\alpha v$$

and $\quad H_0^k = W_0^{k,2}$

Basic properties

Sobolev space $W^{k,p}$ is Banach sp.

$W_0^{k,p}(B)$ is closed subsp. of $W^{k,p}(B)$

H^k, H_0^k are Hilbert.

Ex. Recall from courses (Leb. theory)

that if $f \in L^1(0,1)$ and

$$g(x) = \int_0^x f(t)\,dt \quad \text{then}$$

g is absolutely contin. and $g' = f$ a.e.

So in general if $g \in L^1$ has a weak derivative $g' \in L^1$ then g is absolut. contin. on $[0,1]$.

At this stage several questions arise:

1) Completeness of Sobolev spaces, being Banach spaces (we mentioned this. For proof see Adams-Fournier page 60)

2) Dual spaces $(W^{k,p})^*$, what are they?

we know $(L^p)^* = L^q$ $\quad 1 \le p < \infty, \frac{1}{p}+\frac{1}{q}=1$

what about $(W^{k,p})^*$?

This is more tricky! Bcs

$$W^{k,p} \subset L^p \Rightarrow (W^{k,p})^* \supset (L^p)^*$$

so this might be huge space!

E.g. for $k=2$, $p>n$ we shall see later that $W^{2,p}$ $(p>n)$ has representative in C^0; so formally we can think

$$W^{2,p} \subset C^0 \quad , \text{ and then}$$

$$(W^{2,p})^* \supset (C^0)^*$$

where $(C^0)^*$ already contains all Borel measures! (see Adams-Fournier page 62 -)

For $p=2$, Riesz representation theory gives easier access to $(W^{k,2})^*$

Let us focus on $(W^{1,2})^*$

Thm: This space is represented by functions

$$F = f_0 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \quad , \text{ with } f_0, f_1, \dots f_n \in L^2(B)$$

where $\frac{\partial f_i}{\partial x_i}$ is in the sense of distribution

Also $\|F\|_{(W^{1,2}_{(B)})^*} = \inf_{(L^2(B))^{n+1}} \sum_{i=0}^{n} \|f_i\|^2_{L^2(B)}$

notation $W^{-k,q} := (W^{k,p})^*$ $\frac{1}{p}+\frac{1}{q}=1$

and $H^{-1} := (W^{1,2})^*$

The proof of the thm follows by Riesz repres. theory, since $W^{1,2}$ is a Hilbert space, so that for each

$$F \in (W^{1,2})^* \quad \exists \; g \in H^1 :$$

$$F(u) = \langle g, u \rangle_{H^1(B)}$$

so set $f_0 = g$; $f_i = \frac{\partial g}{\partial x_i}$ $i=1,\dots,n$

The norm is also given by

$$\|F\|_{H^{-1}} = \sup_{\|u\|_{H^1}=1} |F(u)| =$$

$$\sup_{\|u\|_{H^1}=1} \langle g, u \rangle_{H^1} = \left\langle g, \frac{g}{\|g\|_{H^1}} \right\rangle = \|g\|_{H^1} =$$

$$= \left(\int \sum_{i=0}^{n} |f_i|^2 \right)^{1/2}$$

Approximation / density thm

Many times proving properties for $W^{k,p}$ might be hard if the function is not smooth. E.g. certain inequalities for $W^{k,p}$ may require proof through smooth functions and then tending to limits (Embedding thms are among ~~one~~ such properties).

However if we know that smooth functions are dense in $W^{k,p}$ then we may work with them and tend to limit. In particular we have

Thm. $C^\infty(B) \cap W^{k,p}(B)$ is dense in $W^{k,p}(B)$.

The proof uses mollifiers, convolution with smooth function

$$u_h(x) = h^{-n}\int_B \rho\left(\frac{x-y}{h}\right)u(y)\,dy$$

$$\rho(x) = \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right) & |x| \le 1 \\ 0 & |x| \ge 1 \end{cases}$$

c: $\int \rho\, dx = 1$

and showing $\|u_h - u\|_{W^{k,p}(B)} \to 0$.

