Intro:

G acts on X and preserves measure μ

$\varrho : G \times X \to X$

$\forall g$ $\varrho(g)$ preserves μ.

Then $U_{\varrho(g)} \doteq U_g : L^2(X) \to L^2(X)$ (Koopman operator)

$U_g \in \mathcal{U}(\underbrace{L^2(X)}_{\text{Hilbert space}})$ - Unitary operators

So: Measure preserving group action induces a <u>HOMOMORPHISM</u>

$$g \longmapsto U_g$$

$$\underbrace{\pi_\varrho : G \longrightarrow \mathcal{U}(L^2(X))}_{\text{representation}}$$

When is ϱ ergodic? mixing? weak mixing?

Topology on unitary operators :

- Strong operator topology : is the smallest topology which makes all maps $U \to Uf$, $f \in \mathcal{H}$, $\underline{\underline{U\text{-cts}}}$.

$\lim U_n = U \in \mathcal{U}(\mathcal{H})$ iff $\|U_n f - Uf\| \to 0$

for all f in some set D whose linear span is dense in $\mathcal{H}$.

- Weak operator topology :

$U_n \to U$ iff each matrix coefficient converges i.e.

$\lim \langle U_n g, f\rangle = \langle Ug, f\rangle$, $\forall g, f \in \mathcal{H}$.

__Exercise__ : $U_n \in \mathcal{U}(\mathcal{H})$ converges to U in strong operator topology $\iff$ it converges in weak topology.

With either topologies $\mathcal{U}(\mathcal{H})$ is a metrizable second countable top. space.

__Def__ Unitary rep. of G is a __continuous__ homomorphism from $G \to \mathcal{U}(\mathcal{H})$ - group of unitary operators.

UNITARY REPS.

Def A unitary representation of a topological group G in a Hilbert space $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ which is <u>strongly continuous</u> (i.e. continuous wrt strong op. topology) in the sense that the mapping

$$G \to \mathcal{H}, \quad g \mapsto \pi(g)\xi \quad \text{is cts for every } \xi \in \mathcal{H}.$$

Rem: Strong continuity in the above sense is equivalent to:

- $\forall \xi \in \mathcal{H}$ the mapping $g \mapsto \langle \pi(g)\xi, \xi \rangle$, $G \to \mathbb{C}$ is cts.

  <u>Matrix coefficients.</u>

$\Updownarrow$

- The mapping $G \times \mathcal{H} \to \mathcal{H}$, $(g, \xi) \mapsto \pi(g)\xi$ is cts.

Def if K is a closed G-invariant subspace of $\mathcal{H}$, then $\pi^K(g) : K \to K$ is restriction of $\pi(g)$ to K

and $\pi^K \overset{\text{def}}{=}$ <u>subrepresentation</u>

Rem: Feature of <u>unitary</u> reps: if K is a closed inv. subspace, then so is $K^\perp$:

$$\langle \pi(g)\xi, \eta \rangle = \langle \xi, \pi(g)^* \eta \rangle = \langle \xi, \pi(g^{-1})\eta \rangle = 0$$

$\forall g \in G, \ \xi \in K^\perp, \ \eta \in K$

UNITARY REPS cont. ①

Equivalent Reps:

$(\pi, \mathcal{H}) \cong (\rho, K)$ if $\exists$ intertwining operator $T \in B(\mathcal{H}, K)$ s.t. $T\pi(g) = \rho(g)T$, $\forall g \in G$.

s.t. T is onto isometry.

- (ρ, K) is strongly contained (or just contained) in $(\pi, \mathcal{H})$ if ρ is $\cong$ to a subrepresentation of π.

This is denoted by $\rho \subset \pi$

- A unitary rep. is irreducible if the only G-invariant closed subspaces of $\mathcal{H}$ are trivial ($=\{0\}$ or $\mathcal{H}$)

- Unitary dual of G: $\hat{G}$, is the set of equivalence classes of irreducible reps. of G.

- Elementary fact: If (π, K) is irreducible u.rep. strongly contained in $\bigoplus_{i \in I} \pi_i \Rightarrow \pi$ is strongly contained in some π_i

Rem: $\bigoplus \pi_i$ = the direct sum of representation (on $\mathcal{H}_i$) $(\pi_i, \mathcal{H}_i)$ is unitary rep. defined on $\mathcal{H} = \bigoplus_i \mathcal{H}_i$

$$\pi(g)\left(\bigoplus_i \xi_i\right) = \bigoplus_i \pi_i(g)\xi_i$$

Each $(\pi_i, \mathcal{H}_i)$ is a subrepresentation of $(\pi, \mathcal{H})$

...tary Characters

One dimensional reps of G are obviously irreducible.

