Growth estimates through scaling for quasilinear partial differential equations *

Tero Kilpeläinen, Henrik Shahgholian, and Xiao Zhong †

March 6, 2007

Abstract

In this note we use a scaling or blow up argument to obtain estimates to solutions of equations of p-Laplacian type.

1. Introduction

Weak solutions of equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0\,, \quad 1 < p < \infty\,,$$

are called p-harmonic. It is known that p-harmonic functions are in $C^{1,\alpha}$ for some $\alpha > 0$, where for $p \neq 2$ one cannot have $\alpha \geq 1$ in general; see [3] for sharp regularity in the planar case. In this note we present a blow up argument and show that if $0 < \alpha \leq 1$ is such that the class of p-harmonic functions are continuously embedded into $C^{1,\alpha}$, then the only entire p-harmonic functions that grow at infinity slower than $|x|^{1+\alpha}$ are the linear ones.

We formulate the proof and the growth rate result only in the p-Laplacian setting, but the argument is more general. The only ingredients required are the following: there is a class $\mathcal{F}$ of functions so that $\mathcal{F}$ contains certainly rescaled versions of

* *2000 Mathematics Subject Classification.* Primary 35J60; Secondary 35J15
Keywords. p-Laplace, scaling.
† T.K. (partially) and X.Z. supported by the Academy of Finland, H.S. partially supported by Swedish Research Council. The present work is part of the ESF programme *GLOBAL*.

functions and $\mathcal{F}$ can be embedded into $C^{1,\alpha}$. Then nonlinear functions in $\mathcal{F}$ grow at least as fast as $|x|^{1+\alpha}$.

As an application of the growth rate result we show that a nonnegative p-harmonic function in a half space is actually linear if it vanishes on the boundary of the half space. This gives an affirmative answer to a query of Mario Bonk, who also found independently a different proof for this fact.

2. Growth

We prove the following two theorems:

2.1. Theorem. *Let u be p-harmonic in $\mathbf{R}^n$. There is a number $\beta > 0$ depending only on p and n so that if*

$$|u(x)| = o(|x|^{1+\beta}) \quad \textit{as} \quad |x| \to \infty,$$

then u is (affine) linear.

The second is an immediate consequence of the first one.

2.2. Theorem. *Let u be p-harmonic in $\mathbf{R}^n$. If*

$$|u(x)| = o(|x|) \quad \textit{as} \quad |x| \to \infty,$$

then u is constant.

2.3. Remark. It is known that there are no entire harmonic functions (i.e. $p = 2$) with noninteger growth rate. That is, if u is harmonic (i.e. 2-harmonic) in $\mathbf{R}^n$ with

$$\limsup_{|x|\to\infty} \frac{\log|u(x)|}{\log|x|} = \gamma \in]0, \infty[\,,$$

then γ is an integer.

If $p \neq 2$, the situation is different. Then there are entire p-harmonic functions whose growth rate $\gamma \in]1, 2[$. See Krol' [4], Tolksdorff [9] or Aronsson's quasiradial solutions [2].

Hence theorem 2.1 is optimal.

Proof of Theorem 2.1. Choose a sequence $R_j \to \infty$ and write

$$S_j = \sup_{B(0,R_j)} |u| \, .$$

Then the scaled functions

$$u_j(x) = \frac{u(R_j x)}{S_j}$$

are p-harmonic and $|u_j| \le 1$ in $B(0,1)$.

By a well known regularity estimate (see e.g. Lewis [5]), there is a constant $\beta = \beta(n,p) > 0$ so that the $C^{1,\beta}(B(0,1))$ norms of u_j are bounded, independently of j. Hence the quantities

$$C_j(x) = \frac{|Du_j(x) - Du_j(0)|}{|x|^\beta} = \frac{R_j^{1+\beta}}{S_j} \frac{|Du(R_j x) - Du(0)|}{|R_j x|^\beta}$$

are uniformly bounded in $B(0,\frac12)$. Since the growth condition $|u(x)| = o(|x|^{1+\beta})$ implies

$$\lim_{j\to\infty} \frac{R_j^{1+\beta}}{S_j} = \infty \, ,$$

we conclude that

$$\sup_{y\in B(0,\frac{R_j}{2})} \frac{|Du(y) - Du(0)|}{|y|^\beta} = \sup_{x\in B(0,\frac12)} \frac{|Du(R_j x) - Du(0)|}{|R_j x|^\beta} \to 0 \qquad \text{as } j \to \infty \, .$$

But this implies that

$$Du(y) = Du(0) \qquad \text{for all } y \in \mathbf{R}^n \, ,$$

and Theorem 2.1 follows. □

2.4. Remark. Another way to prove Theorem 2.1 for the p-Laplacian goes via the estimate

$$\operatorname*{osc}_{B(x_0,r)} |\nabla u| \le C \sup_{B(x_0,R)} |\nabla u| \left(\frac{r}{R}\right)^\alpha$$

that can be found e.g. in [7, Theorem 3.44]. For more general operators the oscillation estimate might not be available but one can prove the embedding into $C^{1,\alpha}$ by other means. We would like to emphasize here that our method works also in those cases where one can establish bounded embedding to $C^{1,\alpha}$ even though there is no oscillation estimate for the gradient.

