Probabilistic techniques

probability space

$(\Omega, \mathcal{F}, \mathbb{P})$ a prob. space

Ω sample space

$\mathcal{F}$ σ-lagebra

$\mathbb{P}$ probability measure $\mathbb{P}(\Omega)=1$

points $\omega \in \Omega$ are sample points

$A \subset \mathcal{F}$ is an event

a.s. = almost surely refers to properties/statement which are true, exept for events of probability zero

The law or distribution of a random variable (measurable function) X w.r.t.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$

is defined by

$$\mathbb{P}_X(A) = \mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A))$$

this means $\mathbb{P}(\omega : X(\omega) \in A)$

observe that $A \subset \mathbb{R}$ in this note.

so $\mathbb{P}_X$ is a probability/measure

on $\mathbb{R}$.

The distrib. function of X is

denoted by

$$F(x) = \mathbb{P}(X \leq x)$$

where F is non-decreasing

and $F(\infty) = 1$; $F(-\infty) = 0$

$F(x^+) = F(x)$ Right contin.

if $F(x^-) = \lim_{x \to x^-} F(x)$ then $F(x^-) = \mathbb{P}(X < x)$

$$\mathbb{P}(X = x) = F(x) - F(x^-)$$

Check any standard text.

If $\mathbb{P}_X = \mathbb{P}_Y$ then we say

$X \stackrel{d}{=} Y$; have the same distrib.

Observe that, by Radon-Nikodym

$$\frac{d\,\mathbb{P}_X}{dm} = F'(x) \quad ; \quad \underset{(a.e.)}{dm} = \text{Leb.}$$

look for properties of $\mathbb{P}_X$

ex $\mathbb{E}f(X) = \int f(x)\,\mathbb{P}_X(dx)$

use approximation $f_n \to f$; f_n simple

and Leb. bdd conv theorem

Independence: $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

and $\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \mathbb{P}(A_i)$

(cf. Fubini, iterated integrals; $f(x,y) = f_1(x)\,f_2(y)$)

σ-Field $\mathcal{F}$ and $\mathcal{G}$ independent if for each $A \in \mathcal{F}$, $B \in \mathcal{G}$, A, B indep.

Two random variables X, Y indep. if $\sigma(X)$, $\sigma(Y)$ indep.

($\sigma(X)$ = σ-field gener. by X)

From measure theoretic point of view indep. means that X, Y dep on diff. variables:

ex. $\Omega = [0,1] \times [0,1]$; $\mathbb{P}$ Lebesg.

$$X = X(x); \quad Y = Y(y)$$

One can show indep. rand. variables are like this.

ex. X, Y indep. and integr., and XY integr.

(*) then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$

Here $\mathbb{E}(X) = \int_{\Omega} X \underbrace{\mathbb{P}(d\omega)}_{d\mathbb{P}}$ is the expected value of X, i.e. integral of X.

To prove ⊛ again use simple functions

$\sum x_i \chi_{A_i}$, $\sum y_i \chi_{B_i}$ and prove it for them. Then go in lim, by monot. conv. thm.

<u>Ex.</u> of distrib. function $F(x) = \mathbb{P}(X \le x)$

1) $F(x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$ uniform distr.

2) $F(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-x} & x \ge 0 \end{cases}$ exp. distrib

3) Bernoulli distrib. for some $p > 0$

$$\mathbb{P}(X=1)=p \qquad \mathbb{P}(X=0)=(1-p)$$

4) Poisson

$$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!} \qquad k=0,1,2,\ldots$$

5) standard normal distr.

$$\mathbb{P}(X\in A)=\int_A \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

$\forall$ A Borel,

This is $\mathcal{N}(0,1)$ "normal 0,1"-law

X is also called Gaussian Random variable, in this case

<u>Check</u>: $\mathbb{E}X = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\,dx = 0$

$E(f(X)) = \int f(x)\,\mathbb{P}_X(dx)$

Variance: $Var(X) = \mathbb{E}(X - \mathbb{E}X)^2$

For Gaussian with zero mean we have $Var(X) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$