Def. A unitary character of G is a continuous homomorphism $\chi : G \to S^1$, S^1 = multiplicative gp in $\mathbb{C}$ $\{|z|=1\}$

- We identify a one-dimensional rep. π of G with its character $g \mapsto \text{Trace}(\pi(g))$

- unit representation (unit character)

$$\mathbb{1}_G : G \to S^1 , \; g \mapsto 1, \; \forall g$$

Rem: $\mathbb{1}_G$ is contained in a unitary rep $(\pi, \mathcal{H})$

Means : $\exists K \subset \mathcal{H}$ and T unitary : $\mathbb{C} \to K$

s.t. $$\underset{(\xi)}{T \mathbb{1}_G(g)} = \underset{(\xi)}{(\pi(g)|_K)\, T} \quad \forall g \qquad \xi \in \mathbb{C}$$

$$T(\xi) = \pi(g)\big(\overset{\in K}{T(\xi)}\big) \quad \forall g$$

So this means that there is $T(\xi) \in K$ which is invariant under all $\pi(g)$

FACT: $\mathbb{1}_G$ is contained in $(\pi, \mathcal{H})$ iff $\{G\text{-invariant vectors in } \mathcal{H}\}$ is non-zero.

Rem 1 [ergodicity] Given a probab. m. measure preserving G-action on X, it induces

Koopman representation : $G \to \mathcal{U}(L^2(X))$ — Unitary.

Ergodicity $\iff$ every G-inv. set has measure 0 or 1

$\iff$ every G-inv. function f is constant

$\iff$ $\left(\begin{array}{l} L^2_0(X) = (\mathbb{C}\mathbb{1})^\perp \text{ in } L^2(X) \\ \mathcal{U}^0 = \mathcal{U}|_{L^2_0(X)} \end{array} \right)$

$\mathcal{U}^0$ does not contain the unit representation (the trivial representation)

$\iff$ There is no non-zero subspace of $L^2_0(X)$ on which $\underline{\mathcal{U}^0 \text{ is trivial}}$. (= id operator on that subspace)

Rem 2 : [Mixing]

G-action preserving $\mu \in M(X)$ is mixing if

<u>all</u> matrix coefficients $g \longmapsto \langle \pi(g)\xi, \eta \rangle$, $\xi, \eta \in L^2_0(X)$

vanish at ∞ (which means : If $\{g \in G \mid |f(g)| \geq \varepsilon\}$ is compact for any $\varepsilon > 0$, then $f : G \to \mathbb{C}$, G-locally compact, vanishes at ∞. Alternatively: $\forall g_n \to \infty$ (leaves compact sets)

$$\lim \langle \pi(g_n)\xi, \eta \rangle \longrightarrow 0 \quad \forall \xi, \eta$$

Mixing is inherited by non-compact subgroups

Ergodicity is not : Example is R acts on S^1 ergodically by R_α's restriction to Z in R is not ergodic!

FACT : Mixing $\Rightarrow$ ergodicity

$f \in L^2_0(\mu)$ is G-invariant $\Rightarrow \langle \pi(g)f, f \rangle = \langle f, f \rangle = 0 \Rightarrow f = 0$.

Schur's lemma

$(\pi, \mathcal{H})$ unitary rep of G ;

$\pi(G)' \overset{def}{=} \{ T \in \mathcal{L}(\mathcal{H}) \mid T\pi(g) = \pi(g)T \quad \forall g \in G \}$ – The COMMUTANT of $\pi(G)$

- $\pi(G)'$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed in the weak operator topology. [exercise]
- $\pi(G)'$ is also self adjoint (that is, if $T \in \pi(G)'$ then so does T^*

$$\left[T^* \pi(g) = T^* \pi(g^{-1})^* = (\pi(g^{-1})T)^* = (T\pi(g^{-1}))^* = \pi(g)T^* \right]$$

Lemma $(\pi, \mathcal{H})$ – unitary rep. of G. K closed subspace in $\mathcal{H}$. P – orthogonal projection to K.

