1. Spaces

TVS $\supset$ TVS with sufficient family of seminorms $\supset$ TVS with countable sufficient family of seminorms $\subset$ Metrizable TVS.

$\cup$

Frechét Spaces

$\cup$

Banach NVS

$\cup$

Hilbert Spaces

- TVS = topological vector space : vector space with Hausdorff topology for which the vector space operations are continuous.

- NVS = Normed vector space
- Hilbert spaces : Complete inner product spaces
- Family of seminorms on a TVS $\{\|\cdot\|_\alpha \mid \alpha \in I\}$ is SUFFICIENT if $\forall x \in E$, $x \neq 0$, There is some $\alpha \in I$ s.t. $\|x\|_\alpha \neq 0$

Ex1 : Uniform convergence on compact subsets is given by a sufficient family of seminorms $\|f\|_K = \sup\{|f(x)| \mid x \in K\}$ K - compact

Ex2 : X-Hausdorff. One can define topology of pointwise convergence on $C(X)$ by $\|f\|_x = |f(x)|$. Then $\{\|\cdot\|_x \mid x \in X\}$ is a sufficient family of seminorms.

- FRECHÉT Space is TVS which is complete in the topology given by a countable sufficient family of semi-norms. For example C^∞ with C^r-norms
- Ex1 and Ex2 are complete spaces but not Frechét because the family of semi-norms is not countable.

More Spaces . . .

- μ - positive σ-finite measure
  (X, μ) measure space
  $f: X \to \mathbb{R}$ measurable, $1 \le p < \infty$ set $|||f|||_p = \left(\int_X |f|^p d\mu\right)^{1/p}$
  $|||\cdot|||_p$ is a seminorm on $\{f \mid |||\cdot|||_p < \infty\}$
  $L^p(x) = \{f \mid |||f|||_p < \infty\}/_\sim$, $f \sim g$ if $f = g$ μ a.e.
  $\hookrightarrow$ $\|\cdot\|_p$ - the induced norm.
- $f: X \to \mathbb{R}$. $y \in \mathbb{R}$ is in the essential range of f if for any open nbhood U of y, $\mu(f^{-1}(U)) > 0$
  If essential range is bounded $\Rightarrow$ f is essentially bdd
  $\{f$ -measurable & ess. bdd $\}/_\sim = L^\infty(X)$
  $\|f\|_\infty = \sup\{|z| \mid z \in$ ess range of $f\}$
- L^p can be defined for $0<p<1$ but $\|\cdot\|_p$ does not satisfy $\Delta \le$
- Riesz: $L^p(X)$, $1 \le p \le \infty$ are all Banach
- $C^\infty([0,1])$ with the sup norm is not Banach
- C^r spaces $D^\alpha f = \dfrac{\partial^\alpha f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \sum \alpha_i$
  $\| \{f: \Omega \to \mathbb{R} \mid D^\alpha f$ exists & cts for $|\alpha| \le r\}$
  $C^\infty(\Omega) = \bigcap_{r \ge 1} C^r(\Omega)$, $C^0(\Omega) = C(\Omega)$
  $\|f\|_r = \max_{|\alpha| \le r} \sup\{|D^\alpha f(x)| \mid x \in \Omega\}$ or any other norm on $\mathbb{R}^n$
- Ω-open $\Rightarrow$ $\|f\|_{r,K} = \sup_{|\alpha| \le r}\{|D^\alpha f(x)| \mid x \in K\}$ $\|$ ~~Top~~ uniform convergence on cpct sets
  K-compact set

When size of derivatives is measured by L^p-norms

Sobolev spaces.

$|\alpha| \le k$ define $\|f\|_{p,\alpha} = \|D^\alpha f\|_p$

$\{f \in C^\infty(\Omega) \mid \|f\|_{p,\alpha} < \infty$ for all $|\alpha| \le k\}$

with $\{\|\cdot\|_{p,\alpha}, |\alpha| \le k\}$ is a sufficient family

of seminorms in C^∞ for $k \ge 0$. ~~Usually we take~~

$$\|f\|_{p,k} = \sum_{|\alpha| \le k} \|f\|_{p,k} = \sum_{|\alpha| \le k} \|D^\alpha f\|_p$$

Sobolev norm.