Conditional expectation:

cond. probability:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

A, B indep if $\mathbb{P}(A|B) = \mathbb{P}(A)$

Cond. exp $\quad E(X|B) = \frac{1}{P(B)} \int_B X \, \mathbb{P}(dx)$

with $\mathbb{P}(B) \neq 0$

If X sample $= \sum \alpha_j X_{A_j}$ then

$$E(X|B) = \sum \alpha_j \frac{\mathbb{P}(A_j \cap B)}{\mathbb{P}(B)}$$

Cond. exp. w.r.t. σ-fields

$\mathcal{F} \subset \mathcal{G}$ σ-fields, X is integrable and $\mathcal{G}$ measurable.

Define $E(X|\mathcal{F})$ to be the random variable Y s.t.

$$E(Y|A) = E(X|A)$$

for every $A \in \mathcal{F}$.

This gives unique Y almost surely.

<u>Remark</u> If $X \in L^2(\Omega)$ then $E(Y|\mathcal{F})$ is the projection of X onto L^2-space of $\mathcal{F}$ measurable functions. Indeed if

Z is $\mathcal{F}$ meas.

$$E(Z(X - E(X|\mathcal{F}))) = E(ZX) - E(\underbrace{Z E(X|\mathcal{F})}_{E(ZX|\mathcal{F})}) =$$

$$= \text{exercise} = E(ZX) - \underbrace{E(E(ZX|\mathcal{F}))}_{E(ZX)} = E(ZX) - E(ZX) = 0$$

For X integrable $E(X|\mathcal{F})$ is the Radon-Nikodym derivative of the measure $Q(A) := \mathbb{E}(X/A)$ wrt $\mathbb{P}/_{\mathcal{F}}$.

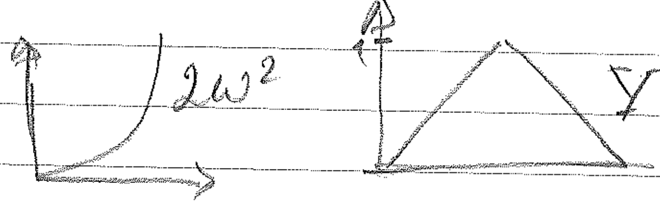
We also have $E(X|Y)$ meaning

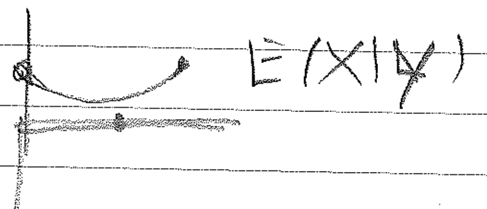
$$E(X|\sigma(Y)).$$

Ex. $\Omega = [0,1]$; $X = 2w^2$;

$Y = 1 - |2w - 1|$. Find $E(X|Y)$

Sol.





you need to find

$\sigma(Y)$ = symmetric around $w = \frac{1}{2}$ intervals

<u>Filtration</u> $(\Omega, \mathcal{F}, \mathbb{P})$ given

$\{\mathcal{F}_t\}_{t \geq 0}$ increasing family of sub-σ-algebras of $\mathcal{F}$

$$s \leq t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t$$

We assume $\mathcal{F}_t$ right contin.

$$\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$$

<u>Def</u> $X = \{X_t\}_{t \geq 0}$ stochastic process

i.e. a family of rand. var. (Think of measurable function $f(X,t)$; $t \geq 0$)

<u>Def</u> $X = \{X_t\}$ is adapted to $\mathcal{F}_t$ $\overset{\text{filtration}}{=}$ if for each $t \geq 0$ X_t is a $\mathcal{F}_t$-measurable function / Rand. var; i.e. $\mathcal{F}^{X_t} \subseteq \mathcal{F}_t$ $\forall t \geq 0$

Def Martingale (sub, super)

i) X_t is integrable for each t

ii) X_t is adapted to $\mathcal{F}_t$ ($\mathcal{F}_t$ is a filtration)

iii) $X_s = E(X_t | \mathcal{F}_s)$ $\begin{pmatrix} \leq & \text{or} & \geq \\ \text{sub} & & \text{super} \end{pmatrix}$

$t \geq s$

Def stopping time:

A random variable $T : \Omega \to [0, \infty)$ is called a stopping time if for each t

$\{T(\omega) < t\} \in \mathcal{F}_t$; i.e. the sub-level sets of T are measurable sets in $\mathcal{F}_t$

meaning of stopping time.