Then: K is G-invariant $\iff$ $P \in \pi(G)'$

[If K is G-invariant, then $\underbrace{\pi(g)P}_{\in K} = P\pi(g)P \quad \forall g \in G$.

$$P\pi(g) = (\pi(g^{-1})P)^* = (P\pi(g^{-1})P)^* = P\underbrace{\pi(g)P}_{= \pi(g)P} \quad \forall g$$

$$\Big(\overset{\shortparallel}{P^*\pi(g)} \Big) \qquad (P = P^*)$$

(P - Orthogonal $\Rightarrow$ P is self-adjoint)

$$\nabla$$

$$\pi(g)P = P\pi(g) \quad \forall g.\ ]$$

Schur's Lemma

A unitary representation $(\pi, \mathcal{H})$ of G is IRREDUCIBLE <u>IFF</u> $\pi(G)'$ consists of scalar multiples of the identity operator.

Corollary G - abelian topological group.
Then any irreducible unitary representation of G is 1-dimensional. Therefore, the unitary dual $\widehat{G}$ can be identified with the set of unitary characters of G.

Pf of the corollary: Let $(\pi, \mathcal{H})$ be an irreducible unitary rep of G. Since G is abelian, $\pi(G) \subset \pi(G)'$. Thus by Schur's Lemma, for every $g \in G$ the unitary operator $\pi(g)$ is of the form $\chi(g) \cdot I$. It is clear that the dimension of $\mathcal{H}$ is 1 and that $g \mapsto \chi(g)$ is a unitary character of G.

For an arbitrary top. group G, the set of unitary characters is a group wrt pointwise multiplication

$$\chi_1\chi_2(g) := \chi_1(g)\chi_2(g), \quad g\in G.$$

The group unit = the unit character
the inverse = the conjugate character.

- If G is abelian $\Rightarrow$ $\hat{G}$ is an abelian group.
$\downarrow$
it is called THE DUAL GROUP.

+ topology of uniform convergence on compact subsets, $\hat{G}$ becomes a topological group.

- Ex1: $\forall y\in\mathbb{R}$ $\rightsquigarrow$ unitary character of $\mathbb{R}$:
$$\chi_y(x) = e^{2\pi i xy}, \quad x\in\mathbb{R}.$$

It is proved that: every unitary character is of this form for some y.

(This generalizes to any local field)

- Ex2) A character χ_y of $\mathbb{R}^n$ factorises through $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ iff $y\in\mathbb{Z}^n$. Hence $\hat{\mathbb{T}^n}$ is isomorphic to $\mathbb{Z}^n$.

Peter - Weyl theorem

given any representation $\pi : G \to \mathcal{U}(\mathcal{H})$, standard approach to understanding π is to:

a) express π in terms of irreducible reps.

b) understand all the irreducible reps.

Def: G locally compact group with left-invariant measure μ which is finite on compact subsets. Left regular representation $\lambda : G \to \mathcal{U}(G)$ is defined by

$$(\pi(g) f)(x) = f(g^{-1}x)$$

[this is a continuous unitary representation]
for the strong operator topology
defined by the family of seminorms $\{\|\cdot\|_x \mid x \in E\}$
where for $T \in \mathcal{B}(E, F)$, $\|T\|_x = \|Tx\|$

<u>Example</u>: $G = S^1$

ONB for $L^2(S^1)$ is $\{e_n \mid n \in \mathbb{Z}\}$, $e_n(z) = z^n$.

For any $a \in S^1$, $(\lambda(a) e_n)(z) = e_n(az) = a^n z^n$ i.e.

$$\lambda(a) e_n = a^n e_n$$

This means that e_n is a simultaneous eigenvector for all of the operators $\pi(a)$, $a \in S^1$,

in other words: $L^2(S^1) = \sum_{n=-\infty}^{\infty} \underbrace{\mathbb{C} \cdot e_n}$

this is a 1-dimensional (hence irreducible) invariant subspace for π

MPLE :

In contrast to the circle, the regular representation for $\mathbb{R}$ has no 1-dim. invariant subspace:

for f in such a subspace, we have for $t \in \mathbb{R}$ that $\pi(t) f = C_t f$ for some C_t

Since π is unitary : $\| \pi(t) f \| = \| f \|$

So $|C_t| = 1$.

