Convex configurations for solutions to semilinear elliptic problems in convex rings

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Abstract

For a given convex ring $\Omega = \Omega_2 \setminus \overline{\Omega}_1$ and an L^1 function $f : \Omega \times \mathbb{R} \to \mathbb{R}_+$ we show, under mild assumptions on f, that there exists a solution (in the weak sense) to

$$\begin{cases} \Delta_p u = f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega_2 \\ u = M & \text{on} \quad \partial\Omega_1, \end{cases}$$

with $\{x \in \Omega : \ u(x) > s\} \cup \Omega_1$ convex, for all $s \in (0, M)$.

1 Introduction and statement of the problem

1.1 The mathematical setting

We start with the mathematical setting of the problem. Let us be given two convex domains $\Omega_1 \subset\subset \Omega_2 \subset \mathbb{R}^N$ and the function $f(x, y)$. We study the following boundary value problem:

$$\begin{cases} \Delta_p u = f(x,u) & \text{in} \quad \Omega := \Omega_2 \setminus \overline{\Omega}_1 \\ u = 0 & \text{on} \quad \partial\Omega_2 \\ u = M & \text{on} \quad \partial\Omega_1, \end{cases} \tag{1}$$

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where M is a given constant and $\Delta_p,\ 1 < p < \infty$ is the p-Laplace operator defined by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

The differential equation in (1) will be understood in the weak sense, i.e. for every $\eta \in C_0^\infty(\Omega)$

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\eta(x)dx = -\int_\Omega f(x,u(x))\cdot\eta(x)dx. \tag{2}$$

The differential equation in problem (1) is the Euler equation for the following minimization problem:

$$\begin{cases} \int_\Omega \Big(|\nabla u(x)|^p + F(x,u(x))\Big)dx \to inf \\ u \in K := \{v \in W^{1,p}(\Omega) : u = 0 \quad \text{on} \quad \partial\Omega_2, \quad u = M \quad \text{on} \quad \partial\Omega_1\}, \end{cases} \tag{3}$$

where

$$F(x,t) := p\cdot\int_0^t f(x,z)dz.$$

Our objective is to prove the existence of a solution, with convex level sets, to problem (1) (with some restrictions on the right hand side of the equation, of course). For our proof, we require convex solutions to the multi-layer free-boundary problem, which occurs in fluid dynamics (see [AHPS]). Our solutions are obtained by passing to the limit as the number oflayers (and free boundaries) becomes infinite.

This approach was realized in [LS], where those authors have proved the existence of the weak solution with convex level lines (in $\mathbb{R}^2$) of the following problem:

$$\begin{cases} \Delta u = f(u) & \text{in} \quad \Omega := \Omega_2 \setminus \overline{\Omega}_1 \\ u = 0 & \text{on} \quad \partial\Omega_2 \\ u = M & \text{on} \quad \partial\Omega_1, \end{cases} \tag{4}$$

where Ω_1 and Ω_2 are as above and the function f satisfies

$$f \in L^1(-\infty,\infty), \quad f(x) \geq 0, \qquad \text{and} \quad f(x) = 0 \quad \text{on} \quad (-\infty, 0).$$

It should be added that recently, the third author and R. Monneau [MS] have constructed a solution u, with non-convex level sets, to the above problem with $f \leq 0$ and smooth.

Definition 1.1 (The class of functions $\mathcal{F}$) *We will always assume, unless otherwise stated, that the function $f(x,y)$ on the right hand side of (1) belongs to the class $\mathcal{F}$ of functions having the following four properties:*

($\mathcal{F}$1): *$f(x,y) : \Omega\times\mathbb{R}\to\mathbb{R}$ is continuous and $f(x,y)\geq 0$ for every $(x,y)\in\Omega\times\mathbb{R}$.*

($\mathcal{F}$2): *For every $\alpha,\beta\in[0,M]$ with $\alpha<\beta$, the function:*

$$g_{\alpha,\beta}(x) := \left(\int_\alpha^\beta f(x,y)dy\right)^{-\frac{1}{p}}$$

is concave in Ω.

($\mathcal{F}$3): *There exists a number C, s.t.*

$$\int_0^M \left(\sup_{x\in\Omega} f(x,y)\right)dy \leq C.$$

($\mathcal{F}$4): *For $y\to 0+$, we have*

$$y^{1-p}\left(\sup_{x\in\Omega} f(x,y)\right) \to 0.$$

1.2 Main result

The body of this paper is devoted to the proof of the following result:

Theorem 1.2 *Let $M > 0$ be a given constant, and let $f(x, y) \in \mathcal{F}$. Then there exists a weak solution $u(x)$ of Problem (1) with convex level sets, for which*

$$0 \leq u(x) \leq M, \quad x \in \Omega.$$

In addition, if $f(x, y)$ is non-decreasing function with respect to its second argument, then the solution is unique.

We remark that an alternate uniqueness result not requiring the monotonicity of f relative to y appears in Section 5.

The continuity of $f(x, y)$ and the last two assumptions in Definition 1.1 of the class $\mathcal{F}$ are actually not critical to the validity of our convex existence results, although they facilitate the proof. In fact Theorem 1.2 directly generalizes by an approximation argument (see [DPS], Section 3) to the following result:

Theorem 1.3 *The assertion of Theorem 1.2 continues to hold when $f(x, y) \in \mathcal{F}'$, where $\mathcal{F}'$ denotes the closure of $\mathcal{F}$ in L^1.*

We observe that $\mathcal{F}'$ consists of the L^1 functions $f(x, y) : \Omega \times \mathbb{R} \to \mathbb{R}$ with properties ($\mathcal{F}1$) and ($\mathcal{F}2$).