It says at any time t, the event $\{T < t\}$ that the stopping time has already occured is observable (through the information $\mathcal{F}_t$)

EX. $T(\omega) = \inf\{t \in \mathbb{R}_+ : X_t(\omega) \geq K\}$

for some real value K and process X_t.

EX. First Exit time: Given a domain $D \subset \mathbb{R}^n$ and a process $X_t(\omega)$ in $\mathbb{R}^n$

$$\tau(\omega) = \inf\{t \in \mathbb{R}_+ \;\; X_t(\omega) \in D^c\}$$

is the first exit time for X_t from D.

Central limit theorem and
law of large numbers

$X_1, X_2, \ldots$ an infinite sequence of
i.i.d. (indep. and ident. distr.) with
$E(X_i) = \mu$, $Var(X_i) = \sigma^2$

(weak) $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow \mu$ in probability

$$\mathbb{P}(|\bar{X}_n - \mu| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$$

(strong)

$$\bar{X}_n \xrightarrow[n\to\infty]{a.s.} \mu \quad \text{(almost surely)}$$

$$\mathbb{P}(\lim_{n\to\infty} \bar{X}_n = \mu) = 1$$

proof (weak)

$$Var(\bar{X}_n) = \frac{\sigma^2}{n}, \quad E(\bar{X}_n) = \mu$$

$$P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \text{Chebyshev's} \leq \frac{\sigma^2}{n\varepsilon^2}$$

as $n \to \infty$ we obtain
the weak law.

proof (strong law)

Assume $X_i \in L^4$, makes it easier

we let $\mu = 0$, otherwise work with

$X_i - \mu$.

Now $n^4 E(\bar{X}_n^4) = \sum_{i,j,k,l} E(X_i X_j X_k X_l)$

$$= \sum_{i=1}^{n} E(X_i^4) + 3 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} E(X_i^2 X_j^2)$$

$$= n E(X_1^4) + 3(n^2 - n)(E(X_1^2))^2$$

$$\leq n^2 C, \quad \text{assuming } E(X_i^2) \leq C_0$$

By Chebyshev's ineq.

$$P(|\bar{X}_n| \geq \varepsilon) \leq \frac{C}{\varepsilon^4 n^2}$$

Now use Borel-Cantelli lemma

$$P\left(|\bar{X}_n| \geq \varepsilon,\ i.o.\right) = 0$$

i.o. = infinitely often

Take $\varepsilon = 1/k$ then the above gives

$$\limsup_{n\to\infty} |\bar{X}_n(\omega)| \leq \frac{1}{k}$$

except possibly for $\omega \in B_k$ with $P(B_k) = 0$

set $B = \bigcup_{k=1}^{\infty} B_k$; Then $P(B) = 0$

and

$$\lim_n \bar{X}_n(\omega) = 0 \quad \forall\, \omega \notin B.$$

so this is convergence almost surely.

Central limit thm

For $X_1, \dots, X_n$ i.i.d., with

$E(X_i) = \mu$; $Var(X_i) = \sigma^2 > 0$

we have

$$\lim_{n \to \infty} \mathbb{P}\left(a \le \frac{S_n - nm}{\sqrt{n}\,\sigma} \le b\right) =$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx$$

where $-\infty < a < b < +\infty$, and

$$S_n = X_1 + \cdots + X_n .$$

The proof uses the characteristic function of the random variables:

$$\phi_X(\lambda) := E\left(e^{i\lambda\cdot X}\right) \; ; \; \lambda \in \mathbb{R}^n$$

when $X \in \mathbb{R}^n$

observe that (for $n=1$) we have

$$\phi_{\frac{S_n}{\sqrt{n}}}(\lambda) = \phi_{\frac{X_1}{\sqrt{n}}}(\lambda) \cdots \phi_{\frac{X_n}{\sqrt{n}}}(\lambda) =$$

$$= \left(\phi_{X_1}\left(\frac{\lambda}{\sqrt{n}}\right)\right)^n \quad \text{for } \lambda \in \mathbb{R}$$

bcs X_i are i.i.d.