For any interval $I \subset \mathbb{R}$:

$$\int_{I-t} |f|^2 = \int_I |\pi(t) f|^2 = \int_I |f|^2$$

because $\pi(t) f = f(x-t)$

If $f \neq 0$, can chose a compact I so that $\int_I |f|^2 = a > 0$. Clearly then we can choose a sequence t_n s.t. $\{I - t_n\}$ are mutually disjoint.

then $$\int_{\mathbb{R}} |f|^2 \geq \sum_n \int_{I-t_n} |f|^2 = \infty$$

which contradicts $f \in L^2(\mathbb{R})$ assumption.

In fact : there are no <u>finite</u> dimensional invariant subspaces of $L^2(\mathbb{R})$ for the regular representation.

d before Peter-Weyl theorem.

G - compact group. Then its Haar measure dg is FINITE and constant functions belong to $L^2(G)$. In particular the unit representation 1_G $(G \mapsto S^1 \ g \mapsto 1)$ is contained in the regular representation λ_G

defined before

Each of these properties actually characterises compact groups.

$\boxed{\text{PROP}}$ For a locally compact group, TFAE:

(i) G is compact

(ii) 1_G is contained in λ_G.

(iii) the Haar measure on G is FINITE.

Pf Exercise

The Peter-Weyl theorem describes the decomposition of the regular representation of a compact group.

$\boxed{\text{Thm}}$ [Peter-Weyl] G-compact group. λ - left regular rep on $L^2(G)$.

Then $L^2(G) = \bigoplus_i H_i$

where $\dim H_i < \infty$ and H_i are ~~invariant~~ invariant and irreducible for π.

Idea of the proof : • $ST = TS$ then eigenspaces of S are T-invariant.

• G-compact $\longrightarrow$ one can construct many ~~bdd~~ compact) integral operators which commute with λ.

• Can also get compact self-adjoint operators commuting with λ.

• Use the spectral theorem for compact bdd self-adjoint operator on $\mathcal{H}$: $\mathcal{L}^2(\mathcal{H}) = \ker T \oplus \sum_{\lambda \neq 0} E_\lambda$, $\dim E_\lambda < \infty$, $\lambda \neq 0$.
↖ direct sum.

$\boxed{\text{Lemma}}$ X-cpct metric space, G acts on X continuously
μ - ~~cc~~ G-invariant probab. measure.
$K \in C(X \times X)$ and G-invariant : $K(gx, gy) = K(x,y)$, $\forall g$.

Let $T_K : L^2(X) \circlearrowleft$ be $(T_K f)(x) = \int_X K(x,y) f(y) d\mu(y)$

If π is given by $(\pi(g)(f))(x) = f(g^{-1}x)$, then

(1) $T_K \cancel{\pi}(g) = \pi(g) T_K$

(2) T_K is a compact operator on $L^2(X)$ $\mathcal{H}$

↳ ($\overline{T_K(\mathcal{H}_1)}$ is compact in $\mathcal{H}$.)
closed unit ball

: For general actions in the lemma there may not be any G-invariant k. But for π - left regular representation on $G=X$ (so that $G=X$ is compact) there are many k's : For every $\varphi \in C(G)$ let $K(x,y) = \varphi(x^{-1}y)$. Then $K(gx,gy)=k(x,y)$.

<u>Proof of Peter-Weyl theorem</u> :

By Zorn's lemma, choose a <u>maximal</u> collection $\{V_j\}$ of mutually orthogonal, finite-dimensional $\pi(G)$-invariant subspaces $V_j \subset L^2(G)$ with V_j irreducible for π.

- Let $W = \oplus V_j$. We claim $W=L^2(G)$.

If not, we can choose $f \in W^\perp$, $f\neq 0$. Since $C(G)$ is dense in $L^2(G)$, we can find $\varphi \in C(G)$ such that $\langle \varphi, \bar{f} \rangle \neq 0$.

Define $K \in C(G\times G)$ by $K(x,y) = \varphi(x^{-1}y)$.

Then FROM LEMMA the integral operator $T=T_K$ is $\boxed{\text{compact}}$ (3.1.5) and $\boxed{\text{commutes}}$ with all $\pi(g)$.