$W^{p,k}(\Omega)$ – completion of this space.

|| can also defined it as

$\{f \mid D_w^\alpha f$ exists for all $|\alpha| \le k$ and $D_w^\alpha f \in L^p(\Omega)\}$

weak derivatives: Def. If f & h are locally integrable functions on Ω, we say that h is the weak α-th partial derivative of f on Ω: $h = D_w^\alpha f$ if for all $\varphi \in C_c^\infty(\Omega)$ we have

$$(f, D^\alpha \varphi) = (-1)^{|\alpha|}(h, \varphi)$$

(where $(f,\varphi) = \int f \bar{\varphi}$)

<u>...t Spaces</u>: E - vector space over a field say $\mathbb{R}$

...r product $\langle \cdot , \cdot \rangle : E \times E \to \mathbb{R}$ s.t.

$\langle \cdot , \cdot \rangle$ is bilinear (for $k = \mathbb{R}$). ~~or for $k = \mathbb{C}$~~

$\langle \cdot , \cdot \rangle \geq 0$ and $(\langle x, x \rangle = 0 \Rightarrow x = 0)$

...y $\langle \cdot , \cdot \rangle$ defines a norm $\|x\| = \langle x, x \rangle^{1/2}$

A <u>complete</u> inner product (therefore <u>normed</u>) space is a <u>Hilbert</u> space.

...: $\mathbb{R}^n$, $L^2(x)$.

...rthogonality : $\langle x, y \rangle = 0$

...orthogonal projections.

...rthonormal basis

Fourier analysis.

E - Hilbert, $A \subset E$ orthonormal set. TFAE

A is maximal (it is not a proper subset of an orthonormal set)

$y \in E$, $x \perp y$ for all $x \in A$ implies $y = 0$

For any $y \in E$, y can be uniquely expressed as

$\sum_{x \in A} c_x \cdot x$ where $c_x \in k$ and $\sum |c_x|^2 < \infty$.

... such sum converges and $c_x = \langle y, x \rangle$

Maps between spaces, functionals, dual spaces . .

- $T: E \to F$ linear is $\{ \text{CTS} \iff \text{bounded} \}$, with E, F NVS
- T is <u>isomorphism</u> of E & F if T, T^{-1} are continuous, T - linear bijection.
- T is <u>isometry</u> if $\|Tx\| = \|x\| \quad \forall x \in E$.

$\boxed{\text{Thm}}$ (Open mapping) $T: E \to F$ cts linear bijection of Banach Spaces $\implies$ T is an <u>isomorphism</u>

- $\|T\| = \sup\{ \|Tx\| \mid \|x\| \le 1 \}$, $B(E,F) = \{ T: E \to F \text{ linear} \mid \|T\| < \infty \}$

$\boxed{\text{Thm}}$ F - Banach $\implies$ $B(E,F)$ is Banach

- $E^* = B(E, \mathbb{R})$ (or $\mathbb{K}$-field) is always Banach, $B(E,E) = B(E)$ is Banach if E is.

$\boxed{\text{Thm}}$ $\boxed{\text{Riesz representation}}$ X - <u>Compact</u> NVS, $\mathcal{M}(X)$ - probab. measures

$$\mathcal{M}(X) \longrightarrow C_{\mathbb{R}}(X)^* \text{ defined by } \mu \to \lambda_\mu$$
$$\lambda_\mu(f) = \int f \, d\mu$$

is a linear <u>bijection</u> of $\mathcal{M}(X)$ with the set $\left\{ \lambda \in C_{\mathbb{R}}(X)^* \;\middle|\; \begin{array}{l} \lambda(f) \ge 0 \text{ for all } f \ge 0 \\ \cancel{\lambda(1) = 1} \end{array} \right\}$ (λ-positive, $\|\lambda\| = 1$)