set $\phi = \phi_{X_1}$ and use Taylor thm

$$\phi(t) = \phi(0) + \phi'(0)t + \frac{1}{2}\phi''(0)t^2 + o(t^2)$$

as $t \to 0$

where $\phi(0) = 1$; $\phi'(0) = iE(X_1) = 0$

$$\phi''(0) = -E(X_1^2) = -1$$

(here we have assumed $E(X_i) = \mu = 0$ and $\sigma = 1$. This can always be achieved by normalization)

Let $t = \frac{\lambda}{\sqrt{n}}$ to obtain

$$\phi_{X_1}\left(\frac{\lambda}{\sqrt{n}}\right) = 1 - \frac{\lambda^2}{2n} + o\left(\frac{\lambda^2}{n}\right)$$

and so

$$\phi_{\frac{S_n}{\sqrt{n}}}(\lambda) = \left(1 - \frac{\lambda^2}{2n} + o\left(\frac{\lambda^2}{n}\right)\right)^n \to e^{-\frac{\lambda^2}{2}}$$

$\forall \lambda$, as $n \to \infty$.

But $e^{-\lambda^2/2}$ is the charact. function of an $N(0,1)$ random variable, which gives the result

Brownian Motion:

Recall symmetric random walk on $\mathbb{R}^n$,

(i.e. Bernoulli random variable)

$$\mathbb{P}(X_i = e) = \frac{1}{2n} \; ; \quad |e| = 1$$

Here e mean going in direction e with one step.

X_i $i = 1, 2, \ldots$ are "assumed" (well they are) to be indep. ident. distr. (i.i.d.) rand. variable (so their distr. functions are the same, and are mutually indep.)

We thus let $\mu = \mathbb{E}(X_i)$; $\sigma^2 = Var(X_i)$

By Central Limit Theorem

$$Z_k = \left(\sum_{i=1}^{k} (X_i - \mu) \right) \Big/ \sqrt{\sigma^2 k} \approx N(0,1)$$

i.e. Z_k, for k large is Gaussian.

The random walk X_i is a process with time steps 1, $\Delta t = 1$; and space distance $\Delta x = 1$. Let us assume $n = 1$, $\mathbb{R}^1$.

Now if we scale space $\Delta x = \frac{1}{\sqrt{k}}$ and time steps to Δt

Then one can work out a relation

If $\delta := \Delta t$; then $\Delta x = \frac{1}{\sqrt{k}} = \sqrt{\delta}$

(In higher dim. this changes.)

If we set $t = k\delta$ and define

$$W_t = W_{k\delta} := Z_k \sqrt{t}$$

Keeping t fixed; $k \to \infty$; $\delta \to 0$

Then $W_{k\delta} \to N(0, \sqrt{t}) = W_t$

which is a process.

Hence W_t has density

$$\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \quad -\infty < x < \infty$$

and is Gaussian.

<u>Observations</u>: Let $\begin{cases} \mu = E(X_i) = 0 \\ \sigma = 1 \end{cases}$, then

$$\sum_{j=k+1}^{m} X_j \qquad k < m \quad \left(\begin{array}{c}\text{this is walk, from} \\ \text{steps } k+1 \text{ to } m\end{array}\right)$$

is independent of walks up to time k

i.e. $\sum_{j=1}^{k} X_j$.