Example. The above existence theory applies to any function of the form

$$f(x, y) = \sum_{i=1}^{n} f_i(x)\phi_i(y) \in L^1(\Omega),$$

where the functions $f_i(x) : \Omega \to \mathbb{R}$, $\phi_i(y) : \mathbb{R} \to \mathbb{R}$ are all non-negative, and $(f_i(x))^{-\frac{1}{p}}$ are concave in Ω.

To show this, it suffices to show that if g, h are non-negative $\mathcal{L}^1$ functions and satisfy the concavity condition $\mathcal{F}_2$, then so does $f = (g + h)$. Observe that a sufficiently regular function g satisfies the concavity condition $\mathcal{F}_2$ if and only if $gg'' \geq C(g')^2$, where $C = (1 + (1/p))$ and g' and g'' refer to 1st and 2nd order directional derivatives at any point and in any direction. Thus, if f, g are sufficiently regular, then

$$ff'' = (g + h)(g'' + h'') = gg'' + gh'' + hg'' + hh'' = (1 + (h/g))gg'' + (1 + (g/h))hh''$$

$$\geq C((1 + (h/g))(g')^2 + (1 + (g/h))(h')^2 \geq C((g')^2 + 2g'h' + (h')^2) = C(f')^2,$$

where we have used the fact that $2f'g' \leq (h/g)(g')^2 + (g/h)(h')^2$. Now an approximation argument gives the result for non-negative L^1-functions.

2 The multilayer free boundary problem

We start with the following multilayer free boundary problem.

Let $T = \{t_0, t_1, ..., t_n\}$ be a partition of $[0, M]$, i.e. $0 = t_0 < t_1 < ... < t_n = M$, and let $\tau_i = M - t_i$, so that $M = \tau_0 > \tau_1 > \cdots > \tau_n = 0$. Also let

$$F(x, t) := p \cdot \int_0^t f(x, z)dz.$$

We consider the following $(n - 1)$-layer problem:

Problem 2.1 *Find convex domains* $K_1, K_2, ..., K_{n-1}$ *such that*

$$K_0 \subset\subset K_1 \subset\subset K_2 \subset\subset .... \subset\subset K_{n-1} \subset\subset K_n,$$

where $K_0 := \Omega_1, K_n := \Omega_2$*, with the property that the p-capacitary potentials* u_i *for each annular convex region* $K_i \setminus \overline{K}_{i-1}$ *satisfies a nonlinear joining Bernoulli condition*

$$|\nabla u_i(x)|^p - |\nabla u_{i+1}(x)|^p = \frac{1}{p-1}[F(x,\tau_{i-1}) - F(x,\tau_i)] \quad \text{on } \partial K_i, \quad i = 1, \ldots, n-1. \quad (5)$$

By p-capacitary potential of the annular region $K_i \setminus \overline{K}_{i-1}$ we mean the solution of the following Dirichlet problem

$$\begin{cases} \Delta_p u_i = 0 & \text{in} \quad K_i \setminus \overline{K}_{i-1} \\ u_i = \tau_{i-1} & \text{on} \quad \partial K_{i-1} \\ u_i = \tau_i & \text{on} \quad \partial K_i. \end{cases} \quad (6)$$

Theorem 2.2 *For every partition* T *of* $[0, M]$ *and every function* $f(x,y) \in \mathcal{F}$ *Problem 2.1 has a (Lipschitz) solution, where the joining condition (5) is satisfied strongly.*

Remark We will define the function $u^T(x) : \overline{\Omega} \to \mathbb{R}$ by

$$u^T(x) := u_i(x), \quad x \in \overline{K}_i \setminus K_{i-1}.$$

Proof of Theorem 2.2

We only need to verify that if $f(x,y) \in \mathcal{F}$ and $\alpha, \beta \in [0, M]$ with $\alpha < \beta$, then the function $q(x,y)$, defined by

$$q(x,y) := \left(\frac{1}{p-1} \int_\alpha^\beta f(x,z)dz + y^p \right)^{\frac{1}{p}} \quad (7)$$

satisfies conditions (A1)-(A4) of Definition 2.3 in the paper [AHPS], which are the following:

(A1): q is continuous and $\exists\ c_0 > 0$ such that $q(x,0) \geq c_0$ for all $x \in \Omega$,

(A2): q is non-decreasing with respect to second argument,

(A3): q satisfies the following concavity property: $x \mapsto \dfrac{1}{q(x,h(x))}$ is concave whenever h is a given function such that $1/h$ is concave, and

(A4): for any given value $y_0 > 0$, there exist constants $0 < C_1 < C_2$ such that $C_1 \leq (q(x,y)/y) \leq C_2$, uniformly for all $x \in \Omega$ and all $y \geq y_0$.

The conditions (A1) and (A2) are obvious, the condition (A3) can be easily verified if we use Lemma 2.1 of the above mentioned paper and the concavity property ($\mathcal{F}2$) of $f \in \mathcal{F}$. The last condition (A4) follows from the fact that

$$\frac{q(x,y)}{y} \to 1, \quad y \to +\infty \quad \text{uniformly in} \quad \Omega.$$

3 Passage To Limit

For a partition $T = \{t_0, t_1, ..., t_n\}$ we denote $|T| := \max\{t_{i+1} - t_i : i = 0, 1, ..., n-1\}$.