$\boxed{\text{Thm}}$ (Riesz) $\frac{1}{p} + \frac{1}{q} = 1$, $L^q(X) \to L^p(X)^*$ is isometric isomorphism for $1 \le p < \infty$ (for $p = \infty$: injective isometry)

$$h \mapsto \lambda_h$$
$$\lambda_h : f \mapsto \int fh$$

$\boxed{\text{Thm}}$ (Hahn-Banach)

E - normed linear space, $F \subset E$ linear subspace
$\forall \lambda \in F^*$, $\exists \tilde{\lambda} \in E^*$ s.t. $\tilde{\lambda}|_F = \lambda$ and $\|\tilde{\lambda}\| = \|\lambda\|$

Corollary $x \in E \Rightarrow \exists \lambda \in E^*$ with $\|\lambda\| = 1$ and $|\lambda(x)| = \|x\|$

The proof of Hahn-Banach works for spaces with semi-norm:

$\boxed{\cancel{\text{Thm}}}$ E - ~~linear sp~~ vector space with a semi-norm $\|\cdot\|$
$F \subset E$ linear subspace.
$\lambda \in F^*$, $B > 0$ s.t. $|\lambda(x)| \le B\|x\|$ ~~$\forall x \in$~~ $\forall x \in F$.
Then there is a linear map $\tilde{\lambda} \in E^*$: $\tilde{\lambda}|_F = \lambda$, $|\tilde{\lambda}(x)| \le B\|x\|$ $\forall x \in E$.

Thm (Hahn-Banach II) If topology on E is defined with suff. family of seminorms $\Rightarrow \forall x \in E$, $x \neq 0$, $\exists \lambda \in E^*$ s.t. $\lambda(x) \neq 0$.

Pf $\|\cdot\|$ - seminorm in the sufficient family s.t. $\|x\| \neq 0$
$\Rightarrow \exists \lambda : E \to \mathbb{R}$ s.t. $\lambda(x) \neq 0$ and $|\lambda(y)| \le \|y\|$.
If $y_\alpha \to 0$ in E then $\|y_\alpha\| \to 0$ hence $|\lambda(y_\alpha)| \to 0$
$\Rightarrow \lambda$ is cts.

C^∞ is Fréchet
$(C^\infty)'$ - cont linear functionals = Distributions
$\longrightarrow$ one possible direction of study.

APPLICATIONS of Distr.
(1) - distributions, tempered d.
~~Four~~ Fourier transform, Space of mod...
weak derivatives etc.

Distributions $\supset$ measures. $\supset$ functions
$\langle R_\mu, \varphi \rangle = \int \varphi \, d\mu$ $\quad$ $\langle T_f, \varphi \rangle = \int_{\mathbb{R}} f(x) \varphi(x) \, dx$ $\quad \varphi \in \mathcal{D}$

Weak & Weak* topologies.

E - TVS defined by a sufficient family of seminorms.

(a) Weak topology on E is given by family of seminorms (which is sufficient in fact)
$$\{\|\cdot\|_\lambda \mid \lambda \in E^*\} \quad \text{where} \quad \|x\|_\lambda = |\lambda(x)|$$
($x_\alpha \xrightarrow{w} x$ on E $\iff$ $\forall \lambda \in E^*$ $\lambda(x_\alpha) \to \lambda(x)$)

(b) weak* topology on E^* is defined by the (sufficient) family of seminorms $\{\|\cdot\|_x \mid x \in E\}$, where $\|\lambda\|_x = |\lambda(x)|$

($\lambda_\alpha \xrightarrow{w^*} \lambda$ on E^* $\iff$ $\lambda_\alpha(x) \to \lambda(x)$ $\forall x \in E$)

$\boxed{\text{Example}}$ $E = C(I)$, $I = [0,1]$. $\mathcal{M}(I)$ - probab. measures. || $\mu_0 = \delta_0$ - measure supported at 0; $\mu_n \in \mathcal{M}(I)$ normalized Lebesgue on $(0, \frac{1}{n})$