Embedding theorems

This is about pointwise properties of Sobolev functions

Ex. if $f = |x|^a$ in $B_1 \subset \mathbb{R}^n$ then

$$|D^k f| \approx |x|^{a-k} \quad \text{near } 0.$$

so $$\int_{B_1} |D^k f|^p \approx \int_{0+} r^{p(a-k)+n-1}\,dr$$

this is bdd, i.e. $f \in W^{k,p}$

if $\quad a > k - \frac{n}{p}$

Now for $k - \frac{n}{p} \geq m \geq 0$ we have $a \geq m$

and f is C^m

~~Similar drills can be made for $W^{k,p}$ with $|x|^a$.~~

This suggests that when p is large

the $W^{k,p}$ functions must have

some pointwise regularity.

Th I if this is true for $W^{k,p}$ functions

for $p > n$, then we say

$$W^{k,p}(B) \underset{\text{embedded}}{\hookrightarrow} C^m(B)$$

We have the following results:

$$(*) \quad W_0^{k,p}(B) \longrightarrow L^{np/(n-kp)}(B) \quad kp < n$$

$$W_0^{k,p}(B) \longrightarrow C^{m}(\bar{B}) \quad 0 \le m < k - \frac{n}{p}$$

For Lipschitz domains B, we can replace $W_0^{k,p}$ with $W^{k,p}$.

$(*)$ follows from the case $k=1$:

$$W_0^{1,p}(B) \longrightarrow L^{np/(n-p)}(B) \quad p < n$$

$$W_0^{1,p}(B) \longrightarrow C^{0}(\bar{B}) \quad p > n$$

and $\exists\, C$:

1) $$\|u\|_{\frac{np}{n-p}} \le C\|Du\|_p \quad p < n$$

2) $$\sup_B |u| \le C\|Du\|_p \quad p > n$$

for all $u \in W_0^{1,p}(B)$

For a proof use density thm

$C_0^\infty \cap W^{1,p}$ is dense in $W^{1,p}$

so it suffices to show the estimates

1), 2) for C_0^1 functions.

You can also do it for $p=1$

and apply the idea to $|u|^p$.

Now

$$|u(x)| \le \int_{-\infty}^{x_i} |D_i u|\, dx_i \qquad 1 \le i \le n$$

so that

$$|u(x)|^{n/(n-1)} \le \left(\prod_{i=1}^{n} \int_{-\infty}^{\infty} |D_i u|\, dx_i \right)^{1/n-1}$$

Integrate this ineq. over x_i - variables

$i=1,\dots,n$, and apply generalized Holder

inequality $\left[\int_B g_1 \cdots g_m \le \|g_1\|_{p_1} \cdots \|g_m\|_{p_m} \right.$

$\left. \frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1 \,; \quad p_i = n-1 \text{ here} \right]$

to obtain

$$\|u\|_{L^{n/(n-1)}} \le \left(\prod_{i=1}^{n} \int_B |D_i u|\,dx \right)^{1/n}$$

$$\le \frac{1}{n} \int_B \sum_i |D_i u| \le \frac{1}{\sqrt{n}} \|Du\|_{L^1}$$

which is the case for $p=1$.

There are several variants of this thm. (see text books)

<u>Compactness</u>: This is like Arzela, Ascoli's theorem, but for $W_0^{1,p}$.

<u>Rellich-Kondrachov</u>:

i) $W^{1,p}(B) \subset\subset L^q(B)$ compactly embedded

$q < \frac{np}{(n-p)}$ for $p < n$

ii) $W^{1,p}(B) \subset\subset C^0(B)$ $p > n$

some conditions on B is forced; regul. of ∂B

This is one of the most central theorem for PDEs, when using Sobolev techniq. It is frequently used to prove properties for sol. of PDEs, or other problems once you have that a seq. of functions (sol. PDE) have uniformly bounded $W^{1,p}$ norm

Let us look at ii) $W^{1,p}(B) \subset\subset C^0(B)$ for $p > n$.

By a theorem of Morrey we have

[Morrey]: $$\|u\|_{C^\alpha(B')} \leq C \|Du\|_{L^p(B)}$$

for $\overline{B'} \subset B$. The proof is simple, see Gilbarg–Trudinger lem 7.16, Thm. 7.17.