Theorem 3.1 *For a given convex annular domain $\Omega := \Omega_2 \backslash \overline{\Omega}_1$, there exist a continuous, (strictly) increasing function $\varphi(s) : [0, \infty) \to \mathbb{R}$ with $\varphi(0) = 0$, and a strictly-positive function $P(x) : \Omega \to \mathbb{R}$, such that for any partition T of $[0, M]$ with $|T|$ sufficiently small, and any solution $u^T(x)$ of the multilayer free boundary problem (2.1), corresponding to T, we have*

$$|\nabla u^T(x)| \leq \frac{\varphi(\mathrm{dist}(x, \partial\Omega_1)) + |T|}{\mathrm{dist}(x, \partial\Omega_1)} \tag{8}$$

and

$$|\nabla u^T(x)| \geq (P(x) - |T|)\mathrm{dist}(x, \partial\Omega_2), \tag{9}$$

both wherever $\nabla u^T(x)$ exists.

Proof. We break the proof into the following 5 steps.
Step 1. (Estimates for a family of multilayer subsolutions) Let the function $V(x)$ solve the Dirichlet problem

$$\begin{cases} \Delta_p V(x) = 0 & \text{in} \quad \Omega \\ V(x) = 0 & \text{on} \quad \partial\Omega_1 \\ V(x) = 1 & \text{on} \quad \partial\Omega_2. \end{cases} \tag{10}$$

For any $\rho \in (0, 1)$, we define

$$V_\rho(x) := \frac{V(x)}{\rho} \quad \text{and} \quad \Omega_\rho := \{x \in \Omega : V(x) < \rho\}, \tag{11}$$

observing that $V_\rho(x)$ is the p-capacitary potential in the domain Ω_ρ. Let $T = \{t_0, t_1, \cdots, t_n\}$ be a given partition of $[0, M]$, and let $A = \{\alpha_0, \alpha_1, \ldots \alpha_n\}$, $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n = 1$, denote a partition of $[0, 1]$ which is to be determined. In terms of $\rho \in (0, 1)$ and A, we define the convex domains $\omega_i := \Omega_1 \cup \{x \in \overline{\Omega}_\rho : V_\rho(x) < \alpha_i\}, \ i = 0, \cdots, n$. Let the functions $U_i(x), \ i = 1, \cdots, n$, solve the Dirichlet problems:

$$\begin{cases} \Delta_p U_i(x) = 0 & \text{in} \quad \omega_i \setminus \overline{\omega}_{i-1} \\ U_i(x) = \tau_{i-1} & \text{on} \quad \partial\omega_{i-1} \\ U_i(x) = \tau_i & \text{on} \quad \partial\omega_i. \end{cases} \tag{12}$$

Then it is clear that

$$U_i(x) = \tau_{i-1} + \frac{\tau_i - \tau_{i-1}}{\alpha_i - \alpha_{i-1}} \cdot (V_\rho(x) - \alpha_{i-1}), \quad i = 1, \cdots, n. \tag{13}$$

Now $(\omega_1, \cdots, \omega_{n-1})$ will be a *subsolution* of the multilayer problem relative to the annular domain Ω_ρ and corresponding to the given partition T (see [AHPS], section 4.2.2), if the partition A is chosen such that

$$|\nabla U_i(x)|^p \geq |\nabla U_{i+1}(x)|^p + \frac{F(x, \tau_{i-1}) - F(x, \tau_i)}{p - 1}, \quad x \in \partial\omega_i \tag{14}$$

for $i = 1, \cdots, n - 1$. Set

$$\hat{\delta}_i := t_i - t_{i-1} = \tau_{i-1} - \tau_i > 0 \quad \text{and} \quad \delta_i := \alpha_i - \alpha_{i-1} > 0$$

for $i = 1, \cdots, n$. Then we must have

$$\hat{\delta}_1 + \cdots + \hat{\delta}_n = M, \tag{15}$$

$$\delta_1 + \cdots + \delta_n = 1. \tag{16}$$

By (13) and (14), $(\omega_1, \cdots, \omega_{n-1})$ is a *subsolution* if

$$\left(\frac{\hat{\delta}_i}{\delta_i}\right)^p \geq \left(\frac{\hat{\delta}_{i+1}}{\delta_{i+1}}\right)^p + \frac{F(x, \tau_{i-1}) - F(x, \tau_i)}{(p - 1) \cdot |\nabla V_\rho(x)|^p}, \quad x \in \partial\omega_i \tag{17}$$

for $i = 1, \cdots, n-1$, where $(1/|\nabla V_\rho(x)|)$ is uniformly bounded from above in Ω_ρ. It is suffices to require for a fixed value $\varepsilon \geq 0$ that

$$\left(\frac{\hat{\delta}_i}{\delta_i}\right)^p \geq \left(\frac{\hat{\delta}_{i+1}}{\delta_{i+1}}\right)^p + \Delta_i + \varepsilon \tag{18}$$

for $i = 1, \cdots, n-1$, where

$$\Delta_i := C_0 \cdot \sup_{x\in\Omega} \left(F(x, \tau_{i-1}) - F(x, \tau_i)\right) \tag{19}$$