$\forall n$, find $f_n \in C(I)$ s.t. $\|f_n\| = 1$, $\int f_n d\mu_n = \frac{1}{2}$, $f_n(0) = 0 = \int f_n d\mu_0$

$\|\mu_n - \mu_0\| \geq \frac{1}{2}$ in the $C(I)^*$ norm

On the other hand, μ_n should $\to \mu_0$ in some sense!
Actually $\mu_n \xrightarrow{w^*} \mu_0$: $\forall f \in C(I)$, $\int f \, d\mu_n \longrightarrow \int f d\mu_0 = f(0)$
By continuity of f at 0, $\forall \varepsilon > 0$ $\exists n$ suff. large s.t.
$|f(x) - f(0)| < \varepsilon$ on $[0, \frac{1}{n})$ so $|\int f \, d\mu_n - f(0)| < \varepsilon$.

Weak* on $\mathcal{M}(X)$ is the weakest topology which makes each evaluation map: $\mu \mapsto \int f d\mu$ cts. for every $f \in C(X)$

$\boxed{\text{Thm}}$ (One of the most useful features of weak*)
E - NVS $\Rightarrow$ E_1^* (unit ball in E^*) is $\boxed{\text{Compact}}$ in weak* top.

$\boxed{\text{Corollary}}$ X - compact ~~metric space~~ $\Rightarrow$ $\mathcal{M}(X)$ is compact with weak* topology.

$\boxed{\text{Rem}}$ E^* in general is not metrizable with weak*, but if E - separable then E_1^* is metrizable. Thus $\mathcal{M}(X)$ is a compact metrizible space.

Why: $\mathcal{M}(X)$ = intersection of a closed (in weak*) set $\{\lambda \mid \lambda(f) \geq 0 \ \forall f\}$ and unit ball in $C(X)^*$, which is compact, so $\mathcal{M}(X)$ is compact.

(8)

L^2, $W^{2,k}$ are Hilbert spaces.

lbert $\Rightarrow$ E identifies with E* naturally and

* top. on E^* defines a topology on E

ich is the same as weak topology on E.

: unit ball in a Hilbert space is COMPACT

in the weak topology.

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Operators.

Translation operators:

X, $\varphi : X \to X$ bijection

Then φ defines a "translation operator" T_φ on the space of functions on X:

$f : X \to \mathbb{R}$, $(T_\varphi f)(x) = f(\varphi^{-1}(x))$

$$T_\varphi f = f \circ \varphi^{-1}$$

T_φ will, under suitable hypothesis be continuous.

Similarly, on the space of $\mathcal{M}(X)$

Ex: ~~(X, μ) - measure space~~

$\varphi : X \to X$ measurable map

$$(\varphi_* \mu)(A) = \mu(\varphi^{-1}(A))$$

push forward of a measure.

φ - continuous $\Rightarrow \varphi_* : \mathcal{M}(X) \to \mathcal{M}(X)$ is continuous in weak* top.

Def: φ is μ-measure preserving if $\varphi_* \mu = \mu$

fixed points for φ_*. δ_x = copies of $\{x\} \in X$

$\mathcal{M}^\varphi(x) = \{\mu, \varphi_* \mu = \mu\}$ - closed convex

will show: φ cts, X cpct $\Downarrow$ Kryloff-Bogoliuboff thm. $\mathcal{M}^\varphi(x) \neq \emptyset$

Example: If $\varphi : X \to X$ is measure preserving bijection with a measurable inverse then

$T_\varphi : L^p(X) \to L^p(X)$ is an isometric isomorphism

for all p $1 \le p \le \infty$:

$$\int |f \circ \varphi|^p d\mu = \int |f|^p \circ \varphi \, d\mu = \int |f|^p d\mu$$

isometric isomorphism given that

$\Downarrow$ topic: Existence of invariant measures for groups of transformations: Kakutani-Markov thm.