Now $$\sum_{j=k+1}^{m} X_j = \sum_{j=1}^{m} X_j - \sum_{j=1}^{k} X_j$$

$$= \sqrt{m}\, Z_m - \sqrt{k}\, Z_k$$

so with $t = k\delta$ as before and $s = m\delta$

we have

$$\sum_{j=k+1}^{m} X_j \approx \frac{\sqrt{m}}{\sqrt{s}} W_s - \frac{\sqrt{k}}{\sqrt{t}} W_t$$

and $\displaystyle\sum_{j=1}^{k} X_j' = \frac{\sqrt{k}}{\sqrt{t}} W_t$.

so

$$\frac{\sqrt{m}}{\sqrt{s}} W_s - \frac{\sqrt{k}}{\sqrt{t}} W_t \quad \text{and} \quad \frac{\sqrt{k}}{\sqrt{t}} W_t$$

are independent, i.e

$$\Big(\underbrace{\frac{\sqrt{m}}{\sqrt{s}}}_{\frac{1}{\sqrt{\delta}}} \underbrace{\frac{\sqrt{t}}{\sqrt{k}}}_{\sqrt{\delta}}\Big) W_s - W_t \quad \text{and} \quad W_t$$

so $W_s - W_t$ and W_t are

indep. when $t < s$.

Since each X_j has normal distrib.

so does $W_s - W_t$.

<u>Formal def.</u> (Brownian Motion)

<u>1-dim.</u> The process $\{W_t\}_{t\geq 0}$ is called Brown. motion starting at 0, and taking values in $\mathbb{R}$, if it has the following properties:

a) If $t_0 < t_1 < \cdots < t_n$ then

$$W(t_0),\quad W(t_1)-W(t_0),\ \ldots,\ W(t_n)-W(t_{n-1})$$

are indep.

b) If $s,t \geq 0$ then

$$P\left(W(s+t)-W(s) \in A\right) = \int_A \frac{e^{\left(-x^2/2t\right)}}{\sqrt{2\pi t}}\,dx$$

c) with probab. 1, $W_0 = 0$ and $t \to W_t$ is contin.

A good source for Brown Motion
is Durrett: probability theory...
chapter 7.

To prove existence of Brownian Motion one has to work out several technical difficulties, see Sec. 7.1 in Durrett, chapter 7.

We shall consider now stochastic integrals, and specially diff. eq. of type (X being random)

$$\begin{cases} X(0) = X_0 \\ dX = a(X,t)\,dt + A(X,t)\,dW \end{cases}$$

where W is the Brownian motion.

dW is called white noise.

So what does this mean.

We mean

$$X(t) = X_0 + \int_0^t a(X,t)\,dt + \int_0^t A(X,t)\,dW.$$

Therefore the integral $\int_0^t A(x,t)\,dW$ needs to be defined.

<u>Remark</u> $t \to W(\omega,t)$ is of infinit variation (it is not Lip) so the integral $\int A\,dW$ is not understood as usual!

First, what do we mean by

$$\int_\alpha^\beta dW \underset{\text{formal def.}}{=\text{means}=} W(\beta) - W(\alpha)$$

We can also define

$$\int_0^T g\,dW := -\int_0^T g' W\,dt$$

for smooth g and then extend it to $L^2(0,T)$ functions.

The problem is to define it for random functions $\int_0^T A(X,t)\,dW$

The idea is to look at step processes

$G(t) \equiv G_k$ for $t_k \le t < t_{k+1}$; $k = 0, \dots, m-1$

for a partition $P = \{0 = t_0 < t_1 < \dots < t_m = T\}$

Then

$$\int_0^T G\,dW = \sum_{k=0}^{m} G_k (W(t_{k+1}) - W(t_k))$$

linearity holds for step processes

and $E\left(\int_0^T G\,dW\right) = 0$

and $E\left(\left(\int_0^T G\,dW\right)^2\right) = E\left(\int_0^T G^2 dt\right)$

observe

($E\left(\int_0^T G\,dw \cdot \int_0^T H\,dw\right) = E\left(\int_0^T G H\,dt\right)$