As an application of Theorem 2.1 we prove the following result.

2.5. Theorem. *If u is a nonnegative p-harmonic function on a half space H and $u = 0$ on ∂H, then u is (affine) linear.*

Theorem 2.5 follows by combining the following lemma with Theorem 2.1, when we observe that u in Theorem 2.5 can be reflected through the hyperplane ∂H and the resulting function is p-harmonic in tho whole space $\mathbf{R}^n$ (this can be easily verified by a direct computation, see [8]).

2.6. Lemma. *Let u be a nonnegative p-harmonic function in the upper half sapce*

$$\mathbf{R}^n_+ = \{(x_1, x_2, \dots, x_n) : x_n > 0\}.$$

If $u = 0$ on $\partial \mathbf{R}^n_+$, then

$$|u(x)| = O(|x|) \quad \text{as} \quad |x| \to \infty.$$

Proof: We first show that there is a constant $c = c(n,p) > 0$ so that

$$u(Re_n) \le c\, R\, u(e_n) \quad \text{for all } R > 2\,; \tag{2.7}$$

here $e_n = (0, 0, \dots, 0, 1)$ is the nth unit vector in $\mathbf{R}^n$. For this, we write $x_0 = Re_n = 2re_n$ and observe that by Harnack's inequality

$$u(x) \approx c\, u(x_0) \quad \text{for all } x \in \bar{B}(x_0, r)\,,$$

where $c = c(n,p) > 0$. Now, let v be the p-capacitary potential in $B(x_0, 2r) \setminus \bar{B}(x_0, r)$, i.e.

$$v(x) = \frac{\int\limits_{|x-x_0|}^{2r} t^{(1-n)/(p-1)}\,dt}{\int\limits_{r}^{2r} t^{(1-n)/(p-1)}\,dt}\,.$$

Then since v is p-harmonic in $B(x_0, 2r) \setminus \bar{B}(x_0, r)$, we have by comparison principle that

$$u(x) \ge c u(x_0) v(x) \quad \text{for all } x \in B(x_0, 2r) \setminus \bar{B}(x_0, r)\,,$$

where $c = c(n,p) > 0$. The claim (2.7) follows from this estimate evaluated at $x = e_n$, for

$$\begin{aligned}
\frac{1}{v(e_n)} &= \frac{\int_r^{2r} t^{(1-n)/(p-1)}dt}{\int_{2r-1}^{2r} t^{(1-n)/(p-1)}dt} \\
&= 1 + \frac{\int_r^{2r-1} t^{(1-n)/(p-1)}dt}{\int_{2r-1}^{2r} t^{(1-n)/(p-1)}dt} \\
&\le 1 + \frac{(r-1)r^{(1-n)/(p-1)}}{(2r)^{(1-n)/(p-1)}} \\
&\le 1 + \frac{r-1}{2^{(1-n)/(p-1)}} \\
&\le c2r = cR\,,
\end{aligned}$$

where $c = c(n,p)$. The estimate (2.7) is proved.

To complete the proof the lemma, we employ the boundary Harnack principle (see [1] or [6]) which states that there is a constant c depending on n and p only so that

$$\frac{u(x)}{x_n} \le c\frac{u(Re_n)}{R} \quad \text{for all } x \in B(0,2R)\cap \mathbf{R}^n_+ \quad \text{and } R > 0\,;$$

here x_n is the nth coordinate of x. Next we combine this with (2.7) and have

$$u(x) \le c\frac{u(Re_n)}{R}x_n \le c\,u(e_n)x_n \le c\,|x|u(e_n)$$

for $x \in B(0,2R)$ and $R > 2$. The lemma follows. □

References

[1] Aikawa, H., Kilpeläinen, T., Shanmugalingam, N., and Zhong, X. Boundary Harnack principle for p-harmonic functions in smooth euclidean domains. to appear in *Potential Analysis*.

[2] Aronsson, G. On certain p-harmonic functions in the plane. *Manuscripta Math. 61*, 1 (1988), 79–101.

[3] Iwaniec, T., and Manfredi, J. J. Regularity of p-harmonic functions on the plane. *Rev. Mat. Iberoamericana 5*, 1-2 (1989), 1–19.

[4] Krol', I. N. The behavior of the solutions of a certain quasilinear equation near zero cusps of the boundary. *Trudy Mat. Inst. Steklov. 125* (1973), 140–146, 233. Boundary value problems of mathematical physics, 8.

[5] Lewis, J. L. Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J. 32*, 6 (1983), 849–858.

[6] Lewis, J. L. and Nyström, K. On a boundary Harnack inequality for p-harmonic functions. *Preprint* (2006), 177–196.

[7] Malý, J. and Ziemer, W. P. *Fine regularity of solutions of elliptic partial differential equations*, vol. 51 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997.

[8] Martio, O. Reflection principle for solutions of elliptic partial differential equations and quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. A I Math. 6*, 1 (1981), 179–187.

[9] Tolksdorf, P. Regularity for a more general class of quasilinear elliptic equations. *J. Differential Equations 51*, 1 (1984), 126–150.

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland.
E-mail: terok@maths.jyu.fi

Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden
E-mail: henriks@kth.se

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland.
E-mail: zhong@maths.jyu.fi