Rem: for $p = \infty$ T_φ will be isometric isomorphism $\mu(\varphi^{-1}(A)) = 0$ iff $\mu(A) = 0$. (μ being invariant is not necessary!)

So for example if $\varphi : X \to X$ is a diffeomorphism $\Rightarrow$ $\boxed{T_\varphi \text{ is isometric isomorphism on } L^\infty(\Omega)}$

Extremal points in $\mathcal{M}^?$ are ERGODIC measures. → Ergodic theory. III

Example $\varphi : \Omega \to \Omega$ diffeomorphism

then $T_\varphi : C^\infty(\Omega) \to C^\infty(\Omega)$ is an isomorphism where $C^\infty(\Omega)$ has the Frechét space topology.

$\boxed{\text{Greenfield-Wallach conjecture}}$ → { $1 \in \operatorname{Spec} T_\varphi$ or not? → ... → $\boxed{\text{G-W conjecture}}$! ~~Cont. Ergodic theory~~. (IV)

- Differential operators $D : W^{p,k} \to W^{p,k-r}$
- embedding operators

$$W^{p,k}(\Omega) \hookrightarrow W^{p,\ell}(\Omega) \qquad k > \ell$$

$$\Downarrow$$

$$C^k(\Omega) \hookrightarrow C^\ell(\Omega)$$

\+ Sobolev embeddings → (V) topics in Sobolev spaces.

→ define.

- Integral operators : (X, μ) – measure space $k \in L^2(X \times X, \mu \times \mu)$. Define $T_k : L^2(X) \circlearrowleft$ by $(T_k f)(x) = \int_X k(x,y) f(y)\, d\mu(y)$

- E – [illegible] : Hilbert space with ONB $\{e_i\}$ ⇒ bdd linear operator T is determined by $\{Te_j\}$ hence by $\{\langle Te_j, e_i \rangle\}$. So to T we associate infinite matrix $\Im$ from which T can be recovered $(Tx)_i = \sum T_{ij} a_j$

Study of Schrödinger operators.

- adjoint operators.

Def E, F TVS, $T: E \to F$ cts linear map.

Define adjoint T^* of T by $T^*: F^* \to E^*$

$$T^*(\lambda) = \lambda \circ T$$

* T^* is linear and cts wrt weak* topologies on F^* and E^*

$\boxed{\text{Lemma}}$ E, F normed $\Rightarrow \|T^*\| = \|T\|$

example: If E is Hilbert and $T \in B(E)$

then $T^* \in B(E^*)$ so T^* defines an operator in $B(E)$ via identification

$$i: E \to E^* \quad i(x)(y) = \langle y, x \rangle.$$

i.e. we obtain operator $T' = i^{-1} \circ T^* \circ i$

$\langle Tx, y \rangle = \langle x, T'y \rangle$.

Then one can show that T^* is conjugate transpose, as in finite dimensions:

$$T_{ij}^* = \langle T^* e_j, e_i \rangle = \langle e_j, Te_i \rangle = \overline{\langle Te_i, e_j \rangle} = \overline{T_{ji}}$$

Def
- If $T = T^*$ then T is self adjoint.
- $U \in B(E)$ is unitary if it is isometric isomorphism,

$\Longleftrightarrow$ U is unitary iff $U^{-1} = U^*$

Operator topologies and groups of operators.

E, F - NVS $\qquad B(E,F)$ has:

- Norm topology: $T_\alpha \to T$ iff $\|T_\alpha - T\| \to 0$

- Strong operator topology:

define family of seminorms $\|\cdot\|_x$ on $B(E,F)$ by

$$\|T\|_x = \|Tx\|$$

Then $T_\alpha \to T$ iff $T_\alpha x \to Tx$ in F for all $x \in E$.

- When $F = \mathbb{R}$ (or k) $\Rightarrow$ strong operator topology on E^* is the same as weak* -topology.