Next: Approximation

$G \in \mathbb{L}^2(0,T)$; G random $G(\omega,t)$ then

$\exists$ a seq G^n of step functions in

$\mathbb{L}^2$: $E\left(\int_0^T |G - G^n|^2 dt\right) \to 0$

(proof for contin. G (in t) use

$G^n(t) = G\left(\frac{k}{n}\right) \quad \frac{k}{n} \le t \le \frac{k+1}{n}$

for $G \in \mathbb{L}^2(0,T)$ use smoothing

$G^m(t) = \int_0^t m e^{m(s-t)} G(s)\,ds$

Finally define

$$\int_0^T G\,dW = \lim_{n\to\infty} \int_0^T G^n\,dW$$

Indefinite integral

$$I(t) := \int_0^t G\,dW \text{, then } I(0) = 0$$

$I(t)$ is a martingale

Thm (Itô's chain rule)

$$dX = F\,dt + G\,dW \quad \text{given}$$

$u = u(x,t)$ smooth. Then

$Y(t) = u(X(t), t)$ has the stoch. diff.

$$du(X,t) = u_t\,dt + u_x\,dX + \frac{1}{2} u_{xx} G^2\,dt$$

$$= \left(u_t + u_x F + \frac{1}{2} u_{xx} G^2\right)dt + u_x G\,dW$$

observe the term u_{xx} in Itô's rule, even though we have first order diff. on the left

This can be seen (too technical) from Riemann sum approxim. of $\int_0^T W dW$.

$$R(P,\lambda) = \sum_{k=0}^{m-1} W(\tau_k)\left(W(t_{k+1}) - W(t_k)\right)$$

with $\tau_k = (1-\lambda)t_k + \lambda t_{k+1} \quad k=0,\dots,m-1$

$$P = \{0 = t_0 < t_1 < \dots < t_m = T\}$$

$$|P| = \max_k |t_{k+1} - t_k| \underset{(m\to\infty)}{\to} 0$$

<u>Quadratic variation</u>:

one obtains

$$Q_P := \sum_{k=0}^{m-1} \left(W(t_{k+1}) - W(t_k)\right)^2 \xrightarrow[|P|\to 0]{\text{in } L^2(\Omega)} (b-a)$$

if P is a partition of $[a,b]$

Apply E to $(Q_P - (b-a))^2$ to arrive at

$$E\left(\left(Q_P - (a-b)\right)^2\right) \le C \sum_{k=0}^{m-1} (t_{k+1} - t_k)^2 \le$$

$$\le C\,|P|\,(b-a) \xrightarrow[|P|\to 0]{} 0$$

The quadratic variation "means" that

$$\left(W(t) - W(s)\right)^2 \approx t-s \quad \text{in } L^2$$

so $\quad W(t) - W(s) \approx \sqrt{t-s}$

i.e. $\quad dW \approx \sqrt{dt}$

Going back to $R(P,\lambda)$ we see that

$$R = \sum_{k=0}^{m-1} W(\tau_k)\left(W(t_{k+1}) - W(t_k)\right) =$$

$$= \frac{W^2(T)}{2} - \frac{1}{2}\sum_{k=0}^{m-1}\left(W(t_{k+1}) - W(t_k)\right)^2 +$$

$$\sum\left(W(\tau_k) - W(t_k)\right)^2 + \sum\left(W(t_{k+1}) - W(\tau_k)\right)\left(W(\tau_k) - W(t_k)\right)$$

$$= \frac{W^2(T)}{2} - A + B + C$$

where as $|P| \to 0$ $A \to \frac{T}{2}$ (quadr. variat.)

and $B \to \lambda T$ (a similar argument) and

$C \to 0$ all in $L^2(\Omega)$

Hence

$$\int_0^T W\,dW = \frac{W^2(T)}{2} - \frac{T}{2}$$

when $\lambda = 0$ (Itô)

and

$$\int_0^T W\,dW = \frac{W^2(T)}{2}$$

when $\lambda = \frac{1}{2}$ (Stratonovich)

<u>Thm</u> Itô's product rule

If $dX_i = F_i\,dt + G_i\,dW$ $i = 1, 2$ then

$$d(X_1 X_2) = X_2\,dX_1 + X_1\,dX_2 + G_1 G_2 \underbrace{dt}_{(dW)^2}$$

This follows easily if G_i, F_i are time independent, measurable . .