and $C_0 = C_0(\rho) = \sup_{x\in\Omega_\rho} (1/((p-1)|\nabla V_\rho(x)|^p))$. Let the values μ_i, $i = 1, \cdots, n$ be chosen such that $\mu_i - \mu_{i+1} = \Delta_i + \varepsilon \geq 0$, $i = 1, \cdots, n-1$, and $\mu_n = 0$. Then

$$0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1 = C_T + (n-1)\varepsilon \leq C^* + (n-1)\varepsilon, \tag{20}$$

where $C_T := \sum_{i=1}^{n-1} \Delta_i$ and $C^* = C^*(\rho) = p \cdot C_0(\rho) \int_0^M (\sup_{x\in\Omega} f(x, y))\, dy$ (due to Assumption $(\mathcal{F}3)$. To define a subsolution $(\omega_1, \cdots, \omega_{n-1})$ satisfying (17), it suffices to choose

$$\delta_i = \hat{\delta}_i \cdot (\mu_i + \lambda)^{-1/p} \qquad \left(\Longleftrightarrow \left(\frac{\hat{\delta}_i}{\delta_i}\right)^p = \mu_i + \lambda\right), \tag{21}$$

for $i = 1, \cdots, n$, where $\lambda > 0$ is a constant determined by (15). Namely, the continuous function $\psi(s) := \sum_{i=1}^{n} \hat{\delta}_i \cdot (\mu_i + s)^{-1/p}$ is such that $\psi'(s) < 0$ for all $s > 0$, $\psi(s) \to \infty$ as $s \downarrow 0$, and $\psi(s) \to 0$ as $s \to \infty$. Therefore, there exists a unique value $\lambda > 0$ such that $\psi(\lambda) = 1$. Assuming (15), we have

$$1 = \psi(\lambda) = \sum_{i=1}^{n} \hat{\delta}_i \cdot (\mu_i + \lambda)^{-1/p} \leq \sum_{i=1}^{n} \hat{\delta}_i \cdot \lambda^{-1/p} = \frac{M}{\lambda^{1/p}}, \tag{22}$$

from which it follows that $\lambda \leq M^p$.

Now let us consider the case $\epsilon = 0$. Due to Property $(\mathcal{F}4)$ of f, we can choose a function $z(y)$ such that $t^{1-p} f(x, t) \leq z(y)$ for all $x \in \Omega$ and $0 < t \leq y$, and such that $z(y) \to 0+$ as $y \to 0+$. For any $m = 1, \cdots, n$, we have $\Delta_i \leq C_0 \eta^{p-1} z(\eta)(\tau_{i-1} - \tau_i)$ for all $i = m+1, \cdots, n-1$, where $\eta = \tau_m$. Thus

$$\mu_k = \sum_{i=k}^{n-1} \Delta_i \leq C_0 \eta^{p-1} z(\eta) \sum_{i=k}^{n-1} (\tau_{i-1} - \tau_i) \leq C_0 \eta^{p-1} z(\eta)(\tau_{k-1} - \tau_{n-1}) \leq C_0 \eta^p z(\eta), \tag{23}$$

for $k = m+1, \cdots, n-1$, with $\eta = \tau_m$. Thus

$$1 \geq \sum_{i=m+1}^{n} \frac{\hat{\delta}_i}{(\lambda + \mu_i)^{1/p}} \geq \sum_{i=m+1}^{n} \frac{\hat{\delta}_i}{(\lambda + C_0 \eta^p z(\eta))^{1/p}} = \frac{\eta}{(\lambda + C_0 \eta^p z(\eta))^{1/p}}. \tag{24}$$

Thus $\lambda \geq \eta^p (1 - C_0 z(\eta))$, from which it follows that $\lambda \geq (1/2)\eta^p$, provided $\eta > 0$ is small enough so that $2C_0 z(\eta) \leq 1$. By (20), (21), and the fact that $(1/2)\eta^p \leq \lambda \leq M^p$, we have

$$\frac{\eta}{2^{1/p}} \leq \lambda^{1/p} \leq \frac{\hat{\delta}_i}{\delta_i} = (\mu_i + \lambda)^{1/p} \leq (C_T + (n-1)\varepsilon + \lambda)^{1/p} \leq (C^* + (n-1)\varepsilon + M^p)^{1/p} \tag{25}$$

for all $i = 1, \cdots, n$, provided that $|T|$ is sufficiently small and $2C_0 z(\eta) \leq 1$.

Step 2. (Inner barriers for multilayer solutions) For any $\rho \in (0, 1)$ and partition $T = \{t_0, t_1, ..., t_n\}$ of $[0, M]$, we let $(\omega_1(\rho), \cdots, \omega_{n-1}(\rho))$ denote the explicit subsolution constructed in **Step 1** for the n-layer problem (Problem 2.1) corresponding to the partition T, the annular domain Ω_ρ, the function V_ρ, and the value $\varepsilon = 0$. Let $(K_1, \cdots, K_{n-1})$ denote any (fixed) solution of the n-layer problem in the original domain Ω corresponding to the same partition T. Then

$$(\omega_1(\rho), \cdots, \omega_{n-1}(\rho)) \subset (K_1, \cdots, K_{n-1}) \tag{26}$$

for any $\rho \in (0, 1)$, where "$\subset$" is interpreted componentwise.