- Weak operator topology:

For $x \in E$, $\lambda \in F^*$ define the seminorm: $\|\cdot\|_{x,\lambda}$ on $B(E,F)$ by $\|T\|_{x,\lambda} = |\lambda(Tx)|$. The topology defined by this family of seminorms $\{\|\cdot\|_{x,\lambda} \mid x \in E, \lambda \in F^*\}$ is Weak operator topology.

$T_\alpha \to T$ iff $\forall x \in E$ $T_\alpha x \to Tx$ in F with the weak topology i.e. $\lambda(T_\alpha x) \to \lambda(Tx)$ for all $\lambda \in F^*$

When $B(E,F) = E^*$, the weak operator top. = weak* top.

- Group representation

13

~~G~~ V - vector space, G - group.

- $G \xrightarrow{\text{homom.}} \{\text{invertible linear maps of } V\}$

- If E is TVS, representation of G on E is $\pi : G \xrightarrow{\text{homom.}} Aut(E)$

- If G is a topological group representation of G on E is continuous if $Aut(E) \subset B(E)$ has a given topology (strong or weak operator top.)

- π is isometric representation if
$$\pi(G) \subset Iso(E) \subset Aut(E)$$

- E - Hilbert, then an isometric representation of G on E is UNITARY REPRESENTATION.

- Any group G action on a space X induces translation operators. $T_g : f \longmapsto f \circ g^{-1}$
$f \in$ some function space on X

T_g satisfy group relations so they form a group G representation into various function spaces.

- Representation $G \to Aut(E)$ is continuous for the strong operator topology iff. it is continuous at $e \in G$.

- If G acts continuously on compact ~~metrizable~~ metric space. If $\exists \mu$ invariant under all G and μ-finite, then for $1 \leq p < \infty$ $\pi : G \to Iso(L^p)$ $(\pi(g)f)(x) = f(g^{-1}x)$ is a cts representation.

$\boxed{\text{KAKUTANI - MARKOV}}$ fixed point theorem.

E - TVS whose topology is defined by a suff. family of seminorms

G - abelian group

$\pi : G \xrightarrow{\text{HOMOM}} Aut(E)$ a representation.

$\quad$ Continuous linear map $E \to E$, bijective with continuous inverse.

$A \subset E$ compact, convex set which is G - invariant;

$$\pi(g) A \subset A \quad \text{for all } g.$$

Then there is a G-fixed point in A.

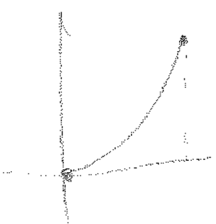
$\boxed{\text{Corollary}}$ G-abelian group acting continuously on a compact metric space X $\Rightarrow$ there is a

G invariant probab. measure.

Example: Not every group of transformations has an invariant measure:

$\varphi_1 : [0,1] \to [0,1]$, $\varphi_1(x) = x^2$. The only inv. probab. measures are supported on $\{0,1\}$

$\downarrow$ project to S^1, to ~~get~~ glue 0 and 1. Then the only inv. probab. measure is at one pt. x_0.



Take φ_2 to be translation on the circle which moves x_0. $\Rightarrow$ No jointly invariant measures for $\langle \varphi_1, \varphi_2 \rangle$.

Suggested topics

- Kakutani - Markov fixed point theorem, Haar measure for compact groups. Krein - Milman theorem. | about spaces of measures, invariant measures etc.
- Compact operators and Hilbert - Schmidt operators, Peter - Weyl theorem.
- General spectral theory, Spectral theorem for self-adjoint operators, Mean ergodic theorem as application
- Distributions, tempered distributions, Fourier transform,
- Regularity for elliptic operators, Spectral theorem for Elliptic operators.
- Greenfield - Wallach papers & conjecture.
- Mixing, weak mixing, ergodicity: dynamical meaning vs. spectral meaning. Multiple mixing conjecture.
- ~~Applications of compact operators~~ (Simon-Reed)
- Wavelets & applications.
- Quantum theory, quantization procedure
- Inverse and implicit function theorems on Frechet spaces (Nash - Moser)