Geometric measure theory

The concept of measuring:

How to measure objects?
specially sets in $\mathbb{R}^n$.

Ex. counting measure H^0

Length, area, volume,

For measuring length we already encounter problems!

length of a unit segment $[0,1]$ is defined to be $\underline{1}$. For any other object in $\mathbb{R}^1$, one has to use this unit of measurement. E.g. the segment $l = [2, 3{,}5]$ is covered by $1{,}5$ unit and hence has length $1{,}5$.

The length of the segment $[0, \frac{1}{2}]$ is $\frac{1}{2}$ since it is covered by half unit.

What is the length of $[0, \frac{1}{2})$ or $[0, 1)$, or $(0,1)$?

so already here we have to extend the notion of measuring length of more complicated objects.

The convention one has made is that points have zero length. so this solves the above problem.

$$\text{length}\,[0,1] = \text{length}\,[0,1)$$

What next? more problems?

Let us see what happens if we use the idea of points have zero length.

<u>Ex.</u> Let $\mathbb{Q}$ be the rational numbers, labeled $\{r_j\}_{j=1}^{\infty}$; then set $I = [0,1]$ and $I_j = I \setminus \{r_1, r_2, \dots, r_j\}$.

According to previous remark and convention

$$\text{length } I_j = 1$$

This is additivity important

Here we have used one further convention that Length of two disjoint segments is the sum of their lengths.

As $j \to \infty$ $\quad I_j \to \overset{I}{\mathbb{R}} \setminus \mathbb{Q}$ = irrational nrs.

so what is $\text{length}(I \setminus \mathbb{Q}) = ?$

This example suggests a rigorous approach to the measureing.

We need thus to extended the notion of length (area, volume) to a wide class of sets in $\mathbb{R}$ ($\mathbb{R}^2$, ...).

Bearing in mind simple properties of regular length: length of disjoint sets is the sum of their lengths (this is called additivity).

So we need the notion of countable additivity, i.e. length of infinit many disjoint segments is the sum of their lengths.

It is also natural that (at least at the begening) lengths, measures should be non-negative. This is later generalized as any other function to have values in another space (see them as set functions)

so one defines measure:

<u>Def.</u> (measure)

μ an extended real valued function is called a measure if

i) $\mu \geq 0$

ii) μ countably/completely additive

we also need $\mu(\phi) = 0$, because empty set $= \phi = A \setminus A$ and we need to work with union and intersection of sets.

E.g. $A \cup B = (A \setminus B) \cup B$

$\mu(A \cup B) = \mu(A \setminus B) + \mu(B)$ additivity

If $A = B$ then $\mu(\underbrace{A \cup A}_{A}) = \mu(\phi) + \mu(A)$

$\Rightarrow \mu(\phi) = 0$

we also need to have a domain of definition for μ (like any other function)

observe that we may allow $\mu(E) = +\infty$.

But what if a set E is <u>not</u> measurable?

what does non-measurability mean?

suppose we are given a set E of $\mathbb{R}^n$ and we have a way of measuring on $\mathbb{R}^n$, we call it μ. Then we see that for any set $A \in \mathbb{R}^n$

$$\mu(A) = \mu((A \cap E) \cup (A \setminus E)) = \text{disjoint}$$

$$= \mu(A \cap E) + \mu(A \setminus E)$$

Is this always true? If yes then we say E is μ-measurable.

<u>Remark</u> $\mu(A) \leq \mu(A \cap E) + \mu(A \setminus E)$

This is subadditivity.

With this starting point we can now work our way up to extend the notion of length, area, volume.

Integrals are weighted length, area, .. so once measure is settled we get integration as well.

Lebesgue integration

Let $\mathbb{K}$ be the class of open intervals including ϕ, $\mathbb{R}^n$.

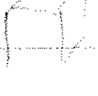
This is called sequential cover.

Define

$$\lambda(\phi) = 0$$

$$\lambda(I_{a,b}) = \prod_{i=1}^{n} (b_i - a_i) \qquad a \neq b$$

$$I_{a,b} = \{ a_i < x_i < b_i \ ; \ i = 1, \cdots, n \}$$



we define <u>outer</u> measure

$$\mu^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \lambda(E_k) \; ; \; E_k \in \mathbb{K} \quad A \subset \bigcup_{k=1}^{\infty} E_k \right\}$$

Define

$$\mu := \mu^* \Big|_{\mu^*\text{-measurable sets}}$$

$\mathcal{O} := \mu^*$-measurable sets $= \sigma$-algebra

we see that this definition extends to more general set functions.

(of course after verifying a few things, as done in books/courses, at undergraduate level.)