For proof of this claim, let $(\omega_1(\rho, \varepsilon), \cdots, \omega_{n-1}(\rho, \varepsilon))$ denote the (strict) subsolution constructed in **Step 1** corresponding to the partition T, the annular domain Ω_ρ, $\rho \in (0, 1)$, and a small value $\varepsilon > 0$. There is an $r \in (0, 1)$ so small that

$$(\omega_1(r, \varepsilon), \cdots, \omega_{n-1}(r, \varepsilon)) \subset (K_1, \cdots, K_{n-1})$$

for all small $\varepsilon > 0$. We assert that the same inequality holds for all $r \in (0, 1)$ and all sufficiently small $\varepsilon > 0$. Clearly, the domains $\omega_i(r, \varepsilon)$ depend continuously on $r \in (0, 1)$ and $\varepsilon \geq 0$. Therefore, if this claim is false, then for some small $\varepsilon > 0$, there is a largest value $\rho = \rho(\varepsilon) \in (0, 1)$ such that $(\omega_1(r, \varepsilon), \cdots, \omega_{n-1}(r, \varepsilon)) \subset (K_1, \cdots, K_{n-1})$ for all $r \in (0, \rho)$. Then $(\omega_1(\rho, \varepsilon), \cdots, \omega_{n-1}(\rho, \varepsilon)) \subset (K_1, \cdots, K_{n-1})$ by continuity, and there exists a point $x_0 \in \partial\omega_i(\rho, \varepsilon) \cap \partial K_i$ for some $i \in \{1, \cdots, n-1\}$. This leads to the following contradiction: By maximum and comparison principles for p-harmonic functions, we have

$$\tau_j \leq u_j(x) \leq U_j(x) \leq \tau_{j-1}$$

in $\omega_j(\rho, \varepsilon) \setminus K_{j-1}$ for $j = i, i+1$, where u_j (resp U_j) solves the Dirichlet problem (6) (resp. (12)) with $i := j$. Thus

$$|\nabla U_i(x_0)| \leq |\nabla u_i(x_0)| \quad \text{and} \quad |\nabla U_{i+1}(x_0)| \geq |u_{i+1}(x_0)|$$

at $x_0 \in \partial\omega_i(\rho, \varepsilon) \cap \partial K_i$, contradicting the fact that $(K_1, \cdots, K_{n-1})$ is a classical solution while $(\omega_1(\rho, \varepsilon), \cdots, \omega_{n-1}(\rho, \varepsilon))$ is a strict C^2-subsoluton. Finally, our assertion follows in the limit as $\varepsilon \to 0+$.

Step 3. (Estimates for multilayer inner solutions.) Let u^T correspond to any solution $(K_1, \cdots, K_{n-1})$ of the n-layer problem (Problem 2.1) in Ω, corresponding to the partition T. Then for any $\rho \in (0, 1)$, there exist positive constants $A = A(\rho)$, $B = B(\rho)$ (independent of the particular partition) such that

$$M - B \cdot V_\rho(x) - |T| \leq u^T(x) \leq M \quad \text{in} \quad \Omega_\rho. \tag{27}$$

$$A \cdot W_\rho(x) - |T| \leq u^T(x) \leq M \quad \text{in} \quad \Omega_\rho, \tag{28}$$

where we define $W_\rho(x) := 1 - V_\rho(x)$.

Proof: Although our development of the multilayer subsolutions in **Step 1** depends on the partition T, the estimate (25) is independent of T (provided only that $|T|$ is sufficiently small to permit a suitable choice of η in (25)). Thus, we have

$$A \cdot \delta_i \leq \hat{\delta}_i \leq B \cdot \delta_i \tag{29}$$

for $i = 1, \cdots, n$, independent of T (with $|T|$ sufficiently small), where $A = A(\rho) > 0$ and $B = B(\rho) := (C^*(\rho) + M^p)^{1/p}$ are independent of T (as follows from (25) with $\varepsilon = 0$). By summation of (29), we have

$$\tau_m = \tau_m - \tau_n = \sum_{i=m+1}^{n} \hat{\delta}_i \geq A \cdot \sum_{i=m+1}^{n} \delta_i = A \cdot (1 - \alpha_m), \tag{30}$$

$$M - \tau_m = t_m = \sum_{i=1}^{m} \hat{\delta}_i \leq B \cdot \sum_{i=1}^{m} \delta_i \leq B \cdot \alpha_m, \tag{31}$$

both for any $m \in \{1, 2, \cdots, n\}$. It follows that:

$$|T| + \inf\{u^T(x) : x \in K_{m+1} \setminus \Omega_1\} \geq \inf\{u^T(x) : x \in K_m \setminus \Omega_1\} \geq u^T(\partial K_m) \quad (32)$$

$$= U^T(\partial\omega_m) = \tau_m \geq A \cdot (1 - \alpha_m) = A \cdot W_\rho(\partial\omega_m) \geq A \cdot \sup\{W_\rho(x) : x \in \Omega_{2,\rho} \setminus K_m\},$$

$$|T| + \inf\{u^T(x) : x \in K_{m+1} \setminus \Omega_1\} \geq \inf\{u^T(x) : x \in K_m \setminus \Omega_1\} = u^T(\partial K_m) \quad (33)$$

$$= U^T(\partial\omega_m) = \tau_m \geq M - B \cdot \alpha_m = M - B \cdot V_\rho(\partial\omega_m) \geq M - B \cdot \inf\{V_\rho(x) : x \in \Omega_{2,\rho} \setminus K_m\},$$

both for any partition T of $[0, M]$ and any $m \in \{0, \cdots, n-1\}$, where $\Omega_{2,\rho} = \mathrm{Cl}(\Omega_1) \cup \Omega_\rho$. Therefore, the asserted estimates (27) and (28) both hold relative to $\Omega_\rho \cap (K_{m+1} \setminus K_m)$ for each $m \in \{1, \cdots, n-1\}$.