Operator T^*T is $\boxed{\text{self-adjoint}}$ $((T^*T)^* = T^*T)$ and $\boxed{\text{compact}}$ ($T\circ S$ is compact if either S or T are compact)

+ T^*T also commutes with $\pi(g)$:

$\hookrightarrow$ $\pi(g)$ is unitary so $\pi(g) = \pi(g^{-1})^*$

so $T^*\pi(g) = T^*\pi(g^{-1})^* = (\pi(g^{-1})T)^* =$
$= (T\,\pi(g^{-1}))^* = \pi(g)T^*$

If X is a compact space with finite measure μ, and $k\in C(X,X)$, then T_k is a compact operator on $L^2(X)$

<u>Def</u> operator T is <u>compact</u> if $\overline{T(E_1)}$ is compact in F.

(3.2.3) - Spectral thm for cpct operators: Suppose $\mathcal{H}$ is Hilbert, $T \in B(\mathcal{H})$ compact and self-adjoint. Then $\mathcal{H}$ has ONB consisting of eigenvectors for T. Furthermore $\forall \lambda \neq 0$, $\dim E_\lambda < \infty$, and $\forall \varepsilon > 0$ $\{\lambda \mid |\lambda| > \varepsilon, \dim E_\lambda \neq 0\}$ is finite.

[T]herefore, by the spectral theorem

$$L^2(G) = \ker(T^*T) \oplus \sum_{\lambda \neq 0}^{\oplus} E_\lambda$$

eigenspaces of T^*T

and for $\lambda \neq 0$ $\dim E_\lambda < \infty$.

By our lemma (π commutes with T^*T) each E_λ is π-invariant.

Now let $P: L^2(G) \to W^\perp$ be the orthogonal projection. Since W is G-invariant by construction, and π is unitary, $W^\perp$ is also G-invariant

[if $y \in W^\perp$, $g \in G$ then $\forall x \in W$ we have

$$\langle \pi(g) y, x \rangle = \langle y, \pi(g)^* x \rangle = \underbrace{\langle y, \pi(g^{-1}) x \rangle = 0}_{\text{since } W \text{ is } \pi\text{-inv.}}\ ]$$

It follows that P also commutes with all $\pi(g)$.

Therefore $PE_\lambda \subset W^\perp$ is also G-invariant

$$[\ \pi(g)(PE_\lambda) = P\pi(g) E_\lambda = PE_\lambda\ ]$$

Since clearly we have $\dim PE_\lambda < \infty$, this contradicts maximality of $\{V_j\}$, if we can show that at least one $PE_\lambda \neq \{0\}$ (for $\lambda \neq 0$)

However, if $PE_\lambda = \{0\}$ for all $\lambda \neq 0$, then $\sum E_\lambda \subset W$ and therefore $W^\perp \subset (\sum E_\lambda)^\perp$

i.e. $W^\perp \subset \operatorname{Ker} T^* T$.

But $T^* T x = 0 \overset{x \in W^\perp}{\Longrightarrow}$

$$0 = \langle T^* T x, x \rangle = \langle Tx, Tx \rangle, \text{ so } Tx = 0.$$

Thus we would have $W^\perp \subset \operatorname{Ker} T$.

However: $\underset{\substack{\uparrow \\ \text{unit in } G}}{(Tf)(e)} = \int K(e,y) f(y)\,dy = \int \varphi(y) f(y)\,dy$

$$= \langle \varphi, \bar{f} \rangle \neq 0.$$

$\boxed{X \text{ compact}, \ \mu(X) < \infty, \ K \in C(X \times X) \text{ then } T_K(L^2(X)) \subset C(X)}$

So we have: $(Tf)(e) \neq 0$

and Tf is continuous

So $Tf \neq 0$ in $L^2(G)$.

However, $f \in W^\perp$, and this contradicts $W^\perp \subset \operatorname{Ker} T$

∎

<u>The general Peter-Weyl</u>: Let G be a compact gp.

(i) <u>Every</u> unitary rep of G is the direct sum of irreducible subrepresentations

(ii) Every irreducible unitary rep. of G is finite dim.

(iii) Every irreducible unitary rep of G is contained in the left regular rep λ of G.

More precisely: $\lambda = \bigoplus_{\pi \in \hat{G}} \sigma_\pi$ — subrepresentations.

$$\sigma_\pi \cong (\dim \pi)\, \pi$$

- A unitary rep. is irreducible if the only G-inv. closed subspaces of $\mathcal{H}$ are $\{0\}$ and $\{\mathcal{H}\}$.
- The set of <u>equivalence classes</u> of irreducible reps. $\hookrightarrow \Gamma$ is called the <u>unitary dual</u> $\hat{G}$ of G. [done before]