<u>Ex.</u> For any set function τ defined on $\mathbb{K}$ we can define

$$\mu_\tau^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \tau(E_k) : E_k \subset \mathbb{K} \quad A \subset \bigcup E_k \right\}$$

Ex. $\tau(E) = d(E) = \text{diameter}(E) =$

$$= \sup\{|x-y| \,;\, x, y \in E\}$$

prove that

$? < \mu^*_\tau([0,1]) \le 1$ in $\mathbb{R}^2$

obviously $(-\varepsilon, 1+\varepsilon) \times (-\varepsilon, \varepsilon) =: I_\varepsilon$

covers $[0,1]$, in $\mathbb{R}^2$ and $d(I_\varepsilon)$

$= \sqrt{1 + 4\varepsilon + 8\varepsilon^2} \to 1$

so $\mu^*_\tau([0,1]) \le 1$

Compute $\mu^*_\tau([0,1])$ exactly?

For this you need a covering argument, which is a fundamental tool in Analysis.

proof. $\mu^*_\tau[0,1] = 1$

suppose $\mu^*_\tau[0,1] < 1$. Then $\exists$ cover

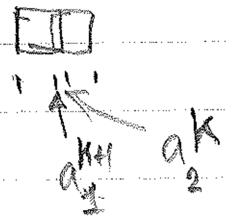
$\cup E_k$ of $[0,1]$ s.t. $\sum d(E_k) < 1 - \varepsilon_0$

still $(\cdot\cdot) < 1 - \varepsilon/2$

make E_k slightly larger and as open boxes, and enumerate them with

$$E_k = \{ a_1^k < x < a_2^k \, ; \; b_1^k < y < b_2^k \}$$

with $a_1^k < a_1^{k+1} < a_2^k$



since $[0,1]$ is compact we may assume that the number of covers are finite N (Heine-Borel). This gives us

$$\sum_{k=1}^{N} d(E_k) \geq \sum_{k=1}^{N} (a_2^k - a_1^k) \geq \sum_{k=1}^{N-1} (a_1^{k+1} - a_1^k) + (a_2^N - a_1^N) \geq a_2^N - a_1^1 > 1$$

contradiction.

observe that we did not use the y-direction

Integration. (Lebesgue)

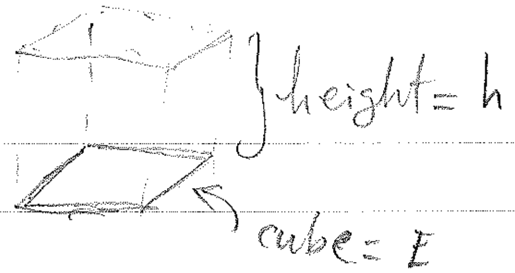
In analogy with length, area, volume one defines integrals, which are weighted versions of measures.

Ex. A unit length wire $[0,1]$ has density $f(x)$ at each $x \in [0,1]$ what is the total mass of the wire?

This is of course done by integration.

Riemann integration works well until we come up with more general objects like $\mathbb{Q} \cap [0,1]$ to find their masses!

To define Lebesgue integral we start again by cubes with one more dimension called height.



$h\chi_E$ = simple function

we can thus define simple functions

$$h(x) = \sum_{i=1}^{k} h_i \chi_{E_i} \quad ; \quad E_i \cap E_j = \emptyset \quad i \neq j$$

and their integrals

$$\int h \, d\mu = \sum h_i \, \mu(E_i) \qquad \left(\text{assume } 0 \cdot \infty = \infty \cdot 0 = 0 \right.$$

For general measurable functions H, i.e. functions whose level sets are measurable, we define

$$\int H \, d\mu = \sup_{h \leq H} \int h \, d\mu \qquad h \text{ simple}$$

(we still have $\mu \geq 0$ and we also assume $H \geq 0$, $h \geq 0$ for simplicity)

Corresponding L^p space is defined by

$$L^p_\mu = \left\{ f : \int |f|^p d\mu < \infty \right\} \qquad 1 \le p < \infty$$

In similar way one defines $\mu = \mu^+ - \mu^-$

or even complex valued, and $f = f^+ - f^-$

or even complex valued.

Check further for product spaces

$X \times Y$; separable metric spaces.