Step 4. Proof of Theorem 3.1, eq. (8). It is easily seen, using the continuity of the function V_ρ (for any fixed $\rho \in (0, 1)$), that $B(\rho) \cdot V_\rho(x) \leq \phi(\mathrm{dist}(x, \partial\Omega_1))$ relative to Ω_ρ, where $\varphi(s) : [0, \infty) \to \mathbb{R}$ denotes a suitable continuous, monotone increasing function such that $\varphi(0) = 0$. In view of this, it follows from Step 3 that

$$M \geq u^T(x) \geq M - B \cdot V_\rho(x) - |T| \geq M - \varphi(\mathrm{dist}(x, \partial\Omega_1)) - |T|, \quad (34)$$

for any partition T of $[0, M]$, any solution u^T (corresponding to T) in Ω, and any $x \in \Omega_\rho$. By enlarging ϕ if necessary, we can assume (34) holds for all $x \in \Omega$. Therefore

$$|\nabla u^T(x)| \cdot \mathrm{dist}(x, \partial\Omega_1) \leq |\nabla u^T(x)| \cdot |\gamma| \leq \int_\gamma |\nabla u^T(y)| ds \quad (35)$$

$$\leq M - u^T(x) \leq \varphi(\mathrm{dist}(x, \partial\Omega_1)) + |T|$$

for any partition T of $[0, M]$, any solution u^T of Problem 2.1 in Ω corresponding to T, and any point $x \in \Omega \setminus (\partial K_1 \cup \cdots \cup \partial K_{n-1})$, where γ is the arc of steepest ascent of u^T joining x to $\partial\Omega_1$. Here, we have used the fact that $|\nabla u^T(x)|$ is weakly increasing (with increasing $u^T(x)$) on γ. The assertion (8) follows.

Step 5. Proof of Theorem 3.1, eq. (9). In view of (28), we have

$$u^T(x) + |T| \geq P(x) := \sup\{A(\rho)W_\rho(x) : \rho \in (0, 1), x \in \Omega_\rho\} > 0, \quad (36)$$

for any $x \in \Omega$. Therefore

$$|\nabla u^T(x)| \geq \int_\gamma |\nabla u^T(y)| ds = u^T(x)|\gamma| \geq u^T(x)\mathrm{dist}(x, \partial\Omega_2) \quad (37)$$

$$\geq (P(x) - |T|)\mathrm{dist}(x, \partial\Omega_2)$$

for any partition T of $[0, M]$, any solution u^T of Problem 2.1 in Ω corresponding to T, and any point $x \in \Omega \setminus (\partial K_1 \cup \cdots \cup \partial K_{n-1})$, where γ is the arc of steepest ascent of u^T joining x to $\partial\Omega_2$. The assertion (9) follows.

Theorem 3.2 *Let T^k be the sequence of partitions of $[0, M]$ such that $|T^k| \to 0$ when $k \to +\infty$. Then there exists a subsequence $\{u^{T^{k_m}}\}$ which converges in $W^{1,p}(\Omega) \cap C^{0,\alpha}(\Omega)$ to a limit $u_0 \in W^{1,p}(\Omega)$. Also, due to equicontinuity, the convergence holds true in $C(\overline{\Omega})$.*

Proof The proof follows from Theorem 3.1.

4 Proof of Theorem 1.2

In case $f(x,z)$ is monotone nondecreasing in z, the uniqueness follows by classical arguments. Next, we will show that the function $u_0(x)$, defined in Theorem 3.2, solves problem (1) in the weak sense, i.e. for every $\eta \in C_0^\infty(\Omega)$

$$\int_\Omega |\nabla u_0(x)|^{p-2}\nabla u_0(x)\cdot\nabla\eta(x)dx = -\int_\Omega f(x,u_0(x))\cdot\eta(x)dx. \tag{38}$$

For simplicity we denote $u^n(x) := u^{T^n}(x)$ and we assume, that $u^n \to u_0$ in $W^{1,p}(\Omega)$. Let $\{K_i^n, i = 0,...,n\}$ be the corresponding solution of the multilayer free boundary problem for the partition $T^n := \{t_0^n, t_1^n, ..., t_n^n\}$, and let $u_i^n := u^n|_{(K_i^n \setminus K_{i-1}^n)}$.

Let $\eta \in C_0^\infty(\Omega)$. By the divergence theorem we have

$$\int_\Omega |\nabla u^n(x)|^{p-2}\nabla u^n(x)\cdot\nabla\eta(x)dx = \sum_{i=1}^{n-1}\int_{\Gamma_i^n}\left(|\nabla u_{i+1}^n(x)|^{p-1} - |\nabla u_i^n(x)|^{p-1}\right)\cdot\eta(x)dx \tag{39}$$

where $\Gamma_i^n = \partial K_i^n$.