So if $\|u_j\|_{W^{1,p}(B)} \le C_0$ for a seq. $u_j \in W^{1,p}(B)$, and C_0 = fixed

then $\|u_j\|_{C^\alpha(\bar{B}')} \underset{\text{Morrey}}{\le} C\|u_j\|_{W^{1,p}(B)} \le CC_0$

By Arzela-Ascoli's Thm $\exists$ subseq. u_{j_k} conv. to some limit u_0 in $C^0(\bar{B}')$, and of course u_0 is contin.

The proof of (i) uses regularization.

For any seq. u_j unif. bdd in $W^{1,p}$, let u_j^ε be a regularization (molify with ε radius, see G.T., page 147)

Assume $q=1$, first. Prove u_j^ε has a subseq. conv. in $L^1(B)$. This is easier since we can show u_j^ε is bdd equicont.

in $C^0(B')$, dep. on ε, so Arzela-Ascoli helps, so u_j^ε has conv. subs. in $L^1(B)$ as well. (More generally if $A \subset W^{1,p}(B)$

then A^h is precompact in $L^1(B)$)

Now work with $\|u_j(x) - u_j^h\|_{L^1}$ to show $\leq \varepsilon C_B$, i.e. u_j^h is unif. close to u_j in $L^1(B)$, and you are done. For $q < \frac{np}{n-p}$ use

Hölder's inequality

$$\|u\|_{L^q} \leq \|u\|_{L^1}^{\lambda} \|u\|_{L^{np/(n-p)}}^{1-\lambda} \leq \|u\|_{L^1}^{\lambda} \|CDu\|_{L^p}^{1-\lambda}$$

etc..

Poincaré ineq.

Let $g \in L^{p^*}(B)$; $\frac{1}{p}+\frac{1}{p^*}=1$

with $\int_B g = 1$. Let $1 \le q < \frac{np}{n-p}$

with $p < n$; $q < \infty$ for $p = n$.

and $1 \le q \le \infty$ for $p > n$.

Then $\exists\ C_0 > 0$ dep. on B, g, p, q.

s.t. for any $f \in W^{1,p}(B)$

$$(+) \quad \|f - \int_B f g\|_{L^q(B)} \le C_0 \|\nabla f\|_{L^p(B)}$$

(For a proof see Lieb-loss book)
page 218

The proof uses compactness theorem

By assuming the contrary one builds

$$h_j := f_j - \int_B g f_j \qquad j = 1, 2, \ldots$$

and s.t. (+) fails and we have

$$\text{(*)} \quad \| \overbrace{f_j - \int_B f_j g}^{h_j} \|_{L^q(B)} \geq j \, \|\nabla f_j\|_{L^p(B)}$$

and $\nabla h_j = \nabla f_j$

now $\tilde{h}_j = \dfrac{h_j}{\|h_j\|_{L^q(B)}}$ and

$$\|\nabla \tilde{h}_j\|_{L^p} = \frac{\|\nabla h_j\|_{L^p}}{\|h_j\|_{L^q}} = \frac{\|\nabla f_j\|_{L^p}}{\|h_j\|_{L^q}} \underset{\text{by (*)}}{\leq} \frac{1}{j}$$

Hence

$\tilde{h}_j$ is bounde in $W^{1,p}(B)$.

~~By compactness (Rellich Kondrachov)~~

There is a subseq. $\tilde{h}_{j_k} \to \tilde{h}_0$ some limit $\tilde{h}_0$ (weakly in $W^{1,p}(B)$) and

$\|\nabla \tilde{h}_j\|_{L^p} \to 0 \Rightarrow \|\nabla \tilde{h}_0\|_{L^p} = 0$

so $\nabla \tilde{h}_0 \equiv 0$. Since B is connected

$\tilde{h}_0 \equiv$ const. But $\int \tilde{h}_0 g = 0$, $\int g = 1 \to \tilde{h}_0 = 0$

Now using Rellich-Kondr. thm to conclude $\tilde{h}_j \to \tilde{h}_0$ strongly in L^q

Since $\|\tilde{h}_j\|_{L^q} = 1$ we have $\|\tilde{h}_0\|_{L^q} = 1$ contradicting $\tilde{h}_0 \equiv 0$. ∎