$\lambda := \mu \times \nu$; μ on X, ν on Y

Riesz Representation

For $f \in L^p_\mu(X)$ we have

$$L : L^q \longrightarrow \mathbb{R}$$
$$g \longrightarrow \int f g \, d\mu$$

is a linear map

The converse (Riesz) says that

for any $L: L^q \to \mathbb{R}$ $\exists$ unique

$f \in L^p_\mu$: $L(g) = \int f g \, d\mu$

so L^p, L^q are conjugates of

each other for $1 < p, q \leq \infty$

also $(L^1)^* = L^\infty$; $\left[\begin{array}{l} * = \text{conjugate space} \\ \quad \text{linear functionals} \end{array} \right.$

$\left(C^\circ(X) \right)^* = ?$ investigate

(Friedman page 176 – 185)
Found. of modern analysis

Differentiation of measures

Recall that for continuous functions f we have $F(x) := \int_0^x f(t)\,dt$ is diff. and

$$F'(x) = f(x).$$

Define now (for reasonable A)

$$\mu(A) = \int_A f(t)\,dt$$

Then for $I_r(x) = (-r+x, r+x)$

$$\frac{\mu(I_r(x))}{|I_r(x)|} = \frac{1}{2r}\int_{-r+x}^{r+x} f(t)\,dt \to f(x)$$

so we may define derivative of the measure μ to be $f(t)\,dt$

i.e. μ' is a nice function

What happens in general, for measures?

We need a definition, first.

Absolute continuity:

μ, λ measures on $\mathbb{R}^n$. we say μ is abs. cont. w.r.t. λ if

$$\lambda(A)=0 \Longrightarrow \mu(A)=0 \quad \forall A \subset \mathbb{R}^n$$

and if so then we use the symbol

$$\mu \ll \lambda$$

Def. upper & lower derivatives

$$\overline{D}(\mu,\lambda,x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}$$

$$\underline{D}(\mu,\lambda,x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}$$

If lim exists at x then we define

$$D(\mu,\lambda,x) = \overline{\lim} = \underline{\lim} \qquad \left(\frac{0}{0}=0 \text{ here}\right)$$

Thm
1) $D(\mu, \lambda, x)$ exists and is finite for λ- a.a. $x \in \mathbb{R}^n$

2) For all Borel sets $B \subset \mathbb{R}^n$

$$\int_B D(\mu, \lambda, x)\, d\lambda_x \leq \mu(B)$$

with equality if $\mu \ll \lambda$

3) $\mu \ll \lambda$ iff $\underline{D}(\mu, \lambda, x) < \infty$ for μ a.a. $x \in \mathbb{R}^n$

Proof ideas?

D is Radon-Nikodym derivative

Various measures

(Invariante measures)

We are familiar that geometric motion rotation, and translation does not change area. specially translation (group action) is something that we may use in other topological spaces, specially topological groups

Ex. Let $\mathbb{R}^{+} = \{x \in \mathbb{R}, x > 0\}$ and assign the group action to it which is multiplication. Define

$$\mu(A) = \int_A \frac{dx}{x}$$

Observe that the group operation is different now, it is not addition:

$$A \longrightarrow Ag$$

$$\int_{xb\in A}\frac{dx}{x} = \int_{xb\in A}\frac{d(xb)}{xb} = \int_{y\in A}\frac{dy}{y} = \int_{\mathbb{R}^+}\chi_A\frac{dx}{x}$$ (19

For $b\in\mathbb{R}^+$: $\displaystyle\int_{\mathbb{R}^+}\chi_A(x\cdot b)\frac{dx}{x} = \int_{\mathbb{R}^+}\chi_A(y)\frac{dy}{y}$

Hence $\mu = \frac{dx}{x}$ is an invariant measure on $\mathbb{R}^+$, with multip. as group action.

So we may now consider general topological groups, i.e. topological spaces with a binary operation that induces a group structure on the sets. We also require continuity of the group operations, (products, inverse) in the given topology.

Ex. $(\mathbb{R}^n,+)$; $(\mathbb{T},\cdot)$ circle group multiplication

$b\cdot x = e^{i\theta_b}e^{i\theta_x} = e^{i(\theta_b+\theta_x)}$

$(O(n), \cdot)$ orthogonal group

$\cdot$ = composition or matrix multiplic.

$(SO(n), \cdot)$ special orthogonal group [orientation preserving transformation.]

Haar measure:

A Borel measure μ in a locally compact topological group X, s.t.

$\mu(B) > 0$ $\forall$ nonempty Borel set B,

and $\mu(xB) = \mu(B)$ $\forall$ Borel sets B, $x \in X$,

is called Haar measure.

~~Thm. For every compact topological group G, there is a unique, invariant~~

Thm In every locally compact topological group X there exists at least one regular Haar measure.

Ex. $\mathcal{O}(2)$.

Since $O(2)$ is rotations and reflection around the origin, or more exactly rotations around the origin and reflection in x-axis (other reflections are composition of these), we can identify ~~[illegible]~~ the invariant Haar measure, which is uniformly distributed, with the normalized Lebesgue measure on $[-2\pi, 2\pi]$. The negative angles correspond to transformations which contain reflection.