¿From the free boundary conditions we get

$$\int_\Omega |\nabla u^n(x)|^{p-2}\nabla u^n(x)\cdot\nabla\eta(x)dx =$$

$$= -\sum_{i=1}^{n-1}\int_{\Gamma_i^n}\frac{F(x,t_i^n)-F(x,t_{i-1}^n)}{p-1}\cdot\frac{|\nabla u_{i+1}^n(x)|^{p-1}-|\nabla u_i^n(x)|^{p-1}}{|\nabla u_{i+1}^n(x)|^p-|\nabla u_i^n(x)|^p}\cdot\eta(x)dx =$$

$$= -\sum_{i=1}^{n-1}\int_{\Gamma_i^n}\int_{t_{i-1}^n}^{t_i^n}\frac{p\cdot f(x,y)}{p-1}\cdot\frac{|\nabla u_{i+1}^n(x)|^{p-1}-|\nabla u_i^n(x)|^{p-1}}{|\nabla u_{i+1}^n(x)|^p-|\nabla u_i^n(x)|^p}\cdot\eta(x)dydx =$$

$$= -\sum_{i=1}^{n-1}\int_{t_{i-1}^n}^{t_i^n}\int_{\Gamma_i^n}\frac{p\cdot f(x,y)}{p-1}\cdot\frac{|\nabla u_{i+1}^n(x)|^{p-1}-|\nabla u_i^n(x)|^{p-1}}{|\nabla u_{i+1}^n(x)|^p-|\nabla u_i^n(x)|^p}\cdot\eta(x)dxdy \tag{40}$$

It is easy to see that for every $\lambda > 0$ there exist $\delta = \delta(p,\lambda) > 0$ and $C = C(p,\lambda) > 0$ such that

$$\left|\frac{b^{p-1}-a^{p-1}}{b^p-a^p} - \frac{p-1}{pa}\right| \le C|b-a| \tag{41}$$

for every $a, b \ge \lambda$ satisfying $|b-a| < \delta$.

We have that $|\nabla u_i^n(x)| = q(x, |\nabla u_{i+1}^n(x)|)$ on Γ_i^n (see (7)). Using the fact, that the function $q(x,y)$ is increasing by y, and the function $\dfrac{1}{q(x,0)}$ is concave, we can claim that $q(x,y)$ is bounded from below, that is, there exists a constant $C_0 > 0$ such that

$$|\nabla u_i^n(x)| \ge C_0, \quad x \in \Gamma_i^n.$$

¿From (5) and (41) we can conclude, that there exists a $\delta > 0$ such that if $|T^n| < \delta$ then on Γ_i^n

$$\left|\frac{|\nabla u_i^n(x)|^{p-1}-|\nabla u_{i+1}^n(x)|^{p-1}}{|\nabla u_i^n(x)|^p-|\nabla u_{i+1}^n(x)|^p} - \frac{p-1}{p|\nabla u_i^n(x)|}\right| \le C\big||\nabla u_i^n(x)| - |\nabla u_{i+1}^n(x)|\big|.$$

Using the last inequality, we can get the following estimate:

$$\left|\int_{\Gamma_i^n}\frac{p\cdot f(x,y)}{p-1}\cdot\frac{|\nabla u_{i+1}^n(x)|^{p-1}-|\nabla u_i^n(x)|^{p-1}}{|\nabla u_{i+1}^n(x)|^p-|\nabla u_i^n(x)|^p}\cdot\eta(x)dx - \int_{\Gamma_i^n}\frac{f(x,y)}{|\nabla u_i^n(x)|}\cdot\eta(x)dx\right| \le$$

$$\leq C_1 \cdot \sup_{x\in\Gamma_i^n\cap\text{supp}\eta} f(x,y)\cdot\sup|\eta(x)|\cdot\int_{\Gamma_i^n}||\nabla u_i^n(x)|-|\nabla u_{i+1}^n(x)||dx \tag{42}$$

for $y\in[t_{i-1}^n,t_i^n]$, where C_1 depends only from p and C_0.

Using the inequality

$$||\nabla u_i^n(x)|-|\nabla u_{i+1}^n(x)||\leq\frac{||\nabla u_i^n(x)|^p-|\nabla u_{i+1}^n(x)|^p|}{\min(|\nabla u_i^n(x)|^{p-1},|\nabla u_{i+1}^n(x)|^{p-1})}\leq\frac{F(x,t_i^n)-F(x,t_{i-1}^n)}{(p-1)\cdot C_0^{p-1}}$$

from (42), we conclude that

$$\left|\int_{\Gamma_i^n}\frac{p\cdot f(x,y)}{p-1}\cdot\frac{|\nabla u_{i+1}^n(x)|^{p-1}-|\nabla u_i^n(x)|^{p-1}}{|\nabla u_{i+1}^n(x)|^p-|\nabla u_i^n(x)|^p}\cdot\eta(x)dx-\int_{\Gamma_i^n}\frac{f(x,y)}{|\nabla u_i^n(x)|}\cdot\eta(x)dx\right|\leq$$

$$\leq C_2(y)\cdot\int_{\Gamma_i^n\cap\text{supp}\eta}\left(\int_{t_{i-1}^n}^{t_i^n}f(x,t)dt\right)dx\leq C_2(y)\cdot|\Gamma_i^n|\cdot\max_{x\in\Gamma_i^n\cap\text{supp}\eta}\int_{t_{i-1}^n}^{t_i^n}f(x,t)dt\leq$$

$$\leq C_2(y)\cdot|\partial\Omega_2|\cdot\int_{t_{i-1}^n}^{t_i^n}f(x_0,t)dt \tag{43}$$

for some $x_0\in\Gamma_i^n\cap\text{supp}\eta$, where $|\Gamma_i^n|$ denotes the length of Γ_i^n, and

$$C_2(y):=C_1\cdot\sup_{x\in\Gamma_i^n\cap\text{supp}\eta}f(x,y)\cdot\sup|\eta(x)|.$$