(one has to use rules such as $\int_0^r W\,dW = \frac{W^2(r)}{2} - r$

and $\int_0^r W\,dt + \int_0^r t\,dW = rW(r)$)

Then allow G_i, F_i to be step processes and after that use approx.

Now the proof of Itô's chain rule will follow using product rule and a few steps.

1) Apply it to X^m : $dX^m = mX^{m-1}dX + \frac{1}{2}m(m-1)X^{m-2}G^2dt$

2) Apply it $u(x,t) = f(x)g(t)$; g, f polynomial and use 1)

3) use approxim. by polynomials

$\underset{\substack{\uparrow \\ \text{polyn.}}}{u^n} \to u$, on $C_x^2 \cap C_t^1$ uniformly

Itô's chain rule for n-dim. Brownian motion $\bar{W}(\cdot)$ with a ODE

$$d\bar{X}(t) = \bar{b}(\bar{X},t)\,dt + B(\bar{X},t)\,d\bar{W}$$

for a C^2 function $u(x,t)$ leads to

$$du(\bar{X}(t),t) = u_t\,dt + \sum_{i=1}^{n} u_i\,dX^i + \sum a_{ij} u_{ij}\,dt$$

where $a_{ij} = \frac{1}{2}\sum_{k=1} b^{ik} b^{jk}$ $\quad B = (b^{ij})_{i,j}$

Let us now take a simple case

$B =$ Identity matrix

$\bar{b} = \bar{0}$

so we have $\bar{X}(\cdot) = \bar{W}(\cdot)$ n-dim. Brownian

Let also $u(x,t) = u(x)$ indep. of t.

then

$$du(\bar{X}(t)) = \sum u_i\,dW^i + \frac{1}{2}\Delta u(\bar{X}(t))$$

i.e.

$$u(\bar{X}(\tau)) - u(\bar{X}(0)) = \int_0^{\tau} \sum u_i\,dW^i + \frac{1}{2}\int_0^{\tau} \Delta u(\bar{X}(t))\,dt$$

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take τ to be any stopping time
then apply Exp. value:

$$\oplus \quad E\left(u(\bar{X}(\tau)) - u(\bar{X}(0))\right) = \frac{1}{2} E\left(\int_0^{\tau} \Delta u(\bar{X}(s))\, ds\right)$$

where we have used $E\left(\int_0^{\tau} \sum u_i \, dw^i\right) = 0$

[indeed
$$E\left(\int_0^{\tau} \cdots\right) = E\left(\int_0^{T} \underbrace{\chi_{\{t \le \tau\}}}_{\in L^2(0,T)} \sum u_i \, dw^i\right) =$$

= now use approxim by time-step functions = 0
(see prev pages.)]

Now if $\tau = \tau_x$ is an exits time for x from
a domain D (bdd and smooth) and
$u = 0$ on ∂D; and $\Delta u = -2$ in D
then we may let $X(\cdot) = W(\cdot) + x$
to have eq. $\oplus$ turn into

$$0 - u(x) + \frac{1}{2} E\left(\int_0^{\tau_x} -2\, ds\right) \Rightarrow u(x) = E(\tau_x)$$

Similarly if $\begin{cases} \Delta u = 0 \text{ in } D; \\ u = g \text{ on } \partial D \end{cases}$ then

$$u(x) = E\left(u(\bar{X}(\tau_x)\right) = E\left(g(\bar{X}(\tau_x)\right)$$

We also observe that if $\Delta u = 0$ in $B(x,r)$

then $\quad u(x) = E(u(\bar{X}(\tau_x))$

where we ~~may take~~ τ_x is the hitting time of Brownian motion starting at x, to $\partial B(x,r)$.

But Brownian motion being isotropic in space, suggest that the r.h.s. above is just the average of u over $\partial B(x,r)$.

This is meanvalue thm for harmonic functions