Ex. $\mathcal{O}(n)$

The invariant measures can be reduced to $(n-1)$-dim. surface measures on S^{n-1} (see Mattila, thm 3.7.)

Ex. Grassmannian: $G(n,m)$ (in $\mathbb{R}^n$)

$G(n,m)$ = all m-dim. linear subsp. of $\mathbb{R}^n$

$G(2,1)$ = all lines through the origin

this can be characterized by the angle of each line with positive x-axis, and hence 1-dim. Leb. measure on $[0,\pi]$ induces a measure on $G(2,1)$. cf. $\mathbb{P}^1(\mathbb{R}^2)$

~~$G(n,1)$ = similar way can be seen for cf. with S^{n-1}, since each plane has a unique normal.~~

$G(n,1)$ = 1-dim. lin. subspaces in $\mathbb{R}^n$ so this too is like $\mathbb{P}^n$ and can be seen as S^{n-1}, where 2 points on the same line through the origin are identified as same objects

Hausdorff measure

Define

$$H^s_\delta := \inf\left\{\sum_{i=1}^{\infty} d^s(E_i) : A \subseteq \bigcup E_i \text{ with } d(E_i) < \delta;\ E_i \in \mathcal{F}\right\}$$

where $\mathcal{F}$ is a family of covering of X (the underlying space, think of $\mathbb{R}^n$). We can assume the are Borel sets.

$0 < s$ ~~$\mathbb{R}_+$~~

Define now

$$H^s(A) = \lim_{\delta \to 0} H^s_\delta(A) = \sup_{\delta > 0} H^s_\delta(A)$$

H^s is a Borel measure (Thm 4.2, Matt.)

H^0 = counting measure

$H^n = c_n \mathcal{L}^n$

H^s; $s = 1, 2, .., n-1$ ~~is~~ is s-dim "surface" measure in $\mathbb{R}^n$

Hausdorff dim

$$\dim_H(A) = \sup\{s : H^s(A) > 0\}$$
$$= \sup\{s : H^s(A) = \infty\}$$
$$= \inf\{t : H^t(A) < \infty\}$$
$$= \inf\{t : H^t(A) = 0\}$$

Observe that for $s < \dim_H A$ we have $H^s(A) = \infty$.

and $t > \dim_H A \implies H^t(A) = 0$

$$\dim_H(\mathbb{R}^n) = n$$

$\dim_H(T) = n-1$; if T = plane in $\mathbb{R}^n$

Ex. (Mattila)

Cantor set: $C(\lambda)$; $0 < \lambda < \frac{1}{2}$

intervals $[0,\lambda]$; $[1-\lambda, 1]$

are subtracted.

Next generation: subtract $[0, \lambda^2]$

$[1-\lambda^2, \lambda]$, ... etc.

Then $\dim_H C(\lambda) = \log 2 / \log \frac{1}{\lambda}$

This follows from the fact that each interval $I_{(k,j)}$ at step k (that remain) has length λ^k. Hence

$$H^s_{\lambda^k}(C(\lambda)) \leq \sum_{j=1}^{2^k} (d(I_{(k,j)}))^s =$$

$$= 2^k \lambda^{ks} = (2\lambda^s)^k$$

The smallest s for which this is bound.

is given by $2\lambda^s = 1$, i.e.

$$s = \log 2 / \log \tfrac{1}{\lambda}$$

Hence $H^s(C(\lambda)) =$

$$= \lim_{k \to \infty} H^s_{\lambda^k}(C(\lambda)) \leq 1$$

so $\dim(C(\lambda)) \leq s$

It suffices now to show

$$H^s(C(\lambda)) > 0 \quad \text{for} \quad s = \log 2 / \log \tfrac{1}{\lambda}$$

(see Mattila)

Ex.

Minkowski dim.

let

$$N(A,\varepsilon) = \min\{k : A \subset \bigcup_{i=1}^{k} B(x_i,\varepsilon),\ x_i \in A\}$$

$$\overline{\dim}_M A = \inf\{s : \overline{\lim_{\varepsilon \to 0}}\, N(A,\varepsilon)\,\varepsilon^s = 0\}$$

$$\underline{\dim}_M A = \inf\{s : \underline{\lim_{\varepsilon \to 0}}\, N(A,\varepsilon)\,\varepsilon^s = 0\}$$

Ex. show

$$\cancel{A}\ \dim_H(A) \le \underline{\dim}_M(A) \le \overline{\dim}_M(A) \le n$$

Ex.