Now from the compactness of the set $\Gamma_i^n\cap\text{supp}\eta$ we can obtain, that for any small $\epsilon>0$ we can choose $\delta_1>0$ such that for all T^n satisfying $|T^n|<\delta_1$ (η is fixed)

$$\int_{t_{i-1}^n}^{t_i^n}f(x_0,t)dt<\epsilon \tag{44}$$

for all $x_0\in\Gamma_i^n\cap\text{supp}\eta$.

Finally, from (40), (43) and (44) we obtain

$$\left|\int_\Omega|\nabla u^n(x)|^{p-2}\nabla u^n(x)\cdot\nabla\eta(x)dx+\sum_{i=1}^{n-1}\int_{t_{i-1}^n}^{t_i^n}\int_{\Gamma_i^n}\frac{f(x,y)}{|\nabla u_i^n(x)|}\cdot\eta(x)dxdy\right|\leq$$

$$\leq\epsilon\cdot|\partial\Omega_2|\cdot\int_0^M C_2(y)dy\leq C_3\cdot\epsilon, \tag{45}$$

where in the last inequality we have used the property 3) of the definition of the class $\mathcal{F}$.

Just like in [LS] (see pp. 494-495) we can prove, that for small $|T^n|$

$$\left|\int_{\Gamma_i^n}\frac{f(x,y)}{|\nabla u_i^n(x)|}\cdot\eta(x)dx-\int_{u^n(x)=y}\frac{f(x,y)}{|\nabla u^n(x)|}\cdot\eta(x)dx\right|\leq\epsilon,\quad y\in[t_{i-1}^n,t_i^n] \tag{46}$$

Combining (45) and (46), we obtain

$$\int_\Omega|\nabla u^n(x)|^{p-2}\nabla u^n(x)\cdot\nabla\eta(x)dx=-\sum_{i=1}^{n-1}\int_{t_{i-1}^n}^{t_i^n}\int_{u^n(x)=y}\frac{f(x,y)\eta(x)}{|\nabla u^n(x)|}dxdy+o(1)=$$

$$=-\int_0^M\int_{u^n(x)=y}\frac{f(x,y)\eta(x)}{|\nabla u^n(x)|}dxdy+o(1)=-\int_\Omega f(x,u^n(x))\cdot\eta(x)dx+o(1). \tag{47}$$

Since $f(x,y)$ is uniformly continuous in $\text{supp}\eta \times [0,M]$, we can claim that

$$\left| \int_\Omega f(x,u^n(x)) \cdot \eta(x)dx - \int_\Omega f(x,u(x)) \cdot \eta(x)dx \right| \leq \epsilon$$

for $n > n_0$. For the first integral in (47), we have, due to local-uniform Lipschitz estimates of u^n, that, for a subsequence, $|\nabla u^n|^{p-2}\nabla u^n$ converges weakly (in $L^{p/(p-1)}$) to $|\nabla u|^{p-2}\nabla u$.

5 A uniqueness result

Theorem 5.1 *Let $u(x) : \overline{\Omega} \to \mathbb{R}$ denote a classical solution of the Dirichlet problem (1) with $f \geq 0$, and let $v(x) : \overline{\Omega} \to \mathbb{R}$ denote a classical solution of the same Dirichlet problem (1) with f replaced by a function $g \geq 0$ (we assume $0 < u, v < M$, from which it follows that $|\nabla u|, |\nabla v| > 0$, both in Ω). Then $u \leq v$ in Ω provided that $0 < g(x,y) \leq f(x,y)$ and that $g(x,y) < t^p g(tx,y)$ for all $x \in \Omega$, $0 < y < M$, and $t > 1$ for which the inequality is meaningfull.*

Proof. (See [A].) We set $v_t(x) = v(x/t)$ in $\Omega_t = t \cdot \Omega$ for any $t > 0$, observing that $\Delta_p v_t = (1/t)^p g(x/t, v_t(x))$ in Ω_t by change of variables. It is easy to see that $v_t > u$ (resp. $v_t < u$) in $\Omega \cap \Omega_t$ for any sufficiently large (small) $t > 0$. Since v_t depends continuously on t, we can choose $t > 0$ to be minimum subject to the requirement that $v_\tau \geq u$ in $\Omega \cap \Omega_\tau$ for all $\tau \geq t$. We claim that $t \leq 1$. Assuming that $t > 1$, it is easy to see that $v_t > u$ on $\partial(\Omega \cap \Omega_t)$ and that $v_t(x_0) = u(x_0)$ for some point $x_0 \in (\Omega \cap \Omega_t)$. Thus $\Delta_p v_t(x_0) \geq \Delta_p u(x_0)$, and we conclude using the final assumption on g that

$$g(x_0, y_0) > (1/t)^p g(x_0/t, v_t(x_0)) = \Delta_p v_t(x_0) \geq \Delta_u(x_0) = f(x_0, u(x_0)) = f(x_0, y_0),$$

where $y_0 = u(x_0) = v_t(x_0)$. However, this violates the assumption that $g \leq